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HYERS-ULAM STABILITY OF FIRST ORDER LINEAR DIFFERENCE OPERATORS ON BANACH SPACE

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Abstract

In this work, the Hyers-Ulam stability of first order linear difference operator T_p defined by

$$(T_p u)(n) = \triangle u(n) - p(n)u(n),$$

is studied on the Banach space $X = l_\infty$, where $p(n)$ is a sequence of reals.
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1 Introduction

Let $X = l_\infty$ be the Banach space of all real valued functions $u(n)$ defined for $n \geq 0$. Let $D(I, X)$ be the linear space of all $X-$valued functions on an open interval $I = (a, b+1) \subset \mathbb{N}(0) = \{0, 1, 2, ...\}$, $a < b$. We define

$$\|f\|_\infty = \sup\{\|f(n)\|, n \in I\}$$

for every $f \in D(I, X)$. Define the linear difference operator $T_p : D(I, X) \to D(I, X)$ by

$$(T_p u)(n) = \triangle u(n) - p(n)u(n), \ \forall u \in D(I, X), \forall\ n \in I. \tag{1.1}$$

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We notice that T_p is onto. Indeed, for every $v \in D(I, X)$

$$u(n) = \prod_{i=n_0}^{n-1}(1+p(i)) \sum_{s=n_0}^{n-1} v(s) \left(\prod_{i=n_0}^{s}(1+p(i))\right)^{-1}$$

satisfies $T_p = v$. Conversely, the general solution of $T_p = v$ is of the form

$$u(n) = \prod_{i=n_0}^{n-1}(1+p(i)) \left[x_0 + \sum_{s=n_0}^{n-1} v(s) \left(\prod_{i=n_0}^{s}(1+p(i))\right)^{-1}\right]$$

for every $n_0, n \in I$ and $x_0 \in X$ is an arbitrary element.

DEFINITION 1.1 *We say that the difference operator T_p has the Hyers-Ulam stability, if there exists a constant $K \geq 0$ with the property:* For every $\epsilon \geq 0$ and $u, v \in D(I, X)$ satisfying $\|T_p u - v\|_\infty \leq \epsilon$ there exists $u_0 \in D(I, X)$ such that $T_p u_0 = v$ and $\|u - u_o\|_\infty \leq K\epsilon$. *We call such K a HUS constant for T_p. If, in addition, minimum of all such $K's$ exists, then we call it the HUS constant for T_p.*

In 1940 [25], S. M. Ulam posed the problem: *When can we assert that approximate solution of a functional equation can be approximated by a solution of the corresponding equation?* before the audience at the University of Wisconsin which was first answered by D. H. Hyers [5] on Banach space. Thereafter, T. Aoki [2], D. H. Bourgin [3] and Th. M. Rassias [20] improved the result of Hyers. For more details, we refer the readers to the books by Hyers et al. [6] and monograph by S. M. Jung [11]. After that many researchers have extended the Ulam's stability problems to other functional equations and generalized Hyer's result in various directions. Recently, the Ulam's stability problem for functional equations has been replaced by stability of differential and difference equations (see for e.g. [[1], [7], [8]- [10], [12], [13], [14], [15], [18], [21], [23]], [19], [24]).

In [24], Tripathy has studied the Hyers-Ulam stability of the following difference equations:

$$\begin{aligned}
&y(n+1) - p(n)y(n) - r(n) = 0, \\
&y(n+2) + \alpha y(n+1) + \beta y(n) = 0, \\
&y(n+2) + \alpha y(n+1) + \beta y(n) = r(n), \\
&y(n+2) - \alpha(n)y(n+1) + \beta(n)y(n) = r(n),
\end{aligned}$$

where α, β, p and r are sequences of reals. In this work, our objective is to study the Hyers-Ulam stability of the operator T_p followed by (1.1) on Banach space in accordance with the necessary and sufficient condition.

