



Argument estimates of certain classes of P-Valent meromorphic functions involving certain operator

A. O. Mostafa¹, M. K. Aouf² and S. M. Madian³

^{1,2}Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt.
³ Basic Sciences Department, Higher Institute of
Engineering and Technology, New Damietta , Egypt
¹adelaeg254@yahoo.com, ²mkaouf127@yahoo.com
³samar_math@yahoo.com

ABSTRACT

In this paper, by making use of subordination , we investigate some inclusion relations and argument properties of certain classes of p-valent meromorphic functions involving certain operator.

Indexing terms/Keywords

Argument estimates, Hadamard product, certain operator, meromorphic functions.

SUBJECT CLASSIFICATION

2010 Mathematics Subject Classification: 30C45.

1. Introduction

For any integer $m > -p$, let $\Sigma_{p,m}$ denote the class of meromorphic functions $f(z)$ of the form:

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \quad (m > -p; p \in \mathbf{N} = \{1, 2, \dots\}), \quad (1.1)$$

which are analytic and p -valent in the punctured open unit disc $\mathbf{U}^* = \{z \in \mathbf{C} : 0 < |z| < 1\} = \mathbf{U} \setminus \{0\}$. For convenience, we write $\Sigma_{p,1-p} = \Sigma_p$ and $\Sigma_{1,0} = \Sigma$. If $f(z)$ and $g(z)$ are analytic in \mathbf{U} , we say that $f(z)$ is subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z)$ ($z \in \mathbf{U}$), if there exists a Schwarz function $w(z)$ in \mathbf{U} with $w(0) = 0$ and $|w(z)| < 1$ ($z \in \mathbf{U}$), such that $f(z) = g(w(z))$ ($z \in \mathbf{U}$). Furthermore, if $g(z)$ is univalent in \mathbf{U} , then the following equivalence relationship holds true (see [7] and [18]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbf{U}) \subset g(\mathbf{U}).$$

For functions $f(z) \in \Sigma_{p,m}$, given by (1.1) and $g(z) \in \Sigma_{p,m}$ defined by

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k \quad (m > -p; p \in \mathbf{N}),$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by



$$(f * g)(z) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k = (g * f)(z).$$

For complex parameters $\alpha_1, \alpha_2, \dots, \alpha_q$ and

$\beta_1, \beta_2, \dots, \beta_s$ ($\alpha_i, \beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}, i = 1, 2, \dots, q, j = 1, 2, \dots, s$), the generalized hypergeometric

function ${}_qF_s(\alpha_1, \alpha_2, \dots, \alpha_q; \beta_1, \beta_2, \dots, \beta_s; z)$ is defined by (see [22])

$${}_qF_s(\alpha_1, \alpha_2, \dots, \alpha_q; \beta_1, \beta_2, \dots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_q)_k}{(\beta_1)_k \dots (\beta_s)_k (1)_k} z^k$$

$$(q \leq s+1; s, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}),$$

where $(\theta)_k$, is the Pochhammer symbol defined in terms of the Gamma function Γ , by

$$(\theta)_v = \frac{\Gamma(\theta+v)}{\Gamma(\theta)} = \begin{cases} 1 & (v=0; \theta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}) \\ \theta(\theta+1)\dots(\theta+v-1) & (v \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases}$$

Liu and Srivastava [16] and Aouf [4] investigated recently the operator $\Upsilon_{p,q,s}(\alpha_1, \alpha_2, \dots, \alpha_q;$

$\beta_1, \beta_2, \dots, \beta_s): \Sigma_{p,m} \rightarrow \Sigma_{p,m}$, defined as follows:

$$\begin{aligned} \Upsilon_{p,q,s}(\alpha_1) &= \Upsilon_{p,q,s}(\alpha_1, \alpha_2, \dots, \alpha_q; \beta_1, \beta_2, \dots, \beta_s; z) = z^{-p} {}_qF_s(\alpha_1, \alpha_2, \dots, \alpha_q; \beta_1, \beta_2, \dots, \beta_s; z), \\ &= z^{-p} + \sum_{k=1-p}^{\infty} \frac{(\alpha_1)_{k+p} \dots (\alpha_q)_{k+p}}{(\beta_1)_{k+p} \dots (\beta_s)_{k+p} (1)_{k+p}} a_k z^k. \end{aligned} \quad (1.2)$$

With aid of the function $\Upsilon_{p,q,s}(\alpha_1)$ given by (1.2), consider the function $\Upsilon_{p,q,s}^*(\alpha_1)$ defined by:

$$\Upsilon_{p,q,s}(\alpha_1) * \Upsilon_{p,q,s}^*(\alpha_1) = \frac{1}{z^p (1-z)^{\lambda+p}} (\lambda > -p; p \in \mathbb{N}; z \in \mathbb{U}^*). \quad (1.3)$$

This function leads us to the following family of linear operators $\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1): \Sigma_{p,m}^{\lambda} \rightarrow \Sigma_{p,m}^{\lambda}$, which are given by:

$$\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) = \Upsilon_{p,q,s}^*(\alpha_1) * f(z) (f \in \Sigma_{p,m}). \quad (1.4)$$

The linear operator $\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)$ was defined by Patel and Patil [20] and Mostafa [17]. If $f(z)$ is given by (1.1), then from (1.4), we deduce that

$$\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) f(z) = z^{-p} + \sum_{k=m}^{\infty} \frac{(\beta_1)_{k+p} \dots (\beta_s)_{k+p} (\lambda+p)_{k+p}}{(\alpha_1)_{k+p} \dots (\alpha_q)_{k+p}} a_k z^k$$



$$(f \in \Sigma_{p,m}; \lambda, m > -p; p \in \mathbf{N}; z \in \mathbf{U}^*). \tag{1.5}$$

