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Uniqueness for entropy solutions to fully nonlinear equations

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ABSTRACT

Let $\mathcal{X}$ be a metric space, $\mathcal{M}(\mathcal{X}, \mathcal{B}, \mu)$ the space of μ–measurable functions, $\Omega\subset\mathbb{R}^N$ be a domain whith boundary $\partial\Omega$ and a(x, ξ) be an operator of Leray-Lions type. If β and γ are nondecreasing continuous function on $\mathbb{R}$ such that $\beta(0) = \gamma(0) = 0$ and $(f,g) \in L^1(\mathcal{X}, \mathcal{B}, \mu)$, then, there exists a unique entropy solution u in $\mathcal{M}(\mathcal{X}, \mathcal{B}, \mu)$ to the problem $-\mathrm{div}[a(., Du)]+\beta(u) = f$ in Ω and $a(.,Du)\nu+\gamma(\tau u) = g$ on $\partial\Omega$.

Keywords:

Entropy solution; Renormalized solution; Elliptic problem; Measure space.

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Introduction

Let $\Omega \subset \mathbb{R}^N$, N ≥ 3, be a domain (not necessarily bounded) whithboundary $\partial\Omega$. For example, to prove existence and uniqueness for a solution to an elliptic or parabolic problem in Ω related with an operator of Leray-Lions type: Au = -div[a(.,Du)] with non linear Neumann condition on the boundary $\partial\Omega$:

$$a(., Du) . \nu + \gamma(\tau u) \ni g, \; g \in L^1(\partial\Omega), \tag{1.1}$$

whereγ is a maximal monotone graph in $\mathbb{R}^2$, ν is the vector field of exterior normal to the boundary $\partial\Omega$, when g = 0, many authors even in the linear case Au = -Δu, (see [1], for example) include the boundary condition (1.1) in the definition of the domain D(A) of the operator A. In the case where g ≠ 0, [2] apply the same process to a family of operators B^g in the elliptic case (rep: $B^{g(t)}$ in the parabolic case).Besides supplementary technical difficulties, even in case where A is linear some notion of multivalued linear operator is needed. This is no more necessary when applying theorems 4.1 of [8] and [9]. More precisely if $\mathcal{X} = \Omega \cup \partial\Omega$, $L^1(\Omega)$ x $L^1(\partial\Omega)$ is identified to $L^1(\mathcal{X})$. Then, let A be The operator defined as follows: $(u,\tau u) \in D(A)$ if there exist $(f,g) \in L^1(\mathcal{X})$ such u is an entropy solution to the next problems:

$$\begin{cases} -\mathrm{div}[a(., Du)] = f \text{ in } \Omega, \\ a(., Du).\nu = g \text{ on } \partial\Omega \end{cases} \tag{1.2}$$

In the sense that, if $T_k(r) = \max\{-k, \min(r,k)\}$, k>0, $r \in \Omega$, $\forall \varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(x, Du) DT_k(u-\varphi) \le \int_\Omega f T_k(u-\varphi) + \int_{\partial\Omega} g T_k(\tau u - \tau\varphi).$$

If A_1 is the restriction of A to $L^1(\mathcal{X})$, then A_1 is said to be accretive in $L^1(\mathcal{X})$, if then the next inequality holds

$$\int_\Omega (f_1 - f_2)\varphi_0 + \int_{\{u_1=u_2\}} |f_1 - f_2| + \int_{\partial\Omega} (g_1 - g_2)\psi_0 + \int_{\{\tau u_1 = \tau u_2\}} |g_1 - g_2| \ge 0, \tag{1.3}$$

For any $F_i = (f_i,g_i) \in L^1(\mathcal{X})$, $U_i=(u_i,\tau u_i) \in D(A_1)$, i=1,2 so that$AU_i = F_i$ and $\varphi_0 = \mathrm{sign}_0(u_1-u_2)$, $\psi_0 = \mathrm{sign}_0(\tau u_1 - \tau u_2)$, where:sign(r)=$\mathrm{sign}_0(r) = \frac{r}{|r|}$, $r \in \mathbb{R}, r \neq 0$, sign(0)=[-1,1] and $\mathrm{sign}_0(0) = 0$. Inequalities (i) in ([8] theorem 4.1) extends (1.3) to the case where $(u_i,\tau u_i) \in D(A)$, and inequality (ii) states that if in addition (ui,□ ui)∈D(A_1) i=1,2, then for every $\varphi \in \mathrm{sign}(u_1-u_2)$ and $\psi \in \mathrm{sign}(\tau u_1 - \tau u_2)$, we have:

$$\int (f_1 - f_2)\varphi + \int_{\partial\Omega} (g_1 - g_2)\psi \ge 0. \tag{1.4}$$

Inequalities (1.3) and (1.4) where applied in [8], and similarly in [9], to prove existence and uniqueness of a solution to the problem: -div[a(., Du)]+ $\beta(u) \ni f$ and $a(., Du).\nu + \gamma(\tau u) \ni g$ on $\partial\Omega$.

