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Investigation of Laplace-Stieltjes transform for the ergodic distribution of the semi-markov random process with positive tendency, negative jumps and delaying boundary at zero

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Abstract: One of the important problems of stochastic processes theory is to define the Laplace-Stieltjes transform for the ergodic distribution of semi-markov random process. With this purpose, we will investigate the semi-markov random processes with positive tendency, negative jumps and delaying boundary at zero in this article. The Laplace transform on time, Laplace-Stieltjes transform on phase of the conditional and unconditional distributions and Laplace-Stieltjes transform of the ergodic distribution are defined. The characteristics of the ergodic distribution will be calculated on the basis of the final results.

Keywords: Laplace-Stieltjes transform; semi-markov random process; ergodic distribution; process with positive tendency and negative jumps.

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Introduction

There are number of works devoted to definition of the distribution of the semi-markov processes and its main boundary functionals. Some authors are used the asymptotic, factorization and etc. methods ([2],[4],[5],[6],[9][12]) But other authors narrowing the class of distributions of walking are found the evident form for Laplace transforms for distributions and its main characteristics. In [7] The Laplace transformation for the distribution of the time of the system sojourn within a given band and its first and second moments are found .In [8] a model of inventory control is considered. It is described by a semi-markov random walk with a negative drift at an angle of 0 < α < 90°, with positive random jumps, a delay, an absorbing screen at zero, and a reflecting screen for a > 0 at an angle α. The Laplace transformation is found for the distribution of the first moment storehouse exhaustion, and the first and the second moments are explicitly obtained. In [9] The Laplace-Stieltjes transform with respect to phase, the Laplace transform with respect to time, the conditional distribution, the unconditional distribution, and the Laplace-Stieltjes transform of the ergodic distribution of the process of semi-markov random walk with negative drift, nonnegative jumps, delays, and boundary screen at zero are obtained. In [10] The first passage of the zero level of the semi-markov process with positive tendency and negative jumps will be included as a random variable. The Laplace transform for the distribution of this random variable is defined. In [11] for the step process of semi-markov random walk with delaying boundary in $a > 0$ the evident form of Laplace transform by time was found.

The presented work explicitly defines the Laplace transform on time, Laplace-Stieltjes transform on phase of the conditional and unconditional distributions and Laplace-Stieltjes transform of the ergodic distribution for the semi-markov random processes with positive tendency, negative jumps and delaying boundary at zero.

1. Problem

Let's assume that in probability space $\{\Omega, F, P(\cdot)\}$ is given the sequence of independent , equally distributed and independent themselves positive random variables ξ_k and ζ_k, $k = \overline{1,\infty}$. Using these random variables we will derive the following semi-markov random process:

$$X_1(t) = z + t - \sum_{i=1}^{k-1} \zeta_i , \qquad \text{if} \qquad \sum_{i=1}^{k-1} \xi_i \le t < \sum_{i=1}^{k} \xi_i , \quad k = \overrightarrow{1,\infty}$$

$X_1(t)$ is called semi-markov random processes with positive tendency and negative jumps.

General form of process semi-markov random walk with delaying boundary is given by A.A. Borovkov[1]

If process $X_1(t)$ is some process without boundary, then process $X(t)$ with delaying boundary at zero is defined following :

$$X(t) = X_1(t) - \inf_{0 \le S \le t} (0, X_1(s)) \qquad \text{or} \qquad X(t) = \max(0, \inf_{0 \le S \le t} (0, X_1(s)))$$

Idea of construction of the process semi-markov random walk is following :

Let $X_1(0) = z \ge 0$. Process $X(t)$ is equally to process $X_1(t)$ until, the process $X_1(t)$ is positive.

Let $X_1(t) \le 0$; then $X(t)$ is equally to zero until, the process $X_1(t)$ will not have positive jump. In moment of jump of the process $X_1(t)$, process $X(t)$ will be have jump, such is equally to jump of the process $X_1(t)$.

