



Cone-Henig Subdifferentials of Set-Valued Maps in Locally Convex Spaces

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ABSTRACT

In locally convex spaces, the concepts of cone-Henig subgradient and cone-Henig subdifferential for the set-valued mapping are introduced through the linear functionals. The theorems of existence for Henig efficient point and cone-Henig subdifferential are proposed, and the sufficient and necessary condition for a linear functional being a cone-Henig subgradient is established.

Keywords

Set-valued mapping; Henig efficiency; Subgradient; Subdifferential; Stability.

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1. INTRODUCTION

In the past several decades there has been lots of emphasis on the study of vector set-valued optimization. It is well known that the subgradient and subdifferential play very important role in nonsmooth analysis and optimization theory. Especially, since Tanino[1] introduced the concept of weak subdifferentials for set-valued mappings, there are a great deal of investigation results relating subdifferential to set-valued optimization, for example: Baier [2], Bigi [3], Flores-Bazan [4], Lin [5], Song [6-7] and Taa [8-10], etc. The (weak) subdifferentials for set-valued maps in above references are the set of a kind of linear operators. On the other hand, the (weak) efficient solution is a kind of extremely efficient

solutions in vector optimization. Since the range of the set of (weak) efficient solutions is often too large, contracting the solution range is a basic topic in vector optimization. For this purpose, many kinds of proper efficiency has been presented, such as Benson [11], Borwein [12], Henig [13], and Luc [14], etc. The Henig proper efficiency is an important proper efficiency. It is worthy to notice that the super efficiency (introduced by Borwein [12]) equals to the Henig efficiency when the convex cone has a bounded base. Recently, Gong [15-16] studied the optimality conditions for Henig proper efficient solutions for vector set-valued optimization.

The aim of this paper is to investigate the Henig proper efficiency in the view of subdifferential in the locally convex spaces. The subdifferential introduced in this note is different from Tanino's in 1992, it is the set of a kind of linear functionals. This paper is organized as follows. In Section 2, we recall some definitions and Lemmas, which are needed in this paper. In addition, a existence theorem for Henig efficient point is proposed. Then, in section 3, the concepts of the generalized gradient and subdifferential in sense of Henig efficiency are introduced, and we obtain the existence condition of the Henig subdifferential to set-valued map.

2. Henig Proper Efficiency

In this note, it is assumed that X and Y are two locally convex spaces with topological duals X^* and Y^* , respectively. The origin of Y is denoted by 0_Y , and the neighborhood family of 0_Y denoted by $N(0_Y)$. When no confusion can arise, we write 0 instead of 0_Y . For a set $A \subset Y$, we write

$$\text{cone}(A) = \{\lambda a : \lambda \geq 0, a \in A\}.$$

The closure and interior of a set A are denoted by $cl(A)$ and $\text{int}(A)$. A set $A \subset Y$ is a cone if $\lambda A \subset A$, $\forall \lambda \geq 0$. In the sequel, we always assume that D is a pointed closed convex cone in Y with $\text{int}(D) \neq \emptyset$. The cone D induces a partially ordering of Y . Let D^+ and D^{+i} be the dual cone and strictly dual cone of convex cone D , defined by

$$D^+ = \{f \in Y^* : f(y) \geq 0 \text{ for all } y \in D\},$$

$$D^{+i} = \{f \in D^+ : f(y) > 0 \text{ for all } y \in D \setminus \{0\}\}.$$

A nonempty convex subset B of the convex cone D is called a base of D if $D = \text{cone}B$ and $0 \notin clB$. In this paper, it is always assumed that B is a base of D . Set

$$D^\Delta(B) = \{f \in D^{+i} : \text{there exists } t > 0 \text{ such that } f(b) \geq t, \text{ for all } b \in B\}.$$

Since $0 \notin clB$, there exists $\varphi \in Y^* \setminus \{0\}$ such that

$$r = \inf\{\varphi(b) : b \in B\} > 0$$

Let

$$V_B = \{y \in Y : |\varphi(y)| < \frac{r}{2}\}.$$

Define the neighborhood family of 0_Y in Y as follows:



$$\Omega = \{U \subset V_B : U \text{ is an open convex circled neighborhood of zero in } Y\}.$$

For each $U \in \Omega$, Let

$$D_U(B) = \text{cone}(B + U).$$

It has been pointed out that for each $U \in \Omega$, $D_U(B)$ is a pointed convex cone in Y with $D \setminus \{0\} \subset \text{int } D_U(B)$. The following results have been proven by Gong [16].

Lemma 2.1 (see Ref. [16]) Assume that D has been a base B .