On Hyers-Ulam stability, several works have been done in the direction of differential equations. Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations(see for e.g[[18], [21]]). After that, Alsina and Ger published

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their work [1], where they have proved the Hyers-Ulam stability of the differential equation $y'(t) = y(t)$. We remark here that absolutely there is no such work on (1.1). We use the following lemma in our next discussion:

LEMMA 1.2 *[22] Let C be a symmetric set, that is, $C = -C$ in a Banach space B. For each $y \in B$, we have*

$$\sup_{x\in C} \|y + x\| \geq \sup_{x\in C} \|x\|.$$

2 Hyers-Ulam Stability Results

In this section, we discuss the necessary and sufficient conditions for Hyers-Ulam stability of the operator T_p followed by (1.1) on the Banach space $X = l_\infty$. We use the following notions for our use in the sequel:

$$1 + p(n) \neq 0, \quad P(n) = \left(\prod_{i=0}^{n-1}(1 + p(i))\right)^{-1}, \; n \in \mathbb{N}(0),$$
$$\alpha_p = \sup_{n\geq 0} \frac{1}{|P(n)|} \sum_{m=n}^{\infty} |P(m+1)|, \quad \beta_p = \sup_{n\geq 0} \frac{1}{|P(n)|} \sum_{m=0}^{n-1} |P(m+1)|.$$

We use the sign convention $\left(\prod_{i=0}^{n-1}(1 + p(i))\right) = 1$ for $n - 1 < 0$.

THEOREM 2.1 *Let $T_p : D(\mathbb{N}(0), X) \to D(\mathbb{N}(0), X)$ be the linear operator defined by*

$$(T_p u)(n) = \triangle u(n) - p(n)u(n), \; \forall u \in D(\mathbb{N}(0), X), \forall\, n \in \mathbb{N}(0). \tag{2.1}$$

If $\inf_{n\geq 0} |P(n)| = 0$, then T_p has the Hyers-Ulam stability with HUS constant α_p if and only if $\alpha_p < \infty$.

Proof. Let $\epsilon \geq 0$ and $u, v \in D(\mathbb{N}(0), X)$ satisfy $\|T_p u - v\|_\infty \leq \epsilon$. Set $w = T_p u - v$. Then $\|w\|_\infty \leq \epsilon$ and $T_p u = v + w$ implies that

$$\begin{aligned} u(n) &= \prod_{i=0}^{n-1}(1 + p(i)) \left[u(0) + \sum_{s=0}^{n-1}(v + w) \left(\prod_{i=0}^{s}(1 + p(i)) \right)^{-1} \right] \\ &= \prod_{i=0}^{n-1}(1 + p(i)) \sum_{s=0}^{n-1} v(s) \left(\prod_{i=0}^{s}(1 + p(i)) \right)^{-1} \\ &+ \prod_{i=0}^{n-1}(1 + p(i)) \left[u(0) + \sum_{s=0}^{n-1} w(s) \left(\prod_{i=0}^{s}(1 + p(i)) \right)^{-1} \right] \\ &= \frac{1}{P(n)} \sum_{s=0}^{n-1} v(s)P(s + 1) + \frac{1}{P(n)} \left[u(0) + \sum_{s=0}^{n-1} w(s)P(s + 1) \right], \end{aligned} \tag{2.2}$$

and $\sum_{s=0}^{n-1} w(s)P(s+1) \in X$ exists for every $n \in \mathbb{N}(0)$.

Now, we consider the case when $\alpha_p < \infty$. For each $n \in \mathbb{N}(0)$, it is easy to see that

$$u(n) = \frac{1}{P(n)} \sum_{s=0}^{n-1} v(s)P(s+1) + \frac{1}{P(n)} \left[u(0) + \sum_{s=0}^{\infty} w(s)P(s+1) \right] - \frac{1}{P(n)} \sum_{s=n}^{\infty} w(s)P(s+1).$$

If we put $x_0 = u(0) + \sum_{s=0}^{\infty} w(s)P(s+1)$ and

$$u_0(n) = \frac{1}{P(n)} \left[x_0 + \sum_{s=0}^{n-1} v(s)P(s+1) \right],$$

then $T_p u_0 = v$ and

$$u(n) = u_0(n) - \frac{1}{P(n)} \sum_{s=n}^{\infty} w(s)P(s+1).$$

Therefore,

$$\|u(n) - u_0(n)\| = \frac{1}{|P(n)|} \| \sum_{s=n}^{\infty} w(s)P(s+1)\|$$

implies that

$$\|u - u_0\|_\infty \leq \sup_{n\geq 0} \frac{\epsilon}{|P(n)|} \sum_{s=n}^{\infty} |P(s+1)| \leq \epsilon\alpha_p.$$