It is easily verified from (1.5) that (see [20] and [17])

$$z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z))' = (\lambda + p)\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z) - (\lambda + 2p)\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z) \tag{1.6}$$

and

$$z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1 + 1)f(z))' = \alpha_1\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z) - (\alpha_1 + p)\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1 + 1)f(z). \tag{1.7}$$

For a function $f \in \Sigma_p$ and $\mu > 0$, let $F_{\mu,p} : \Sigma_p \rightarrow \Sigma_p$ be the integral operator defined by (see [13]):

$$F_{\mu,p}(f)(z) = \frac{\mu}{z^{\mu+p}} \int_0^z t^{\mu+p-1} f(t) dt = z^{-p} + \sum_{k=1-p}^{\infty} \frac{\mu}{\mu+k+p} a_k z^k \tag{1.8}$$

$$(f \in \Sigma_p; \mu > 0; p \in \mathbf{N}; z \in \mathbf{U}^*).$$

It follows from (1.8) that:

$$z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)F_{\mu,p}(f)(z))' = \mu\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z) - (\mu + p)\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)F_{\mu,p}(f)(z). \tag{1.9}$$

We note that:

Putting $\lambda = 1 - p$ ($p \in \mathbf{N}$) in (1.5), then the operator $\mathbf{M}_{p,q,s}^{1-p,m}(\alpha_1)$ reduces to the operator $\mathbf{M}_{p,q,s}^m(\alpha_1)$, defined by:

$$\mathbf{M}_{p,q,s}^m(\alpha_1)f(z) = z^{-p} + \sum_{k=m}^{\infty} \frac{(\beta_1)_{k+p} \dots (\beta_s)_{k+p}}{(\alpha_1)_{k+p} \dots (\alpha_q)_{k+p}} a_k z^k \quad (f \in \Sigma_{p,m}; m > -p; p \in \mathbf{N}; z \in \mathbf{U}^*)$$

Also, by specializing the parameters λ, m, p, α_i ($i = 1, 2, \dots, q$), β_j ($j = 1, 2, \dots, s$), q and s , we have:

- (i) $\mathbf{M}_{p,2,1}^{0,m}(p, p; p)f(z) = \mathbf{M}_{p,2,1}^1(p+1, p; p)f(z) = f(z)$ ($p \in \mathbf{N}$);
- (ii) $\mathbf{M}_{p,2,1}^{1,m}(p, p; p)f(z) = \frac{2pf(z) + zf'(z)}{p}$ ($p \in \mathbf{N}$);
- (iii) $\mathbf{M}_{p,2,1}^{2,m}(p+1, p; p)f(z) = \frac{(2p+1)f(z) + zf'(z)}{p+1}$ ($p \in \mathbf{N}$);
- (iv) $\mathbf{M}_{p,2,1}^{n,0}(a, 1; a)f(z) = D^{n+p-1}f(z)$ ($n > -p, a > 0, p \in \mathbf{N}$) (see Yang [23] and Aouf ([2] and [3])), which for $p = 1$ reduces to the operator $D^n f(z)$ ($n > -1$) (see Cho [8]);
- (v) $\mathbf{M}_{p,2,1}^{0,m}(p+1, p; p)f(z) = \frac{p}{z^{2p}} \int_0^z t^{2p-1} f(t) dt$ ($p \in \mathbf{N}$);
- (vi) $\mathbf{M}_{p,2,1}^{1-p,m}(\mu+1, 1; \mu)f(z) = F_{\mu,p}(f)(z)$ ($p \in \mathbf{N}, \mu > 0$), this integral operator is defined by (1.8);
- (vii) $\mathbf{M}_{p,2,1}^{\lambda,m}(c, p+\lambda; a)f(z) = L_p(a; c)f(z)$ ($p \in \mathbf{N}, a \in \mathbf{R}, c \in \mathbf{R} \setminus \mathbf{Z}_0^-$) (see Liu [15]);



(viii) $\mathbf{M}_{p,2,1}^{\lambda,1-p}(a,1;c)f(z) = L_p^\lambda(a;c)f(z)$ ($\lambda > -p, p \in \mathbf{N}, a, c \in \mathbf{R} \setminus Z_0^-$) (see Aouf et al. [5]), which for $p=1$ reduces to $L^\lambda(a;c)$ (see Aghalary [1]);

(ix) $\mathbf{M}_{p,2,1}^{\eta-p,1-p}(n+p,\eta;\eta)f(z) = I_{n+p-1,\eta}f(z)$ ($\eta > 0, n > -p, p \in \mathbf{N}$) (see Aouf and Xu [6]), which $p=1$ reduces to $I_{n,\eta}$ (see Yuan et al. [24]);

(x) $\mathbf{M}_{1,q,s}^{\sigma-1,0}(\alpha_1)f(z) = H_{\sigma,q,s}(\alpha_1)f(z)$ ($\sigma > 0$) (see Cho and Kim [9]);

(xi) $\mathbf{M}_{1,2,1}^{x-1,0}(a,1;c)f(z) = I_x(a,c)f(z)$ ($x > 0, a, c \in \mathbf{R} \setminus Z_0^-$) (Cho and Noor [10]).

Let M be the class of functions $h(z)$ which are analytic and univalent in \mathbf{U} and for which $h(\mathbf{U})$ is convex with $h(0)=1$ and $\text{Re}\{h(z)\} > 0, z \in \mathbf{U}$.