It is well known from [7] that uniqueness of weak solutions for degenerate problems, is not guaranteed. However, if two kind of solutions u and v in a class of uniqueness, such as entropy solutions or renormalized solutions, are obtained as a limit of a sequence (u_n) of some regular solutions related to some sequence data (f_n), from uniqueness of u_n related to some f_n, the fact that entropy solutions, renormalized solutions etc are particular weak solutions, then by applying (1.3), formally we shall be able to prove that u = v. This seams to be new, since, even that studied separately existence and uniqueness of entropy and renormalized solutions have not proved that it is the same solution (see [6]).

Theorems 4.1 [8] may be extended to general measure spaces $(\mathcal{X}, \mathcal{B}, \mu)$ and the closure in $\mathcal{M}(\mathcal{X}, \mathcal{B}, \mu)$ x $L^1(\mathcal{X}, \mathcal{B}, \mu)$ of the operator A_1 and thus, a larger class of measurable functions that are weak solutions obtained as limit in $\mathcal{M}(\mathcal{X}, \mathcal{B}, \mu)$ with data in $f \in L^1(\mathcal{X})$ of a sequence of entropy solutions, or renormalized solutions that are in $L^1(\mathcal{X})$. It is proved first, that for this particular case, entropy and renormalized solution is the same one. This will be extended, next, for general entropy and renormalized solutions in $\mathcal{M}(\mathcal{X}, \mathcal{B}, \mu)$, that are not necessary integrable, but satisfy some specified conditions of regularity that is required in the definition of this kind of solutions.

Order Preserving Inequalities

Let be given a metric space $\mathcal{X}$ and a complete measure space $(\mathcal{X}, \mathcal{B}, \mu)$ such that $\mathcal{X}$ is σ-infinite and μ is regular, (see [4]). The space of μ-measurable real valued functions $\mathcal{M}(\mathcal{X}, \mathcal{B}, \mu)$, equipped with some distance as in [8] and [9] is a Frechet space and its topology is equivalent to the local convergence in measure. In the sequel, the spaces $\mathcal{M}(\mathcal{X}, \mathcal{B}, \mu)$ and $L^1(\mathcal{X},\mathcal{B},\mu)$ are noted simply $\mathcal{M}$ and L^1.

Definition 2.1.A:$\mathcal{X} \to 2^{\mathcal{X}}$ an operator, possibly multivalued in $\mathcal{X}$, is said to be **accretive** in $\mathcal{X}$, if one of the following equivalent properties is satisfied,

(i) $(x_1-x_2, y_1-y_2) \ge o$, if $x_1,x_2 \in D(A)$, $y_1 \in Ax_1$, $y_2 \in Ax_2$.

(ii) The resolvent$J_\lambda^A = (I + \lambda A)^{-1}$ is a contraction from R(I+λA) to X, for everyλ>0.

Definition 2.2. A is **m-accretive** in $\mathcal{X}$, if the resolvent J_λ^A is a contraction everywhere defined in $\mathcal{X}$, for every $\lambda>0$.

Definition 2.3. The operator A is **m-completely accretive** in X, if A is m-accretive and

$$\int_X (AU_1-AU_2)p(U_1-U_2)\geq 0,\ U_1,\ U_2\in D(A),\ p\in\mathcal{P}_0. \qquad (2.1)$$

$\mathcal{P}_0=\{p: \mathbb{R}\to\mathbb{R}$, p Lipschitz, odd, non decreasing and p' has a compact support$\}$.

The function a of Leray-Lions type is defined as follows,

$$(\mathbf{H.1})\begin{cases} a:\Omega\times\mathbb{R}^N\to\mathbb{R}^N \\ (x,\xi)\mapsto a(x,\xi)\end{cases} \text{is a Carathéodory function in the sense that, a is continous in } \xi,$$

for almost every $x\in\Omega$, and measurable in x for any $\xi\in\mathbb{R}^N$.