- (a) For any $U \in \Omega$, $D_U(B)^+ \setminus \{0\} \subset D^+(B)$.
- (b) For any $\varphi \in D^+(B)$, there exists $\bar{U} \in \Omega$ such that $\varphi \in D_{\bar{U}}(B)^+ \setminus \{0\}$.
- (c) If convex cone D is closed and B is bounded closed, then $\text{int}(D^+) = D^+(B)$.

Definition 2.1. (see Ref. [16]) Let C be a nonempty subset of Y and let B be a base of D . $y_0 \in C$ is said to be a Henig efficient point of C with respect to B , written as $y_0 \in HE[C, B]$, if there exist $U \in \Omega$ such that

$$(C - y_0) \cap (-\text{int}(D_U(B))) = \emptyset.$$

The following Proposition 2.1 will be used in the sequel.

Proposition 2.1. Let C be a nonempty subset of Y . Then $HE[C, B] = HE[C + D, B]$.

Proof Obviously, $HE[C + D, B] \subset HE[C, B]$. Now we prove that

$$HE[C, B] \subset HE[C + D, B].$$

For arbitrary $\bar{y} \in HE[C, B]$, then there exists $\bar{U} \in \Omega$ such that

$$(C - \bar{y}) \cap (-\text{int}(D_{\bar{U}}(B))) = \emptyset. \quad (2.1)$$

If $\bar{y} \notin HE[C + D, B]$, then for all $U \in \Omega$, it holds that

$$(C + D - \bar{y}) \cap (-\text{int}(D_U(B))) \neq \emptyset \quad (2.2)$$

Consequently, there exist $y' \in C$ and $d \in D$ such that

$$y' + d - \bar{y} \in -\text{int}(D_U(B)).$$

When $d = 0$, this contradicts to (2.1). When $d \neq 0$, it yields from $D \setminus \{0\} \subset \text{int}(D_U(B))$ that

$$y' - \bar{y} \in -\text{int}(D_U(B)),$$

which also is a contradiction to (2.1). The proof is completed.

Kothe [17] introduced the concept of weakly countable compactness for a set. The following theorem shows that if a set is weakly countable compact then it must be exist a Henig proper efficient point.

Theorem 2.1. If $C \subset Y$ is weakly countable compact set, then $HE[C, B] \neq \emptyset$.

Proof Since $0 \notin clB$, there exists $\varphi \in Y^* \setminus \{0\}$ such that

$$r = \inf\{\varphi(b) : b \in B\} > 0$$



By the weakly countable compactness of C , it follows that C is bounded set (we refer to Kothe [9], page 310). Hence,

$$\gamma = \inf\{\varphi(y) : y \in C\} > -\infty.$$

Thus, for any $n \in \mathbb{N}$ (\mathbb{N} is positive integer set), there exist $y_n \in C$ such that

$$\gamma \leq \varphi(y_n) < \gamma + \frac{1}{n}.$$

By the weakly countable compactness of C again, there exists $y_0 \in C$ such that y_0 is a weak accumulation point of $\{y_n\}_{n \in \mathbb{N}}$. On the other hand, we obtain that $\varphi(y_0)$ is an accumulation point of $\{\varphi(y_n)\}_{n \in \mathbb{N}}$ from the continuity of φ . In view of above argument, we can set $\varphi(y_0) = \gamma = \inf\{\varphi(y) : y \in C\}$. Hence,

$$\varphi(y - y_0) \geq 0, \quad \forall y \in C. \quad (2.3)$$

Set

$$\bar{U} = \{y \in Y : |\varphi(y)| < \frac{r}{4}\},$$

and let $D_{\bar{U}}(B) = \text{cone}(B + \bar{U})$. It is clear that $\bar{U} \in \Omega$. Then, for any $y \in B + \bar{U}$, there exist $b \in B$ and $\bar{u} \in \bar{U}$ such that $y = b + \bar{u}$. So,

$$\varphi(y) = \varphi(b) + \varphi(\bar{u}) \geq \varphi(b) - \frac{r}{4} \geq \frac{3r}{4} > 0.$$

Consequently, we have

$$\varphi(y) < 0, \text{ for all } y \in -\text{int}(D_{\bar{U}}(B)). \quad (2.4)$$

Combining (2.3) and (2.4), we obtain that

$$(C - y_0) \cap -\text{int}[D_{\bar{U}}(B)] = \emptyset,$$

which follows that $y_0 \in HE[C, B]$.

Let $F : X \rightarrow 2^Y$ be a set-valued map. The set

$$\text{dom}(F) := \{x \in X, F(x) \neq \emptyset\}$$

is called the domain of F . The set

$$\text{graph}(F) := \{(x, y) \in X \times Y : x \in \text{dom}(F), y \in F(x)\}$$

is called the graph of F . The set

$$\text{epi}(F) := \{(x, y) \in X \times Y : x \in \text{dom}(F), y \in F(x) + D\}$$

is called the epigraph of F .