Now, by using the linear operator $\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)$, we define a subclass of $\Sigma_{p,m}$ by

$$\Sigma_{p,q,s}^{\lambda,m}(\alpha_1;h) = \left\{ f : f \in \Sigma_{p,m} \text{ and } -\frac{z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z))'}{p\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)} \prec h(z) (h \in M; z \in \mathbf{U}) \right\} \quad (1.10)$$

We also set

$$\Sigma_{p,q,s}^{\lambda,m}(\alpha_1; \frac{1+Az}{1+Bz}) = \Sigma_{p,q,s}^{\lambda,m}(\alpha_1; A, B) \quad (-1 < B < A \leq 1; z \in \mathbf{U}). \quad (1.11)$$

From (1.10) and (1.11) and by using the result of Silverman and Silvia [21], we observe that a function $f(z)$ is in the class $\Sigma_{p,q,s}^{\lambda,m}(\alpha_1; A, B)$ if and only if

$$\left| \frac{z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z))'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)} + \frac{p(1-AB)}{1-B^2} \right| < \frac{p(A-B)}{1-B^2} \quad (-1 < B < A \leq 1; z \in \mathbf{U}). \quad (1.12)$$

In the present paper, we investigate some inclusion relationships and argument properties of certain meromorphically p -valent functions in \mathbf{U}^* in connection with the linear operator $\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)$.

2. Preliminaries

In order to prove our main results, we need to the following lemmas.

Lemma 1 [12]. Let β and ν be complex constants and let $h(z)$ be convex (univalent) in \mathbf{U} with $h(0) = 1$ and $\text{Re}\{\beta h(z) + \nu\} > 0$. If $q(z) = 1 + q_1z + \dots$ is analytic in \mathbf{U} , then

$$q(z) + \frac{zq'(z)}{\beta q(z) + \nu} \prec h(z),$$



implies

$$q(z) \prec h(z).$$

Lemma 2 [18]. Let $h(z)$ be convex (univalent) in \mathbf{U} and $\psi(z)$ be analytic in \mathbf{U} with $\operatorname{Re}\{\psi(z)\} \geq 0$. If q is analytic in \mathbf{U} and $q(0) = h(0)$, then

$$q(z) + \psi(z)zq'(z) \prec h(z),$$

implies

$$q(z) \prec h(z).$$

Lemma 3 [19]. Let $q(z)$ be analytic in \mathbf{U} , with $q(0) = 1$ and $q(z) \neq 0$ ($z \in \mathbf{U}$). If there exists a point $z_0 \in \mathbf{U}$, such that

$$|\arg q(z)| < \frac{\pi}{2} \tau \quad \text{for } |z| < |z_0| \quad (2.1)$$

and

$$|\arg q(z_0)| = \frac{\pi}{2} \tau \quad (0 < \tau \leq 1). \quad (2.2)$$

Then we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ix\tau, \quad (2.3)$$

where

$$x \geq \frac{1}{2} \left(b + \frac{1}{b}\right) \quad \text{when } \arg q(z_0) = \frac{\pi}{2} \tau, \quad (2.4)$$

$$x \geq -\frac{1}{2} \left(b + \frac{1}{b}\right) \quad \text{when } \arg q(z_0) = -\frac{\pi}{2} \tau \quad (2.5)$$

and

$$q(z_0)^{\frac{1}{\tau}} = \pm ib \quad (b > 0). \quad (2.6)$$

3. Some inclusion relationships

By using Lemma 1, we obtain the following results:



Theorem 1. Let $h(z) \in M$ with $\max_{z \in U} \operatorname{Re}\{h(z)\} < \min\{\frac{\lambda+2p}{p}, \frac{\alpha_1+p}{p}\}$ ($\mathcal{U} \neq \emptyset$, $p \in \mathbb{N}$). Then

$$\Sigma_{p,q,s}^{\lambda+1,m}(\alpha_1; h) \subset \Sigma_{p,q,s}^{\lambda,m}(\alpha_1; h) \subset \Sigma_{p,q,s}^{\lambda,m}(\alpha_1 + 1; h).$$

Proof. To prove the first part, we show that $\Sigma_{p,q,s}^{\lambda+1,m}(\alpha_1; h) \subset \Sigma_{p,q,s}^{\lambda,m}(\alpha_1; h)$. Let $f \in \Sigma_{p,q,s}^{\lambda+1,m}(\alpha_1; h)$ and set

$$R(z) = -\frac{z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z))'}{p\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)} \quad (z \in U), \quad (3.1)$$

where $R(z)$ is analytic with $R(0) = 1$. Using (1.6) in (3.1), we obtain

$$pR(z) - (\lambda + 2p) = -(\lambda + p) \frac{\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z)}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)}. \quad (3.2)$$

Differentiating (3.2) logarithmically with respect to z and multiplying by z , we have

$$R(z) + \frac{zR'(z)}{-pR(z) + \lambda + 2p} = -\frac{z(\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z))'}{p\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z)} < h(z), \quad (3.3)$$

from Lemma 1, it follows that $R(z) < h(z)$ in U , that is, that $f \in \Sigma_{p,q,s}^{\lambda,m}(\alpha_1; h)$.

To prove the second part, let $f \in \Sigma_{p,q,s}^{\lambda,m}(\alpha_1; h)$ and put

$$s(z) = -\frac{z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1 + 1)f(z))'}{p\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1 + 1)f(z)} \quad (z \in U),$$

then, by using the arguments similar to those detailed above and using (1.7) instead of (1.6), it follows that $s(z) < h(z)$ in U , which implies $f \in \Sigma_{p,q,s}^{\lambda,m}(\alpha_1 + 1; h)$. Therefore we complete the proof of Theorem 1.