(H.2) there exist p, $C_1\in\mathbb{R}$, $1<p<N$, and $C_1>0$, so that, $\langle a(x,\xi),\xi\rangle\geq C_1|\xi|^p$, for a.e. $x\in\Omega$ and every $\xi\in\mathbb{R}^N$.

(H.3) $\langle a(x,\xi_1)-a(x,\xi_2),\xi_1-\xi_2\rangle>0$, *if* $\xi_1\neq\xi_2$, *for a.e.* $x\in\Omega$.

(H.4) There exists some $h_0\in L^{p'}(\Omega)$, $p'=\frac{p}{p-1}$ and a positive constant C_2 such that $|a(x,\xi)|\leq C_2(h_0(x)+|\xi|^{p-1})$, for a.e. $x\in\Omega$ and every $\xi\in\mathbb{R}^N$.

If A_1 is an m-accretive operator in L^1**(1)**, its closure $A=\overline{A_1}$ is defined as follows: For any $(u,f)\in\mathcal{M}\times L^1$, then A u = f, if there exist $(u_n,f_n)\in(L^1)^2$ such that, $(u_n,f_n)\to(u,f)$ in $\mathcal{M}\times L^1$ and $A u_n=f_n$.

(1): A_1 is Kato accretive in L^1, in the sense that for any pair $(u_1,f_1),(u_2,f_2)\in D(A_1)\times L^1$, we have:

$$\lim_{t\to 0}\left(\int_X |(f-g)+t(u-v)|-|f-g|\right)=\int_X (f-g)\mathrm{sign}_0(u-v)+\int_{\{u=v\}}|f-g|\geq 0,$$

where $\mathrm{sign}(r)=\mathrm{sign}_0(r)=\frac{r}{|r|}$, if $r\in\mathbb{R}, r\neq 0$, $\mathrm{sign}(0)=[-1,1]$ and $\mathrm{sign}_0(0)=0$.

Theorem 2.1. A_1 and A are defined as previously, let be given $f,g\in L^1$, then we have the following:

(i) If u, v $\in D(A)$ are solutions to Au=f and Av=g, then they satisfy:

$$\int_X (f-g)\mathrm{sign}_0(u-v)+\int_{\{u=v\}}|f-g|\geq 0. \qquad (2.2)$$

(ii) If in addition u, v $\in D(A_1)$, then: $\int_X (f-g)\varphi\geq 0$, for every $\varphi\in\mathrm{sign}(u-v)$ (2.3)

Proof. (i) If $(u_n,f_n),(v_n,g_n)\in(L^1)^2$, $A_1u_n=f_n$ and $A_1v_n=g_n$, $u_n\to u, v_n\to v$ in $\mathcal{M}$ and $f_n\to f$, $g_n\to g$ in L^1. Since A_1 is accretive in L^1, if setting $h_n=f_n-g_n$, $\omega_n=u_n-v_n$, h=f-g, ω=u-v, $\varphi_n=\mathrm{sign}_0(\omega_n)$ and $\varphi=\mathrm{sign}_0(\omega)$, this leads to:

$$\int_X h_n\varphi_n+\int_{\{\omega=0\}}|h_n|\geq 0, \forall n\in\mathbb{N}. \qquad (2.4)$$

Next, if $T_{\frac{1}{k}}(r)=\max\left\{-\frac{1}{k},\min\left(\frac{1}{k},r\right)\right\}, k\in\mathbb{N}^*, r\in\mathbb{R}$, then

$$\left|\int_X h\varphi-\int_X h_n\varphi_n\right|\leq\int_X |h|\left|\varphi-kT_{\frac{1}{k}}(\omega)\right|+\int_X |h|\left|kT_{\frac{1}{k}}(\omega)-kT_{\frac{1}{k}}(\omega_n)\right|+\int_X |h-h_n|\left|kT_{\frac{1}{k}}(\omega_n)\right|+\int_X |h_n|\left|kT_{\frac{1}{k}}(\omega_n)-\varphi_n\right|$$
$$=I_1+I_2+I_3+I_4$$

Then, by appliying the Lebesgue theorem successively to I_1, I_2, I_3, we may assume that:

$\forall\varepsilon>0, \exists k_0, n_0=n_0(k_0), n_1\in\mathbb{N}^*$, s.t. $I_1\leq\varepsilon$, if $k\geq k_0$, $I_2\leq\varepsilon$, if $n\geq n_0$ and $I_3\leq\varepsilon$, if $n\geq n_1$. Thus if n and k are large enough, then: $\left|\int_X h\varphi-\int_X h_n\varphi_n\right|\leq 3\varepsilon+\int_X |h_n|\left|kT_{\frac{1}{k}}(\omega_n)-\varphi_n\right|$, thus $\lim_{n\to+\infty}\left|\int_X h\varphi-\int_X h_n\varphi_n\right|\leq 3\varepsilon+\int_X |h|\left|kT_{\frac{1}{k}}(\omega)-\varphi\right|$.