Hence, T_p has the Hyers-Ulam stability with HUS constant α_p. We claim that u_0 is determined uniquely. If not, let $u_1, u_2 \in D(\mathbb{N}(0), X)$ be such that

$$T_p u_i = v \ \ \textit{and} \ \ \|u - u_i\|_\infty \leq M_i < \infty, \quad (i = 1, 2).$$

Hence for $T_p u_i = v$, we can find x_i for $i = 1, 2$ such that

$$u_i(n) = \frac{1}{P(n)} \left[x_i + \sum_{s=0}^{n-1} v(s)P(s+1) \right], \ \forall\, n \in \mathbb{N}(0).$$

Therefore, it follows that

$$\begin{aligned} \|x_1 - x_2\| = |P(n)|\|u_1(n) - u_2(n)\| &\leq |P(n)|\|u_1 - u_2\|_\infty \\ &\leq |P(n)|(M_1 + M_2), \ \forall\, n \in \mathbb{N}(0), \end{aligned}$$

that is, $\|x_1 - x_2\| \to 0$ due to $\inf_{n\geq 0} |P(n)| = 0$. Consequently, $u_1 = u_2$.

Conversely, let's fix $x_0 \in X$ such that $\|x_0\| = 1$. Set $v(n) = \frac{|P(n+1)|}{P(n+1)} x_0$ for every $n \in \mathbb{N}(0)$. Then for $v \in D(\mathbb{N}(0), X)$ we can find $u \in D(\mathbb{N}(0), X)$ such that

$$u(n) = \frac{1}{P(n)} \sum_{s=0}^{n-1} v(s)P(s+1) = \frac{1}{P(n)} \sum_{s=0}^{n-1} |P(s+1)| x_0$$

for which $T_p u = v$ and hence $\|T_p u\|_\infty = \|v\|_\infty = 1$. Let K be an arbitrary HUS constant for T_p. So, we can find $u^* \in D(\mathbb{N}(0), X)$ such that

$$T_p u^* = 0 \;\; and \;\; \|u - u^*\|_\infty \leq K.$$

It is easy to verify that $u^* = \frac{x_1}{P(n)}$ for every $n \in \mathbb{N}(0)$, where $x_1 = u^*(0) \in X$. Therefore,

$$\|\sum_{s=0}^{n-1} |P(s+1)|x_0 - x_1\| = |P(n)|\|\frac{1}{P(n)} \sum_{s=0}^{n-1} |P(s+1)|x_0 - u^*\| \leq K|P(n)|, \; n \in \mathbb{N}(0). \quad (2.3)$$

As $\inf_{n\geq 0} |P(n)| = 0$, we can find a strictly monotonic increasing set of values $\{n_j\}_{j\in\mathbb{N}} \subset \mathbb{N}(0)$ such that

$$n_j \to \infty \; as \; j \to \infty \;\; and \;\; |P(n_j)| < \frac{1}{j}, \; j \in \mathbb{N}.$$

From (2.3) it follows that

$$\left| \|\sum_{s=0}^{n_j-1} |P(s+1)|x_0\| - \|x_1\| \right| \leq \frac{K}{j},$$

that is,

$$\sum_{s=0}^{n_j-1} |P(s+1)| < \infty \;\; as \;\; j \to \infty.$$

Also, from (2.2) it is immediate that $\sum_{s=0}^{\infty} |P(s+1)|x_0 = x_1$. Consequently,

$$\sum_{s=n}^{\infty} |P(s+1)| = \|\sum_{s=0}^{n-1} |P(s+1)|x_0 - x_1\| = \|\sum_{s=0}^{n-1} |P(s+1)|x_0 - \sum_{s=0}^{\infty} |P(s+1)|x_0\| \leq K|P(n)|$$

implies that $\alpha_p \leq K < \infty$. Since K is an arbitrary HUS constant, then α_p itself is the HUS constant for T_p. This completes the proof of the theorem.