Let us recall some concepts.

Definition 2.2. (see Ref. [14]) Let $F : X \rightarrow 2^Y$ be a set-valued mapping. Let $(x_0, y_0) \in \text{graph}(F)$. F is said to be lower semi-continuous at (x_0, y_0) , if for any neighborhood $N(y_0)$ of y_0 , there exists a neighborhood $N(x_0)$ of x_0 such that $F(x) \cap N(y_0) \neq \emptyset$ for all $x \in N(x_0)$.



Definition 2.3. (see Ref. [14]) Let $S \subset X$ be convex set and $F : S \rightarrow 2^Y$ be a set-valued mapping.

(1) F is said to be (strictly) D -convex on S , if $\forall x_1, x_2 \in S, \forall \lambda \in (0,1)$

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + D.$$

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + \text{int}(D).$$

(2) Let $x_0 \in S$, F is said to be strict D -convex at x_0 , if $\forall x' \in S, \forall \lambda \in (0,1)$

$$\lambda F(x') + (1-\lambda)F(x_0) \subset F(\lambda x' + (1-\lambda)x_0) + \text{int}(D).$$

It is well known that if F is D -convex on S if and only if $\text{epi}(F)$ is a convex subset of $X \times Y$.

3. CONE-HENIG PROPER EFFICIENT SUBDIFFERENTIAL

Now, we introduce the concept of cone-Henig subdifferential for a set-valued map.

Definition 3.1. Let $F : X \rightarrow 2^Y$ be a set-valued map, $(\bar{x}, \bar{y}) \in \text{graph}(F)$, $\varphi \in X^*$ and $p \in \text{int}(D)$. It is said that $\varphi \in X^*$ is a D -Henig efficient subgradient of F at (\bar{x}, \bar{y}) with respect to vector p , if

$$\bar{y} - \varphi(\bar{x})p \in HE[\bigcup_{x \in X} (F(x) - \varphi(x)p), B].$$

The set of all D -Henig efficient subgradient of F at (\bar{x}, \bar{y}) with respect to vector p is called the D -Henig efficient subdifferential of F at (\bar{x}, \bar{y}) with respect to p and is denoted by $\partial_H F(\bar{x}, \bar{y})_p$. It is said that F is D -Henig efficient subdifferentiable at (\bar{x}, \bar{y}) with respect to p , if $\partial_H F(\bar{x}, \bar{y})_p \neq \emptyset$.

Example 3.1. Let $X = Y = R \times L_q[0,1]$, where $0 < q < 1$. We denote

$$D = R_+ \times \{0\} = \{(\alpha, 0) \in R \times L_q[0,1] : \alpha \geq 0\},$$

$$D' = R_+ \times L_q[0,1].$$

It is obviously that the convex pointed cone D is closed, and $\text{int} D' \supset D \setminus \{0\}$ (See E. K. Makarov and N. N. Rachkovski, Density theorems for generalized Henig proper efficiency, Journal of Optimization Theory and Applications, 91(2): 419-437). Since $L_q^*[0,1] = \{0\}$ (See [18] Rudin, W., Functional Analysis, McGraw-Hill Book Company, New York, 1973, Section 1.47), we get $X^* = R \times \{0\}$. Let $p = (1, 0) \in D$, and define $\varphi \in X^*$ by

$$\varphi(\alpha, z) = \alpha, \quad \text{for any } (\alpha, z) \in X.$$

The set-valued mapping $F : X \rightarrow 2^Y$ defined by

$$F(\alpha, z) = R_+ \times L_q[0,1] \quad \text{for any } (\alpha, z) \in X.$$

For $\bar{x} = (0, 0), \bar{y} = (0, 0)$, it is clear that $\varphi \in \partial_H F(\bar{x}, \bar{y})_p$.

The following basic properties of cone-Henig subdifferential can be obtain from Definition 3.1.

Proposition 3.1. Let $F : X \rightarrow 2^Y$ be a set-valued mapping, $(\bar{x}, \bar{y}) \in \text{graph}(F)$, $p \in \text{int}(D)$ and $r > 0$. Then

(1) If $\varphi \in \partial_H F(\bar{x}, \bar{y})_p$ then $\bar{y} \in HE[F(\bar{x}), B]$.



- (2) If $\bar{y} \notin HE[F(\bar{x}), B]$, then $\partial_H F(\bar{x}, \bar{y})_p \neq \emptyset$.
- (3) $\bar{y} \in HE[F(X), B]$ if and only if $0_{X^*} \in \partial_H F(\bar{x}, \bar{y})_p$.
- (4) If $\varphi \in \partial_H F(\bar{x}, \bar{y})_p$, then $\frac{\varphi}{r} \in \partial_H F(\bar{x}, \bar{y})_{rp}$.