Since the left term in the last inequality do not depend to k and $\lim_{k\to+\infty}\int_X |h|\left|kT_{\frac{1}{k}}(\omega)-\varphi\right|=0$, then

$\lim_{n\to+\infty}\int_X h_n\varphi_n=\int_X h\varphi$, hence

$\forall\varepsilon>0, \exists n_\varepsilon\in\mathbb{N}$, such that $\int_X h\varphi\geq\int_X h_n\varphi_n-\varepsilon$ if $n\geq n_\varepsilon$. (2.5)

Now, for the right term in (2.4), denote $E_n=\{\omega_n=0\}$, if C is a compact in $\mathcal{X}$ and $\eta>0$, then by Egorov theorem: $\forall k\in\mathbb{N}^*, \exists N_k\in\mathcal{B}, N_k\subset C$ and $n_k\in\mathbb{N}$ so that $n\geq n_k$, then $\mu(N_k)\leq\eta$, $|u_n-u|\leq\frac{1}{k}$ and $|v_n-v|\leq\frac{1}{k}$ uniformly on $C\backslash N_k$, therefore, after possibly replacing N_k with $\cup_{1\le l\le k} N_l$, we assume that $(N_k)_k$ is increasing and while setting $F_k = \cup_{n\ge n_k} E_n$ and $F_k' = (F_k \cap (C\backslash N_k))_k$, it ensuer that $F_k' \downarrow F_{k\ge1} = \cap F_k'$, if $k \to +\infty$.

Next, for μ almost any $x\in F_k'$, there exist n such that

$$|u(x) - v(x)| \le |u(x) - u_n(x)| + |u_n(x) - v_n(x)| + |v_n(x) - v(x)| \le \frac{2}{k}.$$

Thus u = v, μ.a.e on F,

Therefore $F_k' \downarrow \cap_{k\ge1} F_k' \subset E = \{\omega = 0\}$ and $\int_E |h| \ge \int_{F_k'} |h| - \varepsilon$, of sufficiently large k. (2.6)

Now, with the help of (2.5) and (2.4), if n is large enough, then $\int_X h\varphi + \int_{F_k} |h_n| \ge \int_X h_n\varphi_n - \varepsilon + \int_{E_n} |h_n| \ge -\varepsilon$.

As $\lim_{n\to+\infty} \int_{F_k} |h_n| = \int_{F_k} |h|$, it arises that $\int_X h\varphi + \int_{F_k} |h| \ge -\varepsilon$,

For every ε>0. Therefore $\int_X h\varphi + \int_{F_k} |h| \ge 0$, if $k \ge k_0$. (2.7)

Next, $F_k = F_k' \cup (F_k \cap N_k) \cup (F_k \cap C^c)$. Since $h\in L^1$, then we may suppose that the compact C is sufficiently large and $\mu(N_k)$ is sufficiently small so that $\int_{C^c} |h| \le \varepsilon$ and $\int_{N_k} |h| \le \varepsilon$, therefore $\int_E |h| \ge \int_{F_k'} |h| - \varepsilon \ge \int_{F_k} |h| - 3\varepsilon$ and then, in view of (2.6) and (2.7).

$$\int_X (f-g)\mathrm{sign}_0(u-v) + \int_{\{u=v\}} |f-g| = \int_X h\varphi + \int_E |h| \ge -4\varepsilon, \text{ for any } \varepsilon>0.$$

This completes the *proof* of (2.2).