REMARK 2.2 *We predict that β_p could be infinity when α_p is finite. In this case, $\sum |P(m)| < \infty$. Ultimately, $\inf_{n\geq 0} |P(n)| = 0$. So, we can find a strictly monotonic increasing set of values $\{n_j\}_{j\in\mathbb{N}} \subset \mathbb{N}(0)$ such that*

$$n_j \to \infty \; as \; j \to \infty \;\; and \;\; |P(n_j)| < \frac{1}{j}, \; j \in \mathbb{N}.$$

Consequently,

$$\beta_p \geq \frac{1}{|P(n_j)|} \sum_{m=0}^{n_j-1} |P(m+1)| > j \sum_{m=0}^{n_j-1} |P(m+1)| \to \infty \;\; as \;\; j \to \infty.$$

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THEOREM 2.3 *Let $T_p : D(\mathbb{N}(0), X) \to D(\mathbb{N}(0), X)$ be the linear operator defined by (2.1). If $\inf_{n\geq 0} |P(n)| > 0$, then T_p has the Hyers-Ulam stability with HUS constant β_p if and only if $\beta_p < \infty$.*

Proof. We proceed as in the proof of Theorem 2.1 to obtain (2.2). If we denote

$$u_3(n) = \frac{1}{P(n)}\left[u(0) + \sum_{s=0}^{n-1} v(s)P(s+1)\right],$$

then $T_p u_3 = v$ and

$$u(n) = u_3(n) + \frac{1}{P(n)}\sum_{s=0}^{n-1} w(s)P(s+1)$$

implies that

$$\|u - u_3\|_\infty \leq \sup_{n\geq 0} \frac{\epsilon}{|P(n)|}\sum_{s=0}^{n-1} |P(s+1)| \leq \epsilon\beta_p. \tag{2.4}$$

Hence, T_p has the Hyers-Ulam stability with HUS constant β_p.

Assume that $\inf_{n\geq 0} |P(n)| > 0$. Proceeding as in the converse part of Theorem 2.1, we have $u^*(n) = \frac{x_1}{P(n)}$ for $n \in \mathbb{N}(0)$. Hence,

$$\sup_{n\geq 0} \|u_0(n)\| \leq \frac{\|x_1\|}{\inf_{n\geq 0} |P(n)|} < \infty$$

implies that $\|x_1\| \leq \|u_0\|_\infty |P(n)|$ for every $n \in \mathbb{N}(0)$. Therefore,

$$\sum_{m=0}^{n-1} |P(m+1)| = \left\|\sum_{m=0}^{n-1} |P(m+1)| x_0\right\| \leq (K + \|u_0\|_\infty)|P(n)|,$$

that is, $\beta_p \leq (K + \|u_0\|_\infty) < \infty$. Since K is an arbitrary HUS constant, then β_p is the HUS constant for T_p. Hence, the theorem is proved.

REMARK 2.4 *In Theorem 2.1, we have seen that the uniqueness is true when $\alpha_p < \infty$. However, the same may not be true for the case when $\beta_p < \infty$. In other words, if K is an arbitrary constant with $\beta_p < K$, then for every $\epsilon > 0$ and $u, v \in D(\mathbb{N}(0), X)$ satisfying $\|T_p u - v\|_\infty \leq \epsilon$, we can find infinitely many $w \in D(\mathbb{N}(0), X)$ such that $T_p w = v$ and $\|u - w\|_\infty \leq K\epsilon$.*

Indeed, T_p has the Hyers-Ulam stability due to Theorem 2.2 and (2.4) holds. Due to Theorem 2.4, let's put $\sigma = \inf_{n\geq 0} |P(n)|$. For each $x \in X$ with $\|x - u_3\|_\infty \leq \sigma\epsilon(K - \beta_p)$, we can define $u_x \in D(\mathbb{N}(0), X)$ by

$$u_x(n) = \frac{1}{P(n)}\left[x + \sum_{s=0}^{n-1} v(s)P(s+1)\right]$$

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such that $T_p u_x = v$ *and*

$$\begin{aligned}\|u(n) - u_x(n)\| &\le \|u(n) - u_3(n)\| + \|u_3(n) - u_x(n)\| \\ &\le \epsilon\beta_p + \frac{1}{|P(n)|}\|x - u_3(n)\| \\ &\le \epsilon\beta_p + \frac{\sigma\epsilon}{|P(n)|}(K - \beta_p) \le K\epsilon\end{aligned}$$

for every $n \in \mathbb{N}(0)$. *Hence, continuing in this way we can find many* $w \in D(\mathbb{N}(0), X)$ *such that* $T_p w = v$ *and* $\|u - w\|_\infty \le K\epsilon$.