In order to obtain the existence theorem of cone-Henig efficient subdifferential, we need the following Lemma 3.1.

Lemma 3.1. Let $F : X \rightarrow 2^Y$ be a set-valued mapping and $x_0 \in dom(F)$. If one of the next three conditions is fulfilled, then $int[epi(F)] \neq \emptyset$.

C_1 : There exists $\hat{y} \in F(x_0)$ such that F is lower semi-continuous at (x_0, \hat{y}) .

C_2 : There exists $a \in Y$ such that $F(X) \subset a - D$

C_3 : There is a map $f : X \rightarrow Y$ such that $f(x) \in F(x)$ for any $x \in X$, and f is continuous in a neighborhood $U(x_0)$ of x_0 .

Proof Suppose that condition C_1 holds, taking $p' \in int(D)$, we shall show that $(x_0, \hat{y} + p') \in int[epi(F)]$. In fact, by $p' \in int(D)$, there exists $U \in N(0_Y)$ such that

$$p' + U \subset D. \tag{3.1}$$

Since F is lower semi-continuous at (x_0, \hat{y}) , then for the neighborhood $(\hat{y} - \frac{1}{2}U)$ of \hat{y} there exist $W \in N(x_0)$ (the neighborhood family of x_0) such that

$$F(\omega) \cap (\hat{y} - \frac{1}{2}U) \neq \emptyset, \text{ for all } \omega \in W.$$

Let $V = W - x_0$, then $V \in N(0_X)$ and

$$F(x_0 + v) \cap (\hat{y} - \frac{1}{2}U) \neq \emptyset, \text{ for all } v \in V.$$

Taking $(x, y) \in X \times Y$, then there exists $\delta > 0$, and when $0 \leq \lambda \leq \delta$,

$$\lambda x \in V, \lambda y \in \frac{1}{2}U.$$

Thus, there exists $\bar{y} \in F(x_0 + \lambda x) \cap (\hat{y} - \frac{1}{2}U)$, that is: there is $\tilde{y} \in \frac{1}{2}U$ such that

$$\bar{y} = \hat{y} - \tilde{y} = \hat{y} + p' + \lambda y - (p' + \lambda y + \tilde{y})$$

Combining with (3.1), we get $\bar{y} \in \hat{y} + p' + \lambda y - D$. Thus

$$(x_0 + \lambda x, \hat{y} + p' + \lambda y) = (x_0, \hat{y} + p') + \lambda(x, y) \in epi(F), \lambda \in [0, \delta).$$

It yields that $(x_0, \hat{y} + p') \in int[epi(F)]$.

Assuming that there exists $a \in Y$ such that



$$F(x) \subset a - D, \text{ for all } x \in X. \tag{3.2}$$

For $p' \in \text{int}(D)$, take $\bar{y} = a + p'$, then $\bar{y} - a = p' \in \text{int}(D)$. Consequently, there is $\bar{U} \in N(0_Y)$ such that

$$\bar{U} + \bar{y} - a \subset D. \tag{3.3}$$

By (3.2), for any $x \in X$ and $y_x \in F(x)$, there exists $p_x \in D$ such that $y_x = a - p_x$. Making use of (3.3) and noticing that D is a convex cone, we obtain that

$$\bar{U} + \bar{y} - y_x = \bar{U} + \bar{y} - a + p_x \subset D, \text{ for all } x \in X.$$

Thus, $\bar{U} + \bar{y} \subset y_x + D \subset F(x) + D (\forall x \in X)$. Consequently, we have

$$(x, y) \in \text{epi}(F), \text{ for all } x \in X, y \in \bar{U} + \bar{y},$$

which shows that $\text{int}[\text{epi}(F)] \neq \emptyset$.

It is clear that $\text{int}[\text{epi}(F)] \neq \emptyset$ if the condition C_3 is satisfied. In fact, since f is continuous in the neighborhood $U(x_0)$ of x_0 , it is obviously that $f(U(x_0))$ is an open set. Consequently, $(U(x_0), f(U(x_0))) \subset \text{epi}(F)$. Thus, $\text{int}[\text{epi}(F)] \neq \emptyset$.

Theorem 3.1. Let D be closed convex pointed cone with a bounded closed base B . Let $F : X \rightarrow 2^Y$ be a D -convex set-valued mapping, $(x_0, y_0) \in \text{graph}(F)$, $p \in \text{int}(D)$ and $y_0 \in \text{HE}[F(x_0), B]$. Let F be strict D -convex at x_0 . If one of the three conditions in Lemma 3.1 is fulfilled, then $\partial_H F(x_0, y_0)_p \neq \emptyset$.