(ii) Its *proof* is the same as in [8] and [9], we give an outline for this. For α>0, consider

$W_{i,\alpha} = (\omega_{i,\alpha}, \tau\omega_{i,\alpha}) = J_\alpha^{A_1} u_i = (I + \alpha A_1)^{-1} u_i \in D(A_1)$, i=1, 2,

$$Y_{i,\alpha} = (y_{i,\alpha}, z_{i,\alpha}) = A_1 W_{i,\alpha} = A_{1,\alpha} u_i = \frac{1}{\alpha}(u_i - W_{i,\alpha})$$

Consider $p_n(r) = nT_{\frac{1}{n}}(r)$ and $j_n(r) = \int_0^r p_n(s)ds$, then by definition of the subdifferential $\partial j_n = j_n'$, we have

$j_n(u_1 - u_2) - j_n(\omega_{1,\alpha} - \omega_{2,\alpha}) \ge \alpha p_n(\omega_{1,\alpha} - \omega_{2,\alpha})(y_{1,\alpha} - y_{2,\alpha}) \ge 0$, μ.a.e. on X. Then

$$\int_X j_n(u_1 - u_2) - \int_X j_n(\omega_{1,\alpha} - \omega_{2,\alpha}) \ge \alpha \int_X p_n(\omega_{1,\alpha} - \omega_{2,\alpha})(y_{1,\alpha} - y_{2,\alpha}).$$

Since $|j_n(r)| \uparrow j(r) = |r|$, if $r \to +\infty$, then applying the Lebesgue convergence theorem in $L^1(\mathcal{X}, \mathcal{B}, \mu)$, we obtain:

$$\int_X (|u_1 - u_2| - |\omega_{1,\alpha} - \omega_{2,\alpha}|) \ge 0$$

Next, ∂j(r)=sign(r), then for every $\varphi\in$sign(u_1-u_2), μ.a.e. on X we have:

$|u_1 - u_2| - |\omega_{1,\alpha} - \omega_{2,\alpha}| \le [(u_1 - \omega_{1,\alpha}) - (u_2 - \omega_{2,\alpha})]\varphi = \alpha.(y_{1,\alpha} - y_{2,\alpha})\varphi$, on $\mathbb{R}^N$, then

$$\int_X (y_{1,\alpha} - y_{2,\alpha})\varphi \ge \int_X (|u_1 - u_2| - |\omega_{1,\alpha} - \omega_{2,\alpha}|) \ge 0$$

Since, $Y_{i,\alpha}\to A_1u_i$ in $L^1(\mathcal{X}, \mathcal{B}, \mu)$, if α→0, then (2.3) is proved.

Applications

We consider the problem

$$\begin{cases} -\mathrm{div}[a(.,Du)] = f \text{ in } \Omega \\ a(.,Du)\nu = g \text{ on } \partial\Omega \end{cases} \quad (3.1)$$

Where f, g $\in L^1$. Let $T_k(r)$=max{-k, min(r,k)}, k>0 and r∈Ω. $\mathcal{M}(\Omega)$={u:Ω→ ℝ, u is measurable}.

$\mathcal{L}_0(\Omega)$={u∈$\mathcal{M}(\Omega)$, such that meas{$|u| > k$}<+∞, for every k>0}.

Definition 3.1. u is an entropy solution for the problem (3.1), if u $\in \mathcal{L}_0(\Omega)$, $DT_k(u)\in L^p(\Omega)$, ∀k>0 and $\forall\varphi\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$,

$$\int_{\Omega} a(x,Du)DT_k(u-\varphi) \le \int_{\Omega} fT_k(u-\varphi) + \int_{\partial\Omega} gT_k(\tau u - \tau\varphi).$$

Definition 3.2. u is a renormalized solution for the problem (3.1), if $u \in \mathcal{L}_0(\Omega)$, $DT_k(u) \in L^p(\Omega)$, $\forall k>0$,

$$\lim_{h\to+\infty} \int_{h\le|u|\le k+h} |Du|^p = 0 \text{ and } \forall\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega), \int_{\Omega} S(u)a(x,Du)D\varphi + \int_{\Omega} S'(u)\varphi a(x,Du)Du = \int_{\Omega} f\varphi S(u) + \int_{\partial\Omega} g\tau\varphi S(\tau u)$$

For all regular function S such that has a compact support.