The obtained process is called a process of a semi-markov random walk with positive tendency, negative jumps and delaying boundary at zero.

The aim of the present study is to find an evident form of the Laplace-Stieltjes transform of the ergodic distribution for X(t).

2. Definition of Laplace transform on time for the distribution of the process X(t)

In accordance with formula of total probability for $x \ge 0$ we have

$$P\{X(t) < x \mid X(0) = z\} = P\{X(t) < x; \xi_1 > t \mid X(0) = z\} + P\{X(t) < x; \xi_1 < t \mid X(0) = z\} =$$

$$= P\{z + t < x; \xi_1 > t\} + \int_{s=0}^{\infty} \int_{y=0}^{\infty} P\{\xi_1 \in ds; X(s) \in dy \mid X(0) = z\} \cdot P\{X(t-s) < x \mid X(0) = y\}$$

(1)

We denote

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$$\begin{cases} R(t,x\,|\,z) = P\{X(t) < x\,|\,X(0) = z\}, & x > 0 \\ \widetilde{R}(\theta,x\,|\,z) = \int\limits_{t=0}^{\infty} e^{-\theta t} R(t,x\,|\,z), & \theta > 0 \\ \approx \\ R(\theta,\alpha\,|\,z) = \int\limits_{0}^{\infty} e^{-\alpha x} d_x\,\bar{R}(t,x\,|\,z), & \alpha > 0. \end{cases} ,$$

(2)

In this case equation (1) will be as follows :

$$R(t,x\,|\,z) = P\{z+t-x<0\}P\{\xi_1 > t\} + \int\limits_{y=0}^{\infty}\int\limits_{s=0}^{t} P\{\xi_1 \in ds\}\, d_y P\{\max(0, z+s-\zeta_1) < y\}\, \mathrm{R(t-s,x\,|\,y)}$$

Both sides of this equation we applied Laplace transform by “t”

$$\int\limits_0^{\infty} e^{-\theta t} R(t,x\,|\,z)dt = \int\limits_0^{\infty} e^{-\theta t}\,\varepsilon(x-z-t)P\{\xi_1 > t\}dt + \int\limits_{y=0}^{\infty}\widetilde{R}(\theta,x\,|\,y)\int\limits_{t=0}^{\infty} e^{-\theta t}\, d_y\, \mathrm{P}\{\max(0,z+t-\zeta_1)<y\}\mathrm{dP}\{\xi_1 < t\}$$

where, $\theta > 0$.

Further

$$\widetilde{R}(\theta,x\,|\,z) = \varepsilon(x-z)\int\limits_0^{x-z} e^{-\theta t}P\{\xi_1 > t\}dt + \int\limits_{y=0}^{\infty}\widetilde{R}(\theta,x\,|\,y)\int\limits_{t=0}^{\infty} e^{-\theta t}\, d_y\, \mathrm{P}\{0<y\}\mathrm{P}\{z+t-\zeta_1<y\}\mathrm{d\,P}\{\xi_1 < t\}$$

After some simplifications we will get:

$$\widetilde{R}(\theta,x\,|\,z) = \varepsilon(x-z)\int\limits_0^{x-z} e^{-\theta t}P\{\xi_1 > t\}dt + \int\limits_{y=0}^{\infty}\widetilde{R}(\theta,x\,|\,y)\int\limits_{t=0}^{\infty} e^{-\theta t} d_y \varepsilon(y)[1-\mathrm{P}\{\zeta_1 < z+t-y\}]\mathrm{dP}\{\xi_1 < t\} =$$

$$= \varepsilon(x-z)\int\limits_0^{x-z} e^{-\theta t}P\{\xi_1 > t\}dt + \widetilde{R}(\theta,x\,|\,0)\int\limits_0^{\infty} e^{-\theta t}\,\mathrm{dP}\{\xi_1 < t\} - \int\limits_{y=0}^{\infty}\widetilde{R}(\theta,x\,|\,y)\int\limits_{t=0}^{\infty} e^{-\theta t} d_y \mathrm{P}\{\zeta_1 < z+t-y\}\mathrm{dP}\{\xi_1 < t\} =$$