REMARK 2.5 *Since the uniqueness doesn't hold in case when* $\beta_p < \infty$, *then the simultaneous question is whether the infimum of all HUS constants, that is,*

$$\inf_{x\in X} \sup_{\substack{w\in D(\mathbb{N}(0),X) \\ \|w\|_\infty \le 1}} \sup_{n\ge 0} \left\| \frac{1}{P(n)} \left[x + \sum_{s=0}^{n-1} w(s)P(s+1) \right] \right\|$$

if it exists for T_p *is a HUS constant or not. Indeed, if we denote*

$$L_{T_p} = \inf_{x\in X} \sup_{\substack{w\in D(\mathbb{N}(0),X) \\ \|w\|_\infty \le 1}} \sup_{n\ge 0} \left\| \frac{1}{P(n)} \left[x + \sum_{s=0}^{n-1} w(s)P(s+1) \right] \right\|$$

and

$$L_0(x) = \sup_{\substack{w\in D(\mathbb{N}(0),X) \\ \|w\|_\infty \le 1}} \sup_{n\ge 0} \left\| \frac{1}{P(n)} \left[x + \sum_{s=0}^{n-1} w(s)P(s+1) \right] \right\|,$$

then it is enough to verify that $L_{T_p} = \inf_{x\in X} L_0(x)$.

If $L_{T_p} = \infty$, *then there is nothing to verify. Assume that* $L_{T_p} < \infty$. *Let* K *be an arbitrary HUS constant for* T_p. *Then for any* $w \in D(\mathbb{N}(0), X)$ *with* $\|w\|_\infty \le 1$, *there exists* $u_0 \in D(\mathbb{N}(0), X)$ *such that* $T_p u_0 = w$ *and* $\|u_0\|_\infty \le K$ *with*

$$u_0(n) = \frac{1}{P(n)} \left[x_0 + \sum_{s=0}^{n-1} w(s)P(s+1) \right]$$

for some $x_0 \in X$. *Indeed,* $L_0(x) \le K$. *Since* K *is an arbitrary HUS constant for* T_p, *then it follows that* $L_{T_p} \ge \inf_{x\in X} L_0(x)$. *Conversely, we show that* $L_{T_p} \le \inf_{x\in X} L_0(x)$. *We may assume that* $\inf_{x\in X} L_0(x) < \infty$.

Here, we assert that $L_0(x)$ *is a HUS constant for* T_p, *that is, for any* $\epsilon > 0$ *and* $u, v \in D(\mathbb{N}(0), X)$ *with* $\|T_p u - v\| \le \epsilon$, *there exists* $u_0 \in D(\mathbb{N}(0), X)$ *such that* $T_p u_0 = v$ *and* $\|u - u_0\|_\infty \le \epsilon L_0(x)$.

If we put $\epsilon w = T_p u - v$ for $u, v \in D(\mathbb{N}(0), X)$, then $\|w\|_\infty \leq 1$. Hence for $\epsilon w + v = T_p u$ and for any arbitrary $x_1 \in X$, we have

$$u(n) = \frac{1}{P(n)}\left[x_1 + \epsilon\sum_{s=0}^{n-1} w(s)P(s+1) + \sum_{s=0}^{n-1} v(s)P(s+1)\right]$$

for any $n \in \mathbb{N}(0)$. Let

$$u_0(n) = \frac{1}{P(n)}\left[x_1 - \epsilon x + \sum_{s=0}^{n-1} v(s)P(s+1)\right],\ n \in \mathbb{N}(0).$$

Then $u_0 \in D(\mathbb{N}(0), X)$ and $T_p u_0 = v$. Consequently,

$$\begin{aligned}\|u - u_0\|_\infty &= \sup_{n\geq 0}\|u(n) - u_0(n)\| \\ &= \sup_{n\geq 0}\left\|\frac{\epsilon}{P(n)}\left[x + \sum_{s=0}^{n-1} w(s)P(s+1)\right]\right\| \leq \epsilon L_0(x)\end{aligned}$$

due to the fact that $\|w\|_\infty \leq 1$. Hence, $L_0(x)$ is a HUS constant for T_p. Thus, $L_{T_p} \leq L_0(x)$. Since $x \in X$ is arbitrary, then it follows that $L_{T_p} \leq \inf_{x\in X} L_0(x)$.