Taking $h(z) = \frac{1+Az}{1+Bz}$ ($-1 < B < A \leq 1$) in Theorem 1, we have

Corollary 1. Let $\frac{1+A}{1+B} < \min\{\frac{\lambda+2p}{p}, \frac{\alpha_1+p}{p}\}$ and $-1 < B < A \leq 1$. Then

$$\Sigma_{p,q,s}^{\lambda+1,m}(\alpha_1; A, B) \subset \Sigma_{p,q,s}^{\lambda,m}(\alpha_1; A, B) \subset \Sigma_{p,q,s}^{\lambda,m}(\alpha_1 + 1; A, B).$$

Theorem 2. Let $h(z) \in M$ with $\operatorname{Re}\{h(z)\} < \frac{\mu+p}{p}$ ($\mu > 0$), if $f \in \Sigma_{p,q,s}^{\lambda,m}(\alpha_1; h)$, then

$F_{\mu,p}(f) \in \Sigma_{p,q,s}^{\lambda,m}(\alpha_1; h)$, where $F_{\mu,p}(f)$ is defined by (1.8).

Proof. Let $f \in \Sigma_{p,q,s}^{\lambda,m}(\alpha_1; h)$ and set

$$L(z) = -\frac{z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)F_{\mu,p}(f)(z))'}{p\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)F_{\mu,p}(f)(z)} \quad (z \in U), \quad (3.4)$$

where $L(z)$ is analytic with $L(0) = 1$. Applying (1.9) to (3.4), we get



$$pL(z) - (\mu + p) = -\mu \frac{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)F_{\mu,p}(f)(z)}. \quad (3.5)$$

Differentiating (3.5) logarithmically with respect to z and multiplying by z , we have

$$L(z) + \frac{zL'(z)}{-pL(z) + \mu + p} = -\frac{z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z))'}{p\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)} \prec h(z).$$

Hence, by virtue of Lemma 1, we conclude that $L(z) \prec h(z)$ in \mathbf{U} , which implies

$F_{\mu,p}(f) \in \Sigma_{p,q,s}^{\lambda,m}(\alpha_1; h)$. This completes the proof of Theorem 2.

Taking $h(z) = \frac{1+Az}{1+Bz}$ ($-1 < B < A \leq 1$) in Theorem 2, we have

Corollary 2. Let $\frac{1+A}{1+B} < \frac{\mu+p}{p}$ ($\mu > 0$) and $-1 < B < A \leq 1$, if $f \in \Sigma_{p,q,s}^{\lambda,m}(\alpha_1; A, B)$, then

$F_{\mu,p}(f) \in \Sigma_{p,q,s}^{\lambda,m}(\alpha_1; A, B)$.

4. Some argument properties

Theorem 3. Let $f(z) \in \Sigma_{p,m}$, $0 < \delta \leq 1$, $0 \leq \ell < p$ and

$$\lambda \geq \frac{p(A-B)}{1+B} - p \quad (-1 < B < A \leq 1; p \in \mathbf{N}).$$

If

$$\left| \arg \left(-\frac{z(\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z))'}{\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)g(z)} - \ell \right) \right| < \frac{\pi}{2} \delta,$$

for some $g(z) \in \Sigma_{p,q,s}^{\lambda+1,m}(\alpha_1; A, B)$, then

$$\left| \arg \left(-\frac{z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z))'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z)} - \ell \right) \right| < \frac{\pi}{2} \tau,$$

where τ ($0 < \tau \leq 1$) is the solution of the equation

$$\delta = \tau + \frac{2}{\pi} \tan^{-1} \left(\frac{\tau \cos \frac{\pi}{2} t(A, B)}{\frac{(\lambda+p)(1-B)+p(A-B)}{1-B} + \tau \sin \frac{\pi}{2} t(A, B)} \right) \quad (4.1)$$

and

$$t(A, B) = \frac{2}{\pi} \sin^{-1} \left(\frac{p(A-B)}{(\lambda+2p)(1-B^2) - p(1-AB)} \right). \quad (4.2)$$

Proof. Let



$$q(z) = \frac{1}{p-\ell} \left(-\frac{z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z))'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z)} - \ell \right), \quad (4.3)$$

where $q(z)$ is analytic with $q(0) = 1$. Applying the identity (1.6), we have

$$[-(p-\ell)q(z) - \ell]\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z) = (\lambda+p)\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z) - (\lambda+2p)\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z). \quad (4.4)$$

Differentiating (4.4) with respect to z and multiplying by z , we obtain

$$\begin{aligned} & -(p-\ell)zq'(z)\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z) + [-(p-\ell)q(z) - \ell]z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z))' \\ & = (\lambda+p)z(\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z))' - (\lambda+2p)z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z))'. \end{aligned} \quad (4.5)$$

Then, by using (4.3), (4.4) and (4.5), we have

$$\frac{1}{p-\ell} \left(-\frac{z(\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z))'}{\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)g(z)} - \ell \right) = q(z) + \frac{zq'(z)}{-r(z) + \lambda + 2p},$$

where

$$r(z) = -\frac{z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z))'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z)}.$$

From Corollary 1, since $g(z) \in \Sigma_{p,q,s}^{\lambda+1,m}(\alpha_1; A, B)$, then $g(z) \in \Sigma_{p,q,s}^{\lambda,m}(\alpha_1; A, B)$, which from (1.12) leads to

$$r(z) \prec p \frac{1+Az}{1+Bz}.$$

Letting

$$-r(z) + \lambda + 2p = \rho e^{i\frac{\pi}{2}\phi} \quad (z \in \mathbf{U}),$$

then from (1.12) we have

$$\frac{(\lambda+p)(1+B) - p(A-B)}{1+B} < \rho < \frac{(\lambda+p)(1-B) + p(A-B)}{1-B}$$

and

$$-t(A, B) < \phi < t(A, B),$$

where t is defined by (4.2).