Proof. Firstly, by $y_0 \in \text{HE}[F(x_0), B]$, there exist $\bar{U} \in \Omega$ such that

$$(F(x_0) - y_0) \cap -\text{int}(D_{\bar{U}}(B)) = \emptyset. \tag{3.4}$$

Then, since F is D -convex, $\text{epi}(F)$ is a convex set. Consider the epigraph of F , the proof of this theorem consists of several steps. First, we prove two important properties of $\text{epi}(F)$ and then we apply a separation theorem to obtain the desired result.

Step 1: $\text{int}[\text{epi}(F)] \neq \emptyset$. We obtain this fact from Lemma 3.1.

Step 2: $(x_0, y_0) \notin \text{int}[\text{epi}(F)]$. Otherwise, $(x_0, y_0) \in \text{int}[\text{epi}(F)]$. This follows that there exists

$\tilde{U} \in N(0_Y)$ such that

$$(x_0, y_0 + \tilde{U}) \subset \text{epi}(F).$$

For $p' \in \text{int}(D)$, there exists $\eta > 0$ such that $-\eta p' \in \tilde{U}$. Thus

$$y_0 - \eta p' \in F(x_0) + D,$$

Consequently, there exist $y' \in F(x_0)$ and $d' \in D$ such that

$$y_0 - \eta p' = y' + d'$$

Furthermore, we can see that

$$y' - y_0 = -\eta p' - d' \in -\text{int}(D) \subset -D \setminus \{0\}. \tag{3.5}$$



Since for all $U \in \Omega, D \setminus \{0\} \subset \text{int}(D_U(B))$. So, by (3.5), we get

$$y' - y_0 \in -\text{int}(D_U(B)) \text{ for all } U \in \Omega.$$

Which is a contradiction to (3.4).

Step 3: There exist $(\varphi, \psi) \in X^* \times Y^*$ with $\psi \in D^+ \setminus \{0\}$ such that

$$\varphi(x - x_0) + \psi(y + d - y_0) \geq 0, x \in X, y \in F(x), d \in D. \quad (3.6)$$

In fact, since $\text{int}[epi(F)]$ is an open convex subset of $X \times Y$ with $(x_0, y_0) \notin \text{int}[epi(F)]$ (due to step 2). Now, using a separation theorem, there exists $(\varphi, \psi) \in X^* \times Y^*$, with $(\varphi, \psi) \neq (0_{X^*}, 0_{Y^*})$ such that (3.6) holds. We have $\psi \neq 0_{Y^*}$. Otherwise, it follows that

$$\varphi(x - x_0) \geq 0, x \in X \quad (3.7)$$

For a positive real number $\lambda > 0$, Let $v \in X$ be an arbitrary vector. Taking $x = \pm\lambda v + x_0$ in (3.7), we get $\varphi(\pm\lambda v) \geq 0$. This shows that $\varphi = 0$, a contradiction to the fact $(\varphi, \psi) \neq (0_{X^*}, 0_{Y^*})$. In addition, taking $x = x_0, y = y_0$ in (3.6), we obtain that

$$\psi(d) \geq 0, \text{ for all } d \in D.$$

Thus, $\psi \in D^+ \setminus \{0_{Y^*}\}$.

Step 4: For any $x \in X, y \in F(x), d \in D, y + d \neq y_0$, we have

$$\varphi(x - x_0) + \psi(y + d - y_0) > 0. \quad (3.8)$$

On the contrary, by inequality (3.6), there is $(\overset{\vee}{x}, \overset{\vee}{y} + \overset{\vee}{d}) \in epi(F)$ with $\overset{\vee}{y} + \overset{\vee}{d} \neq y_0$ such that

$$\varphi(\overset{\vee}{x} - x_0) + \psi(\overset{\vee}{y} + \overset{\vee}{d} - y_0) = 0. \quad (3.9)$$

For arbitrary $\lambda \in (0, 1)$, set

$$x_\lambda = \lambda \overset{\vee}{x} + (1 - \lambda)x_0; \quad y_\lambda = \lambda \overset{\vee}{y} + (1 - \lambda)y_0. \quad (3.10)$$

Since F is strict D -convex at x_0 , we have that

$$y_\lambda = \lambda \overset{\vee}{y} + (1 - \lambda)y_0 \in F(\lambda \overset{\vee}{x} + (1 - \lambda)x_0) + \text{int}(D).$$

Setting

$$y_\lambda = \eta + \gamma, \quad \eta \in F(\lambda \overset{\vee}{x} + (1 - \lambda)x_0), \gamma \in \text{int}(D), \quad (3.11)$$