$$= \varepsilon(x-z)\int\limits_0^{x-z} e^{-\theta t}P\{\xi_1 > t\}dt + \widetilde{R}(\theta,x\,|\,0)\int\limits_0^{\infty} e^{-\theta t}\,\mathrm{dP}\{\xi_1 < t\} - \int\limits_{y=0}^{\infty}\widetilde{R}(\theta,x\,|\,y)\int\limits_{t=\max(0,y-z)}^{\infty} e^{-\theta t} d_y \mathrm{P}\{\zeta_1 < z+t-y\}\mathrm{dP}\{\xi_1 < t\}$$

If take into account

$$\max\{0, y-z\} = \begin{cases} 0, & \text{if } \mathrm{y < z} \\ \mathrm{y - z}, & \text{if } \mathrm{y > z} \end{cases}$$

that is, why we get

$$\widetilde{R}(\theta,x\,|\,z) = \varepsilon(x-z)\int\limits_0^{x-z} e^{-\theta t}P\{\xi_1 > t\}dt +$$

$$+\widetilde{R}(\theta,x\,|\,0)\int\limits_0^{\infty} e^{-\theta t}\,\mathrm{dP}\{\xi_1 < t\} -$$

$$-\int_{y=0}^{z}\tilde{R}(\theta,x\,|\,y)\int_{t=0}^{\infty}e^{-\theta t}d_{y}\mathrm{P}\{\zeta_1<z+t-y\}\mathrm{dP}\{\xi_1<t\}-$$

$$-\int_{y=z}^{\infty}\tilde{R}(\theta,x\,|\,y)\int_{t=y-z}^{\infty}e^{-\theta t}d_{y}\mathrm{P}\{\zeta_1<z+t-y\}\mathrm{dP}\{\xi_1<t\} \qquad \textbf{(3)}$$

Both sides of this equation we applied Laplace transform by “x ” [see (2)]

$$\approx\!\!\!\!R(\theta,\alpha\,|\,\mathrm{z})=\int_{x=0}^{\infty}e^{-\alpha x}d_{x}\varepsilon(x-z)\int_{t=0}^{x-z}e^{-\theta t}P\{\xi_1>t\}dt+$$

$$+\approx\!\!\!\!R(\theta,\alpha\,|\,0)\int_{0}^{\infty}e^{-\theta t}\,\mathrm{dP}\{\xi_1<t\}-$$

$$-\int_{y=0}^{z}\approx\!\!\!\!R(\theta,\alpha\,|\,\mathrm{y})\int_{t=0}^{\infty}e^{-\theta t}d_{y}\mathrm{P}\{\zeta_1<z+t-y\}\mathrm{dP}\{\xi_1<t\}-$$

$$-\int_{y=z}^{\infty}\approx\!\!\!\!R(\theta,\alpha\,|\,\mathrm{y})\int_{t=y-z}^{\infty}e^{-\theta t}d_{y}\mathrm{P}\{\zeta_1<z+t-y\}\mathrm{dP}\{\xi_1<t\}$$

Take into consideration

$$\int_{x=0}^{\infty}e^{-\alpha x}d_{x}\varepsilon(x-z)\int_{t=0}^{x-z}e^{-\theta t}P\{\xi_1>t\}dt=\int_{x=0}^{\infty}e^{-\alpha x}\varepsilon(x-z)\,d_{x}\int_{t=0}^{x-z}e^{-\theta t}P\{\xi_1>t\}dt+\int_{x=0}^{\infty}e^{-\alpha x}\int_{t=0}^{x-z}e^{-\theta t}P\{\xi_1>t\}dt\,d_{x}\varepsilon(x-z)$$

$$=\int_{x=z}^{\infty}e^{-\alpha x}d_{x}\int_{t=0}^{x-z}e^{-\theta t}P\{\xi_1>t\}dt=\int_{z}^{\infty}e^{-\alpha x}e^{-\theta(x-z)}P\{\xi_1>x-z\}dx$$