Let h be a function which maps \mathbf{U} onto the angular domain $\{\omega : |\arg \omega| < \frac{\pi}{2} \delta\}$ with $h(0) = 1$.

Applying Lemma 2, for this h with $\psi(z) = \frac{1}{-r(z) + \lambda + 2p}$, we see that $\operatorname{Re}\{q(z)\} > 0$ in \mathbf{U} and hence $q(z) \neq 0$ in \mathbf{U} . If there exists a point $z_0 \in \mathbf{U}$ such that the conditions (2.1) and (2.2) are satisfied, then by Lemma 3, we have (2.3) under the restrictions (2.4) and (2.5).

At first, suppose that $q(z_0)^{\frac{1}{\tau}} = ib$ ($b > 0$). Then

$$\begin{aligned} & \arg \left[-\frac{1}{p-\ell} \left(\frac{z_0 \left(\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1) f(z_0) \right)'}{\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1) g(z_0)} + \ell \right) \right] \\ &= \arg \left(q(z_0) + \frac{z_0 q'(z_0)}{-r(z_0) + \lambda + 2p} \right) = \frac{\pi}{2} \tau + \arg \left(1 + i\ell \tau \left(\rho e^{i\frac{\pi}{2}\phi} \right)^{-1} \right) \\ &= \frac{\pi}{2} \tau + \tan^{-1} \left(\frac{\ell \tau \sin \frac{\pi}{2} (1-\phi)}{\rho + \ell \tau \cos \frac{\pi}{2} (1-\phi)} \right) \\ &\geq \frac{\pi}{2} \tau + \tan^{-1} \left(\frac{\tau \cos \frac{\pi}{2} t(A,B)}{\frac{(\lambda+p)(1-B)+p(A-B)}{1-B} + \tau \sin \frac{\pi}{2} t(A,B)} \right) = \frac{\pi}{2} \delta, \end{aligned}$$

where \mathfrak{A} and $t(A, B)$ are given by (4.1) and (4.2), respectively. This contradicts to the assumption of the theorem.

Next, suppose that $q(z_0)^{\frac{1}{\alpha}} = -ib$ ($b > 0$). Applying the same method as the above, we have

$$\begin{aligned} & \arg \left[-\frac{1}{p-\ell} \left(\frac{z_0 \left(\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1) f(z_0) \right)'}{\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1) g(z_0)} + \ell \right) \right] \\ &\leq -\frac{\pi}{2} \tau - \tan^{-1} \left(\frac{\tau \cos \frac{\pi}{2} t(A,B)}{\frac{(\lambda+p)(1-B)+p(A-B)}{1-B} + \tau \sin \frac{\pi}{2} t(A,B)} \right) = -\frac{\pi}{2} \delta, \end{aligned}$$

where \mathfrak{A} and $t(A, B)$ are given by (4.1) and (4.2), respectively, which contradicts the assumption. This completes the proof of Theorem 3.

Taking $q = 2, s = \lambda = \delta = A = 1, B = 0$ and $\alpha_1 = \alpha_2 = \beta_1 = p$ ($p \in \mathbf{N}$) in Theorem 3, we have the following corollary:

Corollary 3. Let $f(z) \in \Sigma_{p,m}$. If

$$-\operatorname{Re} \left\{ \frac{z(2(p+1)(2p+1)f'(z) + 4(1+p)zf''(z) + z^2f'''(z))}{2p(2p+1)g(z) + 2(2p+1)zg'(z) + z^2g''(z)} \right\} > \ell \quad (0 \leq \ell < p),$$

for some $g(z) \in \Sigma_{p,m}$ satisfying the condition



$$\left| \frac{z(2(p+1))(2p+1)g'(z) + 4(p+1)zg''(z) + z^2g'''(z)}{2p(2p+1)g(z) + 2(2p+1)zg'(z) + z^2g''(z)} + p \right| < p,$$

then

$$-\operatorname{Re} \left\{ \frac{z((2p+1)f'(z) + zf''(z))}{zf'(z) + 2pf(z)} \right\} > \ell.$$

Taking $q=2, s=\delta=A=1, \lambda=2, B=0, \alpha_1=p+1$ and $\alpha_2=\beta_1=p$ ($p \in \mathbf{N}$) in Theorem 3, we have the following corollary:

Corollary 4. Let $f(z) \in \Sigma_{p,m}$. If

$$-\operatorname{Re} \left\{ \frac{z(2((p+1))(2p+3)f'(z) + 2(2p+3)zf''(z) + z^2f'''(z))}{2(p+1)(2p+1)g(z) + 4(p+1)zg'(z) + z^2g''(z)} \right\} > \ell \quad (0 \leq \ell < p),$$

for some $g(z) \in \Sigma_{p,m}$ satisfying the condition

$$\left| \frac{z(2((p+1))(2p+3)g'(z) + 2(2p+3)zg''(z) + z^2g'''(z))}{2(p+1)(2p+1)g(z) + 4(p+1)zg'(z) + z^2g''(z)} + p \right| < p,$$

then

$$-\operatorname{Re} \left\{ \frac{z(2(p+1)f'(z) + zf''(z))}{(2p+1)g(z) + zg'(z)} \right\} > \ell.$$