Takin $d = 0, y = \eta$ and $x = x_\lambda$ in inequality (3.6), we get that

$$\varphi(x_\lambda - x_0) + \psi(\eta - y_0) \geq 0. \quad (3.12)$$

By (3.11) and (3.9), we get



$$\begin{aligned} \varphi(x_\lambda - x_0) + \psi(y_\lambda - y_0) &= \varphi(\lambda \overset{\vee}{x} - \lambda x_0) + \psi(\lambda \overset{\vee}{y} - \lambda y_0) \\ &= \lambda(\varphi(\overset{\vee}{x} - x_0) + \psi(\overset{\vee}{y} - y_0)) \\ &= -\lambda\psi(\overset{\vee}{d}). \end{aligned}$$

Thus,

$$\psi(y_\lambda - y_0) + \lambda\psi(\overset{\vee}{d}) = -\varphi(x_\lambda - x_0).$$

By inequality (3.12), and noticing that $\psi(\overset{\vee}{d}) \geq 0$, we can see that

$$\psi(y_\lambda - y_0) \leq \psi(y_\lambda - y_0) + \lambda\psi(\overset{\vee}{d}) \leq \psi(\eta - y_0). \tag{3.13}$$

On the other hand, from $\psi \in D^+ \setminus \{0_{Y^*}\}$ and $\gamma \in \text{int}(D)$, we have $\psi(\gamma) > 0$. So,

$$\psi(y_\lambda - y_0) = \psi(\eta - y_0) + \psi(\gamma) > \psi(\eta - y_0)$$

which contradicts to (3.13).

Step 5: $\psi \in D^\Delta(B)$. Setting $x = x_0$ and $y = y_0$ in inequality (3.8), and taking $d \in D \setminus \{0\}$ arbitrary. We can get that $y_0 + d \neq y_0$ and

$$\psi(d) > 0, d \in D \setminus \{0\}$$

This yields that $\psi \in \text{int}(D^+)$. (in fact $\psi \notin \text{int}(D^+)$ implies that there is $g \in Y^*$ such that $\psi + \frac{g}{n} \notin D^+$, for all natural number n . Consequently, there is $0 \neq \overset{\cdot}{d} \in D$ such that $(\psi + \frac{g}{n})(\overset{\cdot}{d}) < 0$. Taking $n \rightarrow \infty$ in above inequality, we obtain that $\psi(\overset{\cdot}{d}) \leq 0$: a contradiction.) by Lemma 2.1, we obtain $\psi \in D^\Delta(B)$.

Step 6: $-\frac{\varphi}{\psi(p)} \in \partial_H F(x_0, y_0)_p$. Since $\psi \in D^\Delta(B)$, by Lemma 2.1, there is $U' \in \Omega$ such that $\psi \in D_{U'}(B)^+ \setminus \{0_{Y^*}\}$.

$$\psi(y) < 0, \text{ for all } y \in -\text{int}(D_{U'}(B)) \tag{3.14}$$

On the other hand, from inequality (3.6), it is easy to obtain that

$$\varphi(x - x_0) + \psi(y - y_0) \geq 0, x \in X, y \in F(x) \tag{3.15}$$

Suppose that $-\frac{\varphi}{\psi(p)} \notin \partial_H F(x_0, y_0)_p$, then we have that $\bigcup_{x \in X} (F(x) + \frac{\varphi(x - x_0)}{\psi(p)} p - y_0) \cap -\text{int}(D_{U'}(B)) \neq \emptyset$

Consequently, there exist $\tilde{x} \in X$ and $\tilde{y} \in F(\tilde{x})$ such that

$$\tilde{y} - y_0 + \frac{\varphi(\tilde{x} - x_0)}{\psi(p)} p \notin -\text{int}(D_{U'}(B)).$$

Noticing that $\psi \in D^\Delta(B)$, it follows that



$$\psi(\tilde{y} - y_0 + \frac{\varphi(\tilde{x} - x_0)}{\psi(p)} p) = \varphi(\tilde{x} - x_0) + \psi(\tilde{y} - y_0) < 0,$$

a contradiction to inequality (3.15). The proof is completed.

Theorem 3.2 Let $F : X \rightarrow 2^Y$ be a set-valued mapping $(\bar{x}, \bar{y}) \in graph(F)$, and $p \in \text{int}(D)$. Then $\bar{y} \in HE[F(X), B]$ if and only if $0_{X^*} \in \partial_H F(\bar{x}, \bar{y})_p$.

Proof The argument is easy. Since $\bar{y} \in HE[F(X), B]$, there exist $\bar{U} \in \Omega$ such that

$$(F(x) - \bar{y}) \cap -\text{int}(D_{\bar{U}}(B)) = \emptyset,$$

this equivalent to

$$\bigcup_{x \in X} (F(x) - 0(x - \bar{x})p - \bar{y}) \cap -\text{int}(D_{\bar{U}}(B)) = \emptyset$$

thus, $0_{X^*} \in \partial_H F(\bar{x}, \bar{y})_p$.