At last we received the following integral equation for $\approx\!\!\!\!R(\theta,\alpha\,|\,\mathrm{z})$ when ξ_k and ζ_k, $k=\overline{1,\infty}$ equally distributed and independent themselves positive random variables

$$\approx\!\!\!\!R(\theta,\alpha\,|\,\mathrm{z})=e^{\theta z}\int_{z}^{\infty}e^{-(\alpha+\theta)x}P\{\xi_1>x-z\}dx+$$

$$+\approx\!\!\!\!R(\theta,\alpha\,|\,0)\int_{0}^{\infty}e^{-\theta t}\mathrm{P}\{\zeta_1>z+t\}\mathrm{dP}\{\xi_1<t\}-$$

$$-\int_{y=0}^{z}\approx\!\!\!\!R(\theta,\alpha\,|\,y)\int_{t=0}^{\infty}e^{-\theta t}d_{y}\mathrm{P}\{\zeta_1<z+t-y\}\mathrm{dP}\{\xi_1<t\}-$$

$$-\int_{y=z}^{\infty}\approx\!\!\!\!R(\theta,\alpha\,|\,y)\int_{t=y-z}^{\infty}e^{-\theta t}d_{y}\mathrm{P}\{\zeta_1<z+t-y\}\mathrm{dP}\{\xi_1<t\}$$

(4)

We will solve this integral equation in special case.

Let’s assume that ξ_1 random variable has the Erlangian distribution of n order, while ζ_1 random variable has the single order Erlangian distribution:

$$P\{\xi_1(\omega)<t\}=\left\{1-\left[1+\mu t+\frac{(\mu t)^2}{2!}+\cdots+\frac{(\mu t)^{n-1}}{(n-1)!}\right]e^{-\mu t}\right\}\varepsilon(t)\,, \qquad \mu>0$$

$$\mathrm{P}\{\zeta_1(\omega)<t\}=\left[1-e^{-\lambda t}\right]\varepsilon(t)\,, \qquad \lambda>0$$

where

$$\varepsilon(t)=\begin{cases}0, & t<0\\ 1, & t>0\end{cases}$$

In this case equation (4) will be as follows:

$$\overset{\approx}{R}(\theta,\alpha\,|\,z)=\frac{(\alpha+\mu+\theta)^n-\mu^n}{(\alpha+\mu+\theta)^n\,(\alpha+\theta)}e^{-\alpha z}+\left[\frac{\mu}{\lambda+\mu+\theta}\right]^n e^{-\lambda z}\,\overset{\approx}{R}(\theta,\alpha\,|\,0)-$$
$$-\lambda\left[\frac{\mu}{\lambda+\mu+\theta}\right]^n e^{-\lambda z}\int_0^z e^{\lambda y}\,\overset{\approx}{R}(\theta,\alpha\,|\,y)dy-$$
$$-\frac{\lambda\mu^n}{(n-1)!}e^{-\lambda z}\int_{y=z}^{\infty}e^{\lambda y}\,\overset{\approx}{R}(\theta,\alpha\,|\,y)\int_{t=y-z}^{\infty}e^{-(\lambda+\mu+\theta)t}\,t^{n-1}dtdy \qquad \textbf{(5)}$$