Lemma3.1. *A renormalized solution in L^1 is an entropy solution.*

Proof. If u is a renormalized solution of (3.1),$\forall\psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$

$$\int_{\Omega} S(u)a(x,Du)\nabla\psi + \int_{\Omega} S'(u)\psi a(x,Du)Du = \int_{\Omega} f\psi S(u) + \int_{\partial\Omega} g\tau\psi S(\tau u).$$

Let $\psi = T_k(u-\varphi), \varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, and S=S$_n$ with S$_n$ regular, $0 \le S_n \le 1, S(x) = 0$ if $|x| \ge n+1, S(x) = 1$ if $|x| \le n$ and radial piecewise linear. Then

$$\int_{\Omega} S_n(u)a(x,Du)DT_k(u-\varphi) + \int_{\Omega} S_n'(u)T_k(u-\varphi)a(x,Du)Du \le \int_{\Omega} fT_k(u-\varphi)S_n(u) + \int_{\partial\Omega} gT_k(\tau u - \tau\varphi)S_n(\tau u).$$

If $n\to\infty$, $S_n\to 1$, then $|fS_nT_k(u-\varphi)| \le |fT_k(u-\varphi)|$ and $|gS_nT_k(\tau u-\tau\varphi)| \le |gT_k(\tau u-\tau\varphi)|$.

By dominated convergence

$$\int_{\Omega} fS_nT_k(u-\varphi) \to \int_{\Omega} fT_k(u-\varphi) \text{ and } \int_{\partial\Omega} gS_nT_k(\tau u-\tau\varphi) \to \int_{\partial\Omega} gT_k(\tau u-\tau\varphi).$$

Since $DT_k(u-\varphi) \in L^p(\Omega)$ and $a(x,Du) \in L^{p'}(\Omega)$ then

$\int_{\Omega} S_n(u)a(x,Du)DT_k(u-\varphi) \to \int_{\Omega} a(x,Du)DT_k(u-\varphi)$ and $\left|\int_{\Omega} S_n'(u)a(x,Du)T_k(u-\varphi)Du\right| = \left|\int_{n\le|u|\le n+1} a(x,Du)T_k(u-\varphi)Du\right|$.

By (H4)$\left|\int_{n\le|u|\le n+1} a(x,Du)T_k(u-\varphi)Du\right| \le c\|T_k(u-\varphi)\|_\infty \int_{n\le|u|\le n+1}|Du|^p$ for k=1 by definition of renormalized solution:

$$\lim_{h\to\infty} \int_{h\le|u|\le h+1} |Du|^p = 0 \text{ then}$$

$$\int_{\Omega} S_n'(u)a(x,Du)T_k(u-\varphi)Du \to 0 \text{ finally}$$

$$\int_{\Omega} a(x,Du)DT_k(u-\varphi) \le \int_{\Omega} fT_k(u-\varphi) + \int_{\partial\Omega} gT_k(\tau u-\tau\varphi).$$

Then the renormalized solution is an entropy solution.

Lemma3.2. *An entropy solution is a renormalized solution.*

Proof. From the uniqueness of entropy and renormalized solutions and by Lemma 3.1 we can conclude that an entropy solution is a renormalized solution.

Theorem 3.1. If u, v $\in\mathcal{M}$ are two entropy solutions to (3.1) then u=v.

Proof. If f and g$\in L^1$, u$\in\mathcal{M}$ is an entropy solution to (3.1) and v$\in\mathcal{M}$ is a renormalized solutions to (3.1) (entropy solution). There exists $v_n \in L^1$ is a renormalized solution to (3.1) with $v_n\to v$ in $\mathcal{M}$ and $(f_n,g_n)\to(f,g)\in L^1$. Consider then, for a fixed k,

$$S_1(h) = \{|u-v_n| < k\} \cap [\{|u| < h\} \cup \{|v_n| < h\}]$$

$$S_2(h) = \{|u-v_n| < k\} \cap [\{|u| \ge h\} \cup \{|v_n| < h\}]$$

$$S_2'(h) = \{|u-v_n| < k\} \cap [\{|v_n| \ge h\} \cup \{|u| < h\}],$$

We select $\varphi = T_hv_n$ in the equation related to u. Then, taking into account that

$$\int_{S_2'} \langle a(.,Du),Du\rangle \ge 0, \text{and} \int_{S_2} \langle a(.,Du),-Dv_n\rangle \le \int_{S_2} \langle a(.,Du),Du-Dv_n\rangle, \text{we have}$$

$$-\int_{S_2}\langle a(.,Du),-Dv_n\rangle+\int_{S_1}\langle a(.,Du),Du-Dv_n\rangle\le\int_{\Omega} fT_k(u-T_hv_n)+\int_{\partial\Omega} gT_k(\tau u-T_h\tau v_n).$$