Taking $q=2, s=1, \lambda=\eta-p, m=1-p, \alpha_1=n+p$ and $\alpha_2=\beta_1=\eta$ ($\eta > 0, n > -p, p \in \mathbf{N}$) in Theorem 3, we have the following corollary:

Corollary 5. Let $f(z) \in \Sigma_p, 0 < \delta \leq 1, 0 \leq \ell < p$ and

$$\eta \geq \frac{p(A-B)}{1+B} \quad (-1 < B < A \leq 1; p \in \mathbf{N}).$$

If

$$\left| \arg \left(- \frac{z(I_{n+p-1, \eta+1} f(z))}{I_{n+p-1, \eta+1} g(z)} - \ell \right) \right| < \frac{\pi}{2} \delta,$$

for some $g(z) \in \Sigma_{p,m}$ satisfying the condition

$$-\frac{z(I_{n+p-1, \eta} g(z))}{I_{n+p-1, \eta} g(z)} \prec p \frac{1+Az}{1+Bz}, \tag{4.6}$$

then



$$\left| \arg \left(- \frac{z(I_{n+p-1,\eta} f(z))'}{I_{n+p-1,\eta} g(z)} - \ell \right) \right| < \frac{\pi}{2} \tau,$$

where $\tau(0 < \tau \leq 1)$ is the solution of the equation (4.1) with $\lambda = \eta - p$ ($\eta > 0$).

Taking $p = 1$ in Corollary 5, we have the following corollary:

Corollary 6. Let $f(z) \in \Sigma$, $0 < \delta \leq 1$, $0 \leq \ell < 1$ and

$$\eta \geq \frac{A-B}{1+B} \quad (-1 < B < A \leq 1).$$

If

$$\left| \arg \left(- \frac{z(I_{n,\eta+1} f(z))'}{I_{n,\eta+1} g(z)} - \ell \right) \right| < \frac{\pi}{2} \delta,$$

for some $g(z) \in \Sigma$ satisfying the condition

$$- \frac{z(I_{n,\eta} g(z))'}{I_{n,\eta} g(z)} \prec \frac{1+Az}{1+Bz}, \quad (4.7)$$

then

$$\left| \arg \left(- \frac{z(I_{n,\eta} f(z))'}{I_{n,\eta} g(z)} - \ell \right) \right| < \frac{\pi}{2} \tau,$$

where τ is the solution of the equation (4.1) with $\lambda = \eta - 1$ ($\eta > 0$).

The proof of the next theorem is akin to that of Theorem 3 and so, we omit it.

Theorem 4. Let $f(z) \in \Sigma_{p,m}$, $0 < \delta \leq 1$, $\ell > p$ and

$$\lambda \geq \frac{p(A-B)}{1+B} - p \quad (-1 < B < A \leq 1; p \in \mathbf{N}).$$

If

$$\left| \arg \left(\frac{z(\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1) f(z))'}{\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1) g(z)} + \ell \right) \right| < \frac{\pi}{2} \delta$$

for some $g(z) \in \Sigma_{p,q,s}^{\lambda+1,m}(\alpha_1; A, B)$, then

$$\left| \arg \left(\frac{z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) f(z))'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) g(z)} + \ell \right) \right| < \frac{\pi}{2} \tau,$$



where $\tau(0 < \tau \leq 1)$ is the solution of equation (4.1).

Theorem 5. Let $f(z) \in \Sigma_{p,m}$, $0 < \delta \leq 1$, $0 \leq \ell < p$ and

$$\alpha_1 \geq \frac{p(A-B)}{1+B} \quad (-1 < B < A \leq 1; p \in \mathbb{N}).$$

If

$$\left| \arg \left(- \frac{z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z))'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z)} - \ell \right) \right| < \frac{\pi}{2} \delta,$$

for some $g(z) \in \Sigma_{p,q,s}^{\lambda,m}(\alpha_1; A, B)$, then

$$\left| \arg \left(- \frac{z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)f(z))'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)g(z)} - \ell \right) \right| < \frac{\pi}{2} \tau,$$

where $\tau(0 < \tau \leq 1)$ is the solution of equation (4.1).

Proof. Let

$$X(z) = \frac{1}{p-\ell} \left(- \frac{z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)f(z))'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)g(z)} - \ell \right), \quad (4.8)$$

where $X(z)$ is analytic with $X(0) = 1$. Using (1.7), we have

$$[-(p-\ell)X(z) - \ell]\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)g(z) = \alpha_1\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z) - (\alpha_1+p)\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z). \quad (4.9)$$

Differentiating (4.9) with respect to z and multiplying by z , we obtain

$$\begin{aligned} & -(p-\ell)zX'(z)\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)g(z) + [-(p-\ell)X(z) - \ell]z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)g(z))' \\ & = \alpha_1z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z))' - (\alpha_1+p)z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z))'. \end{aligned} \quad (4.10)$$

Then, by using (4.8), (4.9) and (4.10), we have

$$\frac{1}{p-\ell} \left(- \frac{z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z))'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z)} - \ell \right) = X(z) + \frac{zX'(z)}{-j(z) + \alpha_1 + p},$$

where

$$j(z) = - \frac{z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)g(z))'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)g(z)}.$$



The remaining part of the proof is similar to that of Theorem 3 and so we omit it.