We will get differential equation from this integral equation. For this purpose, we will multiply both sides of equation (5) by $e^{\lambda z}$ and derive on z . Then we will multiply both sides of last equation by $e^{-(\lambda+\mu+\theta)z}$ and derive on z . If repeat this process (n-1) time we have following differential equation**:**

$$\sum_{k=o}^{n}C_n^k\left[\lambda\,\overset{\approx}{R}^{(k)}(\theta,\alpha\,|\,z)+\overset{\approx}{R}^{(k+1)}(\theta,\alpha\,|\,z)\right](-1)^{n-k}(\mu+\theta)^{n-k}+(-1)^{n-1}\lambda\mu^n\,\overset{\approx}{R}(\theta,\alpha\,|\,z)=$$
$$=(-1)^{n-1}\frac{[(\alpha+\mu+\theta)^n-\mu^n](\lambda-\alpha)}{(\alpha+\theta)}e^{-\alpha z} \qquad \textbf{(6)}$$

3. The general solution of the differential equation (6)

The general solution of this differential equation will be

$$\overset{\approx}{R}(\theta,\alpha\,|\,z)=C_1(\theta,\alpha)e^{k_1(\theta)z}+C_2(\theta,\alpha)e^{k_2(\theta)z}+\cdots+C_n(\theta,\alpha)e^{k_n(\theta)z}+\overset{\approx}{R}_{sp}(\theta,\alpha\,|\,z) \qquad \textbf{(7)}$$

where

$k_i(\theta)\,,\quad i=1,2,...n,$ -are the roots of characteristic equation of (6)

$\overset{\approx}{R}_{sp}(\theta,\alpha\,|\,z)$ - is the special solution of the equation (5)

$$\overset{\approx}{R}_{sp}(\theta,\alpha\,|\,z)=A\,e^{-\alpha z}$$

where

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$$A=\frac{(\lambda-\alpha)[(\alpha+\mu+\theta)^{n}-\mu^{n}]}{(\alpha+\theta)\prod_{i=1}^{n}[\alpha+k_i(\theta)]}$$

$$\begin{cases}
\overset{\approx}{R}(\theta,\alpha\,|\,0)=\dfrac{(\alpha+\mu+\theta)^{n}-\mu^{n}}{(\alpha+\mu+\theta)^{n}\,(\alpha+\theta)}+\left[\dfrac{\mu}{\lambda+\mu+\theta}\right]^{n}\overset{\approx}{R}(\theta,\alpha\,|\,0)-\dfrac{\lambda\mu^{n}}{(n-1)!}\displaystyle\int_{y=0}^{\infty}e^{\lambda y}\,\overset{\approx}{R}(\theta,\alpha\,|\,y)\int_{t=y}^{\infty}e^{-(\lambda+\mu+\theta)t}t^{n-1}dtdy\\
\overset{\approx}{R}'(\theta,\alpha\,|\,0)=-\alpha\dfrac{(\alpha+\mu+\theta)^{n}-\mu^{n}}{(\alpha+\mu+\theta)^{n}\,(\alpha+\theta)}-\lambda\left[\dfrac{\mu}{\lambda+\mu+\theta}\right]^{n}\overset{\approx}{R}(\theta,\alpha\,|\,0)+\dfrac{\lambda^{2}\mu^{n}}{(n-1)!}\displaystyle\int_{y=0}^{\infty}e^{\lambda y}\,\overset{\approx}{R}(\theta,\alpha\,|\,y)\int_{t=y}^{\infty}e^{-(\lambda+\mu+\theta)t}t^{n-1}dtdy-\\
\qquad-\dfrac{\lambda\mu^{n}}{(n-1)!}\displaystyle\int_{0}^{\infty}e^{-(\mu+\theta)y}\,\overset{\approx}{R}(\theta,\alpha\,|\,y)\,y^{n-1}dy\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\displaystyle\sum_{k=0}^{n-1}C_n^k\left[\lambda\,\overset{\approx}{R}^{(k)}(\theta,\alpha\,|\,0)+\overset{\approx}{R}^{(k+1)}(\theta,\alpha\,|\,0)\right]=(-1)^{n-k}\dfrac{[(\alpha+\mu+\theta)^{n}-\mu^{n}](\lambda-\alpha)}{(\alpha+\theta)}-(-1)^{n-1}\lambda\mu^{n}\displaystyle\int_{z}^{\infty}e^{-(\mu+\theta)y}\,\overset{\approx}{R}(\theta,\alpha\,|\,y)dy
\end{cases}$$