The following Theorem 3.3 shows that a linear functional is a cone-Henig subgradient under the suitable conditions.

Theorem 3.3 Let $F : X \rightarrow 2^Y$ be a set-valued mapping, $(\bar{x}, \bar{y}) \in graph(F)$, $\varphi \in X^*$ and $p \in \text{int}(D)$. Assuming that F is D -convex in X , then $\varphi \in \partial_H F(\bar{x}, \bar{y})_p$ if and only if there exist $\psi \in D^\Delta(B)$ such that

$$\psi(y - \bar{y}) \geq \varphi(x - \bar{x})\psi(p), y \in F(x) \quad x \in X \tag{3.16}$$

bf Necessity. By $\varphi \in \partial_H F(\bar{x}, \bar{y})_p$, there is $\bar{U} \in \Omega$ such that

$$\left(\bigcup_{x \in X} F(x) - \varphi(x - \bar{x})p - \bar{y} \right) \cap -\text{int}(D_{\bar{U}}(B)) = \emptyset$$

(3.17)

By proposition 2.1, we get

$$\left(\bigcup_{x \in X} F(x) - \varphi(x - \bar{x})p - \bar{y} + D \right) \cap -\text{int}(D_{\bar{U}}(B)) = \emptyset$$

(3.18)

On the other hand, from F be D -convex, it yields that $\bigcup_{x \in X} F(x) - \varphi(x - \bar{x})p + D$ is a convex set. Making use of a

separation theorem, there is $\psi \in Y^* \setminus \{0_{Y^*}\}$ such that

$$\psi(y - \varphi(x)p - \bar{y} + \varphi(\bar{x})p + d) \geq \psi(z), \quad x \in X, y \in F(x), z \in -\text{int}(D_{\bar{U}}(B)), d \in D$$

Noticing that D and $D_{\bar{U}}(B)$ are convex cones, it is clear that

$$\psi(y - \varphi(x)p - \bar{y} + \varphi(\bar{x})p + d) \geq 0, \quad x \in X, y \in F(x), d \in D. \tag{3.19}$$

$$\psi(z) > 0, \forall z \in D_{\bar{U}}(B).$$

Thus, $\psi \in (D_{\bar{U}}(B))^+ \setminus \{0_{Y^*}\}$, this follows that $\psi \in D^\Delta(B)$ from Lemma 2.1. Takin $d = 0$ in inequality (3.19), we prove that there exist $\psi \in D^\Delta(B)$ such that inequality (3.16) hold.



Sufficiency. Suppose that there is $\psi \in D^\Delta(B)$ such that (3.16) hold, If $\varphi \notin \partial_H F(\bar{x}, \bar{y})_p$, then for any $U \in \Omega$, we have

$$\left(\bigcup_{x \in X} F(x) - \varphi(x - \bar{x})p - \bar{y}\right) \cap -\text{int}(D_U(B)) \neq \emptyset \tag{3.20}$$

Since $\psi \in D^\Delta(B)$, by Lemma 2.1, there is $\bar{U} \in \Omega$ such that $\psi \in (D_{\bar{U}}(B))^+ \setminus \{0_{Y^*}\}$. So, for $D_{\bar{U}}(B)$, we also have

$$\left(\bigcup_{x \in X} F(x) - \varphi(x - \bar{x})p - \bar{y}\right) \cap -\text{int}(D_{\bar{U}}(B)) \neq \emptyset \tag{3.21}$$

Consequently, there exist $\tilde{x} \in X$ and $\tilde{y} \in F(\tilde{x})$ such that

$$\tilde{y} - \varphi(\tilde{x} - \bar{x})p - \bar{y} \in -\text{int}(D_{\bar{U}}(B)),$$

then

$$\psi(\tilde{y} - \varphi(\tilde{x} - \bar{x})p - \bar{y}) < 0,$$

which contradicts to inequality (3.16)

As the end of section, we deal with the basic properties of come-Henig subdifferentials. The following Proposition 3.2 and Proposition 3.3 are direct consequences of Definition 3.1 and Theorem 3.2, and their proofs are omitted.

Proposition 3.2. Let $F : X \rightarrow 2^Y$ be a D -convex set-valued mapping, $(\bar{x}, \bar{y}) \in \text{graph}(F)$ and $p_1, p_2 \in \text{int}(D)$. If $\varphi \in \alpha \partial_H F(\bar{x}, \bar{y})_{p_1}$, then there is $\psi \in D^\Delta(B)$ such that $\alpha \varphi \in \partial_H F(\bar{x}, \bar{y})_{p_2}$,

where $\alpha = \frac{\psi(p_1)}{\psi(p_2)}$.