Taking $q = 2, s = 1, \lambda = \eta - p, m = 1 - p, \alpha_1 = n + p$ and $\alpha_2 = \beta_1 = \eta$ ($\eta > 0, n > -p, p \in \mathbf{N}$) in Theorem 5, we have the following corollary:

Corollary 7. Let $f(z) \in \Sigma_{p,m}$, $0 < \delta \leq 1, 0 \leq \ell < p$ and

$$n \geq \frac{p(A-B)}{1+B} - p \quad (-1 < B < A \leq 1; p \in \mathbf{N}).$$

If

$$\left| \arg \left(-\frac{z(I_{n+p-1,\eta} f(z))'}{I_{n+p-1,\eta} g(z)} - \ell \right) \right| < \frac{\pi}{2} \delta,$$

for some $g(z) \in \Sigma_{p,m}$ satisfying (4.6), then

$$\left| \arg \left(-\frac{z(I_{n+p,\eta} f(z))'}{I_{n+p,\eta} g(z)} - \ell \right) \right| < \frac{\pi}{2} \tau,$$

where $\tau (0 < \tau \leq 1)$ is the solution of equation (4.1).

Taking $p = 1$ in Corollary 7, we have the following corollary:

Corollary 8. Let $f(z) \in \Sigma$, $0 < \delta \leq 1, 0 \leq \ell < 1$ and

$$n \geq \frac{(A-B)}{1+B} - 1 \quad (-1 < B < A \leq 1).$$

If

$$\left| \arg \left(-\frac{z(I_{n,\eta} f(z))'}{I_{n,\eta} g(z)} - \ell \right) \right| < \frac{\pi}{2} \delta,$$

for some $g(z) \in \Sigma$ satisfying (4.7), then

$$\left| \arg \left(-\frac{z(I_{n+1,\eta} f(z))'}{I_{n+1,\eta} g(z)} - \ell \right) \right| < \frac{\pi}{2} \tau,$$

where $\tau (0 < \tau \leq 1)$ is the solution of equation (4.1).

The proof of the next theorem is akin to that of Theorem 5 and so, we omit it.

Theorem 6. Let $f(z) \in \Sigma_{p,m}$, $0 < \delta \leq 1, \ell > p$ and



$$\alpha_1 \geq \frac{p(A-B)}{1+B} \quad (-1 < B < A \leq 1; p \in \mathbf{N}).$$

If

$$\left| \arg \left(\frac{z \left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) f(z) \right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) g(z)} + \ell \right) \right| < \frac{\pi}{2} \delta,$$

for some $g(z) \in \Sigma_{p,q,s}^{\lambda,m}(\alpha_1; A, B)$, then

$$\left| \arg \left(\frac{z \left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1 + 1) f(z) \right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1 + 1) g(z)} + \ell \right) \right| < \frac{\pi}{2} \tau,$$

where $\tau (0 < \tau \leq 1)$ is the solution of equation (4.1).

Theorem 7. Let $f(z) \in \Sigma_{p,m}$, $0 < \delta \leq 1$, $0 \leq \ell < p$ and

$$\mu \geq \frac{p(A-B)}{1+B} \quad (-1 < B < A \leq 1; p \in \mathbf{N}).$$

If

$$\left| \arg \left(- \frac{z \left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) f(z) \right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) g(z)} - \ell \right) \right| < \frac{\pi}{2} \delta,$$

for some $g(z) \in \Sigma_{p,q,s}^{\lambda,m}(\alpha_1; A, B)$, then

$$\left| \arg \left(- \frac{z \left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(f)(z) \right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(g)(z)} - \ell \right) \right| < \frac{\pi}{2} \tau,$$

where $\tau (0 < \tau \leq 1)$ is the solution of equation (4.1).

Proof. Let

$$k(z) = \frac{1}{p-\ell} \left(- \frac{z \left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(f)(z) \right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(g)(z)} - \ell \right), \quad (4.11)$$

where $k(z)$ is analytic with $k(0) = 1$. Using (1.9), we have

$$[-(p-\ell)k(z) - \ell] \mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(g)(z) = \mu \mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) f(z) - (\mu+p) \mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(f)(z). \quad (4.12)$$

Differentiating (4.12) with respect to z and multiplying by z , we obtain

$$-(p-\ell)zk'(z) \mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(g)(z) + [-(p-\ell)k(z) - \ell] z \left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(g)(z) \right)'$$



$$= \mu z (\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) f(z))' - (\mu + p) z (\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(f)(z))'. \quad (4.13)$$

Then, by using (4.11), (4.12) and (4.13), we have

$$\frac{1}{p - \ell} \left(- \frac{z (\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) f(z))'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) g(z)} - \ell \right) = k(z) + \frac{z k'(z)}{-\rho(z) + \mu + p},$$

where

$$\rho(z) = - \frac{z (\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(g)(z))'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(g)(z)}.$$

The remaining part of the proof is similar to that of Theorem 3 and so we omit it.

The proof of the next theorem is akin to that of Theorem 7 and so, we omit it.

Theorem 8. Let $f(z) \in \Sigma_{p,m}$, $0 < \delta \leq 1$, $\ell > p$ and

$$\mu \geq \frac{p(A - B)}{1 + B} \quad (-1 < B < A \leq 1; p \in \mathbf{N}).$$

If

$$\left| \arg \left(\frac{z (\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) f(z))'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) g(z)} + \ell \right) \right| < \frac{\pi}{2} \delta$$

for some $g(z) \in \Sigma_{p,q,s}^{\lambda,m}(\alpha_1; A, B)$, then

$$\left| \arg \left(\frac{z (\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(f)(z))'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(g)(z)} + \ell \right) \right| < \frac{\pi}{2} \tau,$$

where $\tau (0 < \tau \leq 1)$ is the solution of equation (4.1).

Remark 1. Specializing the parameters p, q, s, m and λ in the above results, we obtain the results for the corresponding operators defined in the introduction.