(8)

By finding $C_1(\theta,\alpha)$ …… $C_n(\theta,\alpha)$ from equation (5) we will get the following system of algebraic equations:

By

exploitation of equation (7), equation (8) becomes

$$\begin{cases}
\displaystyle\sum_{i=1}^{n}C_i(\theta,\alpha)+A=\dfrac{(\alpha+\mu+\theta)^{n}-\mu^{n}}{(\alpha+\mu+\theta)^{n}\,(\alpha+\theta)}+\left[\dfrac{\mu}{\lambda+\mu+\theta}\right]^{n}\left[\sum_{i=1}^{n}C_i(\theta,\alpha)+A\right]-\\
\qquad-\dfrac{\lambda\mu^{n}}{(n-1)!}\displaystyle\int_{y=0}^{\infty}e^{\lambda y}\left[\sum_{i=1}^{n}C_i(\theta,\alpha)e^{k_i(\theta)y}+Ae^{-\alpha y}\right]\int_{t=y}^{\infty}e^{-(\lambda+\mu+\theta)t}t^{n-1}dtdy\\
\displaystyle\sum_{i=1}^{n}C_i(\theta,\alpha)k_i(\theta)-\alpha A=-\alpha\dfrac{(\alpha+\mu+\theta)^{n}-\mu^{n}}{(\alpha+\mu+\theta)^{n}\,(\alpha+\theta)}-\lambda\left[\dfrac{\mu}{\lambda+\mu+\theta}\right]^{n}\left[\sum_{i=1}^{n}C_i(\theta,\alpha)+A\right]+\\
\qquad+\dfrac{\lambda^{2}\mu^{n}}{(n-1)!}\displaystyle\int_{y=0}^{\infty}e^{\lambda y}\left[\sum_{i=1}^{n}C_i(\theta,\alpha)e^{k_i(\theta)y}+Ae^{-\alpha y}\right]\int_{t=y}^{\infty}e^{-(\lambda+\mu+\theta)t}t^{n-1}dtdy-\\
\qquad-\dfrac{\lambda\mu^{n}}{(n-1)!}\displaystyle\int_{0}^{\infty}e^{-(\mu+\theta)y}\left[\sum_{i=1}^{n}C_i(\theta,\alpha)e^{k_i(\theta)y}+Ae^{-\alpha y}\right]y^{n-1}dy\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\displaystyle\sum_{k=0}^{n}C_n^k\left\{\lambda\left[\sum_{i=1}^{n}k^{n}C_i(\theta,\alpha)+\alpha^{n}A\right]+\left[\sum_{i=1}^{n}k^{n+1}C_i(\theta,\alpha)+\alpha^{n+1}A\right]\right\}=(-1)^{n-1}\alpha^{n-1}\dfrac{[(\alpha+\mu+\theta)^{n}-\mu^{n}](\lambda-\alpha)}{(\alpha+\theta)}-\\
\qquad-(-1)^{n-1}\lambda\mu^{n}\displaystyle\int_{z}^{\infty}e^{-(\mu+\theta)y}\left[\sum_{i=1}^{n}C_i(\theta,\alpha)e^{k_i(\theta)y}+Ae^{-\alpha y}\right]dy
\end{cases}$$

(9)

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Now we proof linear dependence of this algebraic system