Proposition 3.3. Let $F : X \rightarrow 2^Y$ be a D -convex set-valued mapping, $(\bar{x}, \bar{y}) \in \text{graph}(F)$ and $p \in \text{int}(D)$. If $\alpha \geq 0$ then $\partial_H(\alpha F)(\bar{x}, \bar{y})_p = \alpha \partial_H F(\bar{x}, \bar{y})_p$.

Proposition 3.4. Let $F : X \rightarrow 2^Y$ be a D -convex set-valued mapping, $(\bar{x}, \bar{y}) \in \text{graph}(F)$, and $p \in \text{int}(D)$. Assuming that $\partial_H F(\bar{x}, \bar{y})_p \neq \emptyset$, then $\partial_H F(\bar{x}, \bar{y})_p$ is convex and closed.

Proof. Convexity. Taking $\varphi_1, \varphi_2 \in \partial_H F(\bar{x}, \bar{y})_p$ and $\lambda \in (0, 1)$ arbitrary. Setting $\varphi_\lambda := \lambda \varphi_1 + (1 - \lambda) \varphi_2$. We claim that $\varphi_\lambda \in \partial_H F(\bar{x}, \bar{y})_p$. In fact, otherwise $\varphi_\lambda \notin \partial_H F(\bar{x}, \bar{y})_p$. By Definition 3.1, there are $x' \in X$, $y' \in F(x')$ with $(x', y') \neq (\bar{x}, \bar{y})$ and $\bar{U} \in \Omega$ such that

$$\bar{y} - \varphi_\lambda(\bar{x})p - (y' - \varphi_\lambda(x')p) \in \text{int}[D_{\bar{U}}(B)] \tag{3.22}$$

since $\varphi_1, \varphi_2 \in \partial_H F(\bar{x}, \bar{y})_p$, by theorem 3.2, there exist $\psi_1, \psi_2 \in D^\Delta(B)$ such that

$$\psi_1(y' - \bar{y}) \geq \varphi_1(x' - \bar{x})\psi_1(p) \tag{3.23}$$

$$\psi_2(y' - \bar{y}) \geq \varphi_2(x' - \bar{x})\psi_2(p) \tag{3.24}$$

Noticing that $\psi_1, \psi_2 \in D^\Delta(B)$, we obtain from (3.22) that

$$\psi_1(\bar{y} - y') + \varphi_\lambda(x' - \bar{x})\psi_1(p) > 0 \tag{3.25}$$



$$\psi_2(\bar{y} - y') + \varphi_\lambda(x' - \bar{x})\psi_2(p) > 0 \quad (3.26)$$

From inequality (3.23) and (3.25), it follows that

$$\varphi_\lambda(x' - \bar{x}) > \varphi_1(x' - \bar{x}) \quad (3.27)$$

Respectively, by inequality (3.24) and (3.26), we get

$$\varphi_\lambda(x' - \bar{x}) > \varphi_2(x' - \bar{x}) \quad (3.28)$$

Consequently, by inequality (3.27) and (3.28), it yields that

$$\varphi_\lambda(x' - \bar{x}) > \varphi_\lambda(x' - \bar{x}),$$

a contradiction. Closeness. Suppose that there is $\{\varphi_n\} \subset \partial_H F(\bar{x}, \bar{y})_p$ with $\varphi_n \rightarrow \varphi (n \rightarrow \infty)$ such that $\varphi \notin \partial_H F(\bar{x}, \bar{y})_p$. So, there exist $x' \in S$, $y' \in F(x')$ and $\bar{U} \in \Omega$ such that $\bar{y} - y' + \varphi(x' - \bar{x})p \in \text{int}[D_{\bar{U}}(B)]$. Since $\bar{y} - y' + \varphi_n(x' - \bar{x})p \rightarrow \bar{y} - y' + \varphi(x' - \bar{x})p$, there is the sufficient large n' such that

$$\bar{y} - y' + \varphi_{n'}(x' - \bar{x})p \in \text{int}[D_{\bar{U}}(B)],$$

which yield $\varphi_{n'} \notin \partial_H F(\bar{x}, \bar{y})_p$, a contradiction.

4. CONCLUDING REMARKS

In 1993, Borwein and Zhuang [12] introduced the concept of super efficiency in vector optimization. They have proven that if the closed convex pointed cone has a bounded base, then the set of Henig efficient points for a nonempty set are equal to its super efficient points (see Proposition 3.5 of Ref. [12]). This together with the results obtained in this note, we can investigate the super efficiency in the view of subdifferential.

The globally proper efficiency has the similar properties to Henig efficiency in many respects (see Gong [15-16], Yu [19]). Of course, the concept of cone-subdifferential in sense of globally proper efficiency can be defined in the same way. Making use of the method of this paper, we claim that the similar results for globally proper efficiency can be obtained.

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