On the other hand by (H4),

$$\left|\int_{S_2}\langle a(.,Du),-Dv_n\rangle\right|\le C\|Dv_n\|_{L^p(\{h-k\le|v|<h\})}\times\left(\|h_0\|_{L^{p'}(\{h\le|u|<h+k\})}+\||Du|^{p-1}\|_{L^{p'}(\{h\le|u|<h+k\})}\right)$$

$$\text{or}\lim_{h\to+\infty}\int_{h\le|v_n|<h+k}|Dv_n|^p=0 \text{ then } \lim_{h\to+\infty}\int_{S_2}\langle a(.,Du),-Dv_n\rangle=0$$

Next, we do the same for the equation related to v_n, with test function $\varphi=T_hu$ and add the two inequalities.

$$\int_{\Omega}\lim_{h\to+\infty}\langle a(.,Du)-a(.,Dv_n),Du-Dv_n\rangle\mathbf{1}_{S_1(h)}+\lim_{h\to+\infty}\int_{S_2(h)}\langle a(.,Du),-Dv_n\rangle+\lim_{h\to+\infty}\int_{S_2'(h)}\langle a(.,Dv_n),-Du\rangle$$
$$\le\lim_{h\to+\infty}\int_{\Omega} f\big(T_k(u-T_hv_n)\big)+f_n\big(T_k(v_n-T_hu)\big)+\lim_{h\to+\infty}\int_{\partial\Omega} g\big(T_k(\tau u-\tau T_hv)\big)+g_n\big(T_k(\tau v_n-\tau T_hu)\big)$$

Then, by applying the Lebesgue dominated convergence on the right, and letting $n\to\infty$, we obtain

$$\int_{\{|u-v|<k\}}\langle \mathbf{a}(.,Du)-\mathbf{a}(.,Dv),Du-Dv\rangle=0, k>0.$$

It arises from H3 (Leary-Lions), that Du=Dv, a.e in Ω (if $n\to\infty$) and therefore u-v=0, a.e. in Ω. This leads to $\tau T_ku=\tau T_kv$ a.eon $\partial\Omega$, for any k>0. Thus $\tau u=\tau v$ a.eon $\partial\Omega$.

Theorem 3.2. Ifβ, γ are nondecreasing continuous function on $\mathbb{R}$ such that $\beta(0)=\gamma(0)=0$, $(f,g)\in L^1$, then, there exists a unique entropy solution u in $\mathcal{M}$ to the problem:

$$\begin{cases}-\mathrm{div}[a(.,Du)]+\beta(u)=f \text{ in } \Omega\\ a(.,Du)\nu+\gamma(\tau u)=g \text{ on } \partial\Omega\end{cases}\qquad(3.2)$$

Proof. If u and v are two entropy solutions to (3.2) in $\mathcal{M}$ with the same data $(f,g)\in L^1$, then applying (2.2), since $\beta(u)=\beta(v)$ a.e. on {u=v} and $\gamma(\tau u)=\gamma(\tau v)d\sigma$–a.e. on {u=v}, one obtain:

$$-\int_{\Omega}|\beta(u)-\beta(v)|-\int_{\partial\Omega}|\gamma(\tau u)-\gamma(\tau v)|\ge 0 \text{ thus } \beta(u)=\beta(v)\text{a.e. on } \Omega \text{ and } \gamma(\tau u)=\gamma(\tau v) \text{ on } \partial\Omega.$$

If $\beta(u)$ = **h** and $\gamma(\tau u)$ = **k**, then u and v are two entropy solutions in $\mathcal{M}$ to the problem, $A(u,\tau u)$ = (f-**h**,g-**k**). Then, the uniqueness of the entropy solution u to (3.2) derives from Theorem 3.1.

Corollary 3.1. If λ and $\tilde{\lambda}$ are bounded measure on Ω and $\partial\Omega$, then $-\mathrm{div}[a(.,Du)]+\beta(u)\ni\lambda$ on Ω and $a(.,Du).\nu+\gamma(\tau u)\ni\tilde{\lambda}$ on $\partial\Omega$ has at least a weak solution.

Proof. Set $AU=(-\mathrm{div}[a(.,Du)],a(.,Du).\nu)$, if $U_n,F_n\in L^1(\Omega\cup\partial\Omega)$ is some approximative sequence of solutions to the equation $AU_n+H_n\ni F_n, H_n\in BU_n$ previous arguments is that (H_n) is a Cauchy sequence in L^1, then the classical methods are applied.

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