If to consider the following substitutions:

$$\prod_{i=1}^{n}\left[\mu+\theta-k_i(\theta)\right]=(-1)^{n+1}\lambda\mu^n$$

$$\prod_{i=1}^{n}\left[\lambda+k_i(\theta)\right]=(-1)^{n+1}\lambda\mu^n$$

$$\prod_{i=1}^{n}\left[\mu+\theta-k_i(\theta)\right]=(-1)^{n+1}\left[\lambda+k_j(\theta)\right]\left[\mu+\theta-k_j(\theta)\right]^n=(-1)^{n+1}\lambda\mu^n$$

$$\prod_{i=1}^{n}\left[\alpha+k_i(\theta)\right]=(\lambda-\alpha)(\mu+\theta+\alpha)^n-\lambda\mu^n$$

$$\int_y^{\infty}t^{n-1}e^{-(\lambda+\mu+\theta)t}dt=\frac{y^{n-1}}{\lambda+\mu+\theta}+\sum_{k=2}^{n}\frac{(n-1)!}{(n-k)!}\frac{y^{n-k}}{(\lambda+\mu+\theta)^k}e^{-(\lambda+\mu+\theta)y}$$

(10)

equation (9) becomes:

$$\begin{cases}\sum_{i=1}^{n}\left\{\left[\mu+\theta-k_i(\theta)\right]^n-\mu^n\right\}C_i(\theta,\alpha)=\dfrac{\alpha\mu^n\left[(\alpha+\mu+\theta)^n-\mu^n\right]}{(\alpha+\theta)\left[\lambda\mu^n-(\lambda-\alpha)(\mu+\theta+\alpha)^n\right]}=\alpha\mu^n\ \mathrm{A}\\ \sum_{i=1}^{n}\left\{\left[\mu+\theta-k_i(\theta)\right]^n-\mu^n\right\}C_i(\theta,\alpha)=\dfrac{\alpha\mu^n\left[(\alpha+\mu+\theta)^n-\mu^n\right]}{(\alpha+\theta)\left[\lambda\mu^n-(\lambda-\alpha)(\mu+\theta+\alpha)^n\right]}=\alpha\mu^n\ \mathrm{A}\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \sum_{i=1}^{n}\left\{\left[\mu+\theta-k_i(\theta)\right]^n-\mu^n\right\}C_i(\theta,\alpha)=\dfrac{\alpha\mu^n\left[(\alpha+\mu+\theta)^n-\mu^n\right]}{(\alpha+\theta)\left[\lambda\mu^n-(\lambda-\alpha)(\mu+\theta+\alpha)^n\right]}=\alpha\mu^n\ \mathrm{A}\end{cases}$$

(11)

Thus, (11) is a linear dependence equations system, as

$$C_2(\theta,\alpha)=C_3(\theta,\alpha)=\cdots=C_n(\theta,\alpha)=0$$

Then we have

$$C_1(\theta,\alpha)=\frac{\alpha\mu^n\ \left[(\alpha+\mu+\theta)^n-\mu^n\right]}{(\alpha+\theta)(\left[\mu+\theta-k_1(\theta)\right]^n-\mu^n)\left[\lambda\mu^n-(\lambda-\alpha)(\mu+\theta+\alpha)^n\right]}\tag{12}$$

Then the general solution of integral equation (5) will be as follows:

$$\overset{\approx}{R}(\theta,\alpha\mid z)=\frac{\alpha\mu^n\ \left[(\alpha+\mu+\theta)^n-\mu^n\right]}{(\alpha+\theta)(\left[\mu+\theta-k_1(\theta)\right]^n-\mu^n)\left[\lambda\mu^n-(\lambda-\alpha)(\mu+\theta+\alpha)^n\right]}e^{k_1(\theta)z}+$$

$$+\frac{(\lambda-\alpha)\left[(\alpha+\mu+\theta)^n-\mu^n\right]}{(\alpha+\theta)\left[\left[\lambda\mu^n-(\lambda-\alpha)(\mu+\theta+\alpha)^n\right]\right]}e^{-\alpha z}$$

(13)

This expression is the Laplace transform on time, Laplace-Stieltjes transform on phase for **conditional** distribution of the process $X(t)$

4. Ergodic distribution of the process.

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We will need to find Laplace transform on time, Laplace-Stieltjes transform on phase for **unconditional** distribution of the process $X(t)$.