Remark 2. (i) Putting $q = 2, s = 1, m = 1 - p$, $\alpha_1 = \beta_1 = a (a > 0, p \in \mathbf{N})$ and $\alpha_2 = 1$ in Theorems 3 and 4, respectively, we obtain the results obtained by Cho and Owa [11, Theorems 2.1 and 2.2, respectively];

(ii) Putting $q = 2, s = 1, m = 0$, $\alpha_1 = a, \alpha_2 = p = 1$ and $\beta_1 = a (a > 0)$ in Theorems 3 and 4, respectively, we obtain the results obtained by Cho [8, Theorems 2.1 and 2.2, respectively];

(iii) Putting $q = 2, s = 1, m = 1 - p$, $\alpha_1 = a, \alpha_2 = 1$ and $\beta_1 = c (a, c \in \mathbf{R} \setminus \mathbf{Z}_0^-, p \in \mathbf{N})$ in the above results, we obtain the results obtained by Aouf et al. [5];



(iv) Putting $q=2, s=1, m=1-p$, $\alpha_1=c, \alpha_2=p+\lambda$ and $\beta_1=a$ ($a \in \mathbb{R}, c \in \mathbb{R} \setminus \mathbb{Z}_0^-, \lambda > -p, p \in \mathbb{N}$) in Theorem 4, we obtain the results obtained by Lashin [14, Theorem 2.2] .

References

- [1] R. Aghalary, Some properties of a certain family of meromorphically univalent functions defined by integral operator, *Kyungpook Math. J.*, 48 (2008), 379-385.
- [2] M. K. Aouf, New criteria for multivalent meromorphic starlike functions of order alpha, *Proc. Japan Acad., Ser. A, Math. Sci.*, 69 (1993), 66-70.
- [3] M. K. Aouf, A new criterion for meromorphically p-valent convex functions of order alpha, *Math. Sci. Research Hot-Line*, 1 (1997), no. 8, 7-12.
- [4] M.K. Aouf, Certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function, *Comput. Math. Appl.*, 55 (2008), 494-509.
- [5] M.K. Aouf, A. Shamandy, A.O. Mostafa and F.Z. El-Emam, Argument estimates of certain meromorphically p -valent functions associated with a family of linear operator, *Math. Slovaca*, 61 (2012), no. 6, 907-920.
- [6] M. K. Aouf and N.- E. Xu, Some inclusion relationships and integral-preserving properties of certain subclasses of p-valent meromorphic functions, *Comput. Math. Appl.*, 61 (2011), 642-650.
- [7] T. Bulboaca, *Differential Subordinations and Superordinations, Recent Results*, House of Scientific Book Publ., Cluj-Napoca, 2005.
- [8] N. E. Cho, Argument estimates of certain meromorphic functions, *Comm. Korean Math. Soc.*, 15 (2000), no. 2, 263-274.
- [9] N. E. Cho and I. H. Kim, Inclusion properties of certain classes of meromorphic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.*, 187 (2007), 115-121.
- [10] N. E. Cho and K. I. Noor, Inclusion properties for certain classes of meromorphic functions associated with the Choi-Saigo-Srivastava operator, *J. Math. Anal. Appl.*, 320, (2006), 779-786.
- [11] N. E. Cho and S. Owa, Argument estimates of meromorphically multivalent functions, *J. Inequal. Appl.*, 5 (2000), 419-432.
- [12] P. Eenigenberg, S. S. Miller, P. T. Mocanu and M. O. Reade, On a Briot--Bouquet differential subordination, *General Inequal.*, 3 (1983), 339-348.
- [13] V. Kumar and S. L. Shukla, Certain integrals for classes of p-valent meromorphic functions, *Bull. Austral. Math. Soc.*, 25 (1982), 85-97.
- [14] A. Y. Lashin, Argument estimates of certain meromorphically p-valent functions, *Soochow J. Math.*, 33 (2007), no. 4, 803-812.
- [15] J.-L. Liu, A linear operator and its applications on meromorphic p-valent functions, *Bull. Inst. Math. Acad. Sinica*, 31 (2002), no. 1, 23-32.



- [16] J.-L. Liu and H. M. Srivastava, Classes of meromorphically multivalent functions associated with generalized hypergeometric function, *Math. Comput. Modelling*, 39 (2004), 21-34.
- [17] A. O. Mostafa, Applications of differential subordination to certain subclasses of p-valent meromorphic functions involving certain operator, *Math. Comput. Modelling*, 54 (2011), 1486-1498.
- [18] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, *Michigan Math. J.*, 28 (1981), 157-171.
- [19] M. Nunokawa, On the order of strongly starlikeness of strongly convex functions, *Proc. Japan Acad., Ser. A, Math. Sci.*, 69 (1993), 234-237.
- [20] J. Patel and A. K. Patil, On certain subclasses of meromorphically multivalent functions associated with the generalized hypergeometric function, *J. Inequal. Pure Appl. Math.*, 10 (2009), no. 1, Art 13, 1-33.
- [21] H. Silverman and E. M. Silvia, Subclasses of starlike functions subordinate to convex functions, *Canad. J. Math.*, 37 (1985), 48-61.
- [22] H. M. Srivastava and P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press, Ellis Horwood Limited, Chichester, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1985.
- [23] D. -G. Yang, Certain convolution operators for meromorphic functions, *South. Asian Bull. Math.*, 25 (2001), 175-186.
- [24] S.-M. Yuan, Z. -M. Liu and H. M. Srivastava, Some inclusion relationships and integral-preserving properties of certain subclasses of meromorphic functions associated with a family of integral operators, *J. Math. Anal. Appl.*, 337 (2008), 505-515.