From construction process $X(t)$ is seen that

$$X(0) = X_1(0) = \xi_1(\omega)$$

Then we will get

$$\overset{\approx}{R}(\theta,\alpha) = \int_0^\infty \overset{\approx}{R}(\theta,\alpha \mid z)\,dP\{X(0) < z\}$$

Therefore

$$\overset{\approx}{R}(\theta,\alpha) = \int_{z=0}^{\infty} [\frac{\alpha\mu^n\ [(\alpha+\mu+\theta)^n - \mu^n]}{(\alpha+\theta)\left([\mu+\theta-k_1(\theta)]^n - \mu^n\right)\left[\lambda\mu^n - (\lambda-\alpha)(\mu+\theta+\alpha)^n\right]} e^{k_1(\theta)z} + \\ + \frac{(\lambda - \alpha)[(\alpha+\mu+\theta)^n - \mu^n]}{(\alpha+\theta)[\left[\lambda\mu^n - (\lambda-\alpha)(\mu+\theta+\alpha)^n\right]]} e^{-\alpha z}]\, d[1 - e^{-\mu z}]$$

or

$$\overset{\approx}{R}(\theta,\alpha) = \frac{\alpha\mu^n\ [(\alpha+\mu+\theta)^n - \mu^n]}{(\alpha+\theta)\left([\mu+\theta-k_1(\theta)]^n - \mu^n\right)\left[\lambda\mu^n - (\lambda-\alpha)(\mu+\theta+\alpha)^n\right]} \frac{\mu}{[\mu - k_1(\theta)]} + \\ + \frac{(\lambda - \alpha)[(\alpha+\mu+\theta)^n - \mu^n]}{(\alpha+\theta)\left[\lambda\mu^n - (\lambda-\alpha)(\mu+\theta+\alpha)^n\right]} \frac{\mu}{[\alpha+\mu]}$$

This expression is Laplace transform on time, Laplace-Stieltjes transform on phase for **unconditional** distribution of the process $X(t)$.

Now, we will find Laplace-Stieltjes transform for ergodic distribution of the process $X(t)$.

In [3] (see p.363) proved a general theorem on the ergodicity of the process semi-markov random walk. The process described in this article a special case of this process.

Process $X(t)$ will be ergodic, if $E\xi_1 < E\zeta_1$, or

$$\frac{1}{\mu} < \frac{n}{\lambda} \Rightarrow \lambda < n\mu$$

If process $X(t)$ ergodic,then we can use Tauber's theorem [4]

$$Ee^{-\alpha X(\omega)} = \tilde{R}(\alpha) = \lim_{\theta\to 0} \theta \overset{\approx}{R}(\theta,\alpha)$$

We obtined

$$\tilde{R}(\alpha) = \frac{1}{(n-1)!} \frac{[(\alpha+\mu)^n - \mu^n]\,\mu}{[\lambda\mu^n + (\alpha-\lambda)(\alpha+\mu)^n]} \qquad \textbf{(14)}$$

Expression (14) is Laplace-Stieltjes transform for ergodic distribution of the process $X(t)$.Respectively, we will get the following characteristics for $\lambda < n\mu$:

$$\tilde{R}'(0) = -EX(\omega) = \frac{n}{\lambda - n\mu}, \qquad \lambda < n\mu$$

$$\tilde{R}''(0) - [\tilde{R}'(0)]^2 = DX(\omega) = \frac{2n}{(\lambda - n\mu)^n}, \qquad \lambda < n\mu$$

5. Conclusions

In this article we have defined Laplace transforms on time, Laplace-Stieltjes transforms on phase for conditional and unconditional distributions and Laplace-Stieltjes transform for the ergodic distribution**.**

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