THEOREM 2.6 *Let $T_p : D(\mathbb{N}(0), X) \to D(\mathbb{N}(0), X)$ be the linear operator defined by* (2.1). *If $\beta_p < \infty$, then T_p has the Hyers-Ulam stability with HUS constant L_{T_p}.*

Proof. Suppose that $\beta_p < \infty$. Then by Theorem 2.2, T_p has the Hyers-Ulam stability with HUS constant β_p. Because the uniqueness doesn't hold in case when $\beta_p < \infty$ due to Remark 2.4 and $L_{T_p} < \infty$ exists due to Remark 2.5, then it is sufficient to show that $L_{T_p} = \beta_p$. By definition, $L_{T_p} \leq \beta_p$. Hence, we need to show that $L_{T_p} \geq \beta_p$ only. Define a linear operator $S : D(\mathbb{N}(0), X) \to D(\mathbb{N}(0), X)$ by

$$(Sw)(n) = \frac{1}{P(n)}\sum_{s=0}^{n-1} w(s)P(s+1),\ \ \forall\, n \in \mathbb{N}(0),\ \ w \in D(\mathbb{N}(0), X).$$

Then for all $w \in D(\mathbb{N}(0), X)$,

$$\begin{aligned}\|Sw\|_\infty = \sup_{n\geq 0}\|(Sw)(n)\| &= \sup_{n\geq 0}\left\|\frac{1}{P(n)}\sum_{s=0}^{n-1} w(s)P(s+1)\right\| \\ &\leq \sup_{n\geq 0}\frac{\|w\|_\infty}{|P(n)|}\sum_{s=0}^{n-1}|P(s+1)| = \beta_p\|w\|_\infty < \infty.\end{aligned}$$

Hence, S is a bounded linear operator with $\|S\| \leq \beta_p$. Moreover, if x_0 is a unit element of X and $u_0 = \frac{|P(n)|}{P(n)}x_0$ for $n \in \mathbb{N}(0)$, then $u_0 \in D(\mathbb{N}(0), X)$ and $\|u_0\|_\infty = 1$. Consequently, $\|Su_0\|_\infty = \beta_p$ and hence $\|S\| \geq \beta_p$. Therefore, $\|S\| = \beta_p$.

Since $|P(0)| = 1$, then we can find $n_1 > 0$ such that $|P(n)| \geq \frac{1}{2}$ for $n \geq n_1 + 1$. Thus,

$$\beta_p \geq \frac{1}{|P(n)|}\sum_{s=0}^{n-1}|P(s+1)| \geq \frac{1}{|P(n)|}\sum_{s=n_1}^{n-1}|P(s+1)| \geq \frac{(n-n_1)}{2|P(n)|},$$

that is, $\frac{1}{|P(n)|} \leq 2\beta_p < \infty$. Therefore, if we choose $x \in X$ arbitrary, then it follows that $\frac{x}{|P(n)|} \in D(\mathbb{N}(0), X)$. We notice that the set $(\{w \in D(\mathbb{N}(0), X) : \|w\|_\infty \leq 1\})$ is a symmetric set of $D(\mathbb{N}(0), X)$. Applying Lemma 1.2, we obtain

$$\sup_{\substack{w\in D(\mathbb{N}(0),X)\\ \|w\|_\infty\leq 1}} \sup_{n\geq 0} \left\| \frac{1}{P(n)}\left[x + \sum_{s=0}^{n-1} w(s)P(s+1)\right]\right\| = \sup_{\substack{w\in D(\mathbb{N}(0),X)\\ \|w\|_\infty\leq 1}} \sup_{n\geq 0} \left\| \frac{x}{P(n)} + (Sw)(n)\right\|$$

$$= \sup_{\substack{w\in D(\mathbb{N}(0),X)\\ \|w\|_\infty\leq 1}} \left\| \frac{x}{P} + (Sw)\right\|_\infty \geq \sup_{\substack{w\in D(\mathbb{N}(0),X)\\ \|w\|_\infty\leq 1}} \|(Sw)\|_\infty = \|S\|.$$

which holds for all $x \in X$. Ultimately,

$$\inf_{x\in X} \sup_{\substack{w\in D(\mathbb{N}(0),X)\\ \|w\|_\infty\leq 1}} \sup_{n\geq 0} \left\| \frac{1}{P(n)}\left[x + \sum_{s=0}^{n-1} w(s)P(s+1)\right]\right\| \geq \|S\| = \beta_p.$$

This completes the proof of the theorem.

EXAMPLE 2.7 *Consider*

$$(T_p u)(n) = \triangle u(n) - (1 + (-1)^n)u(n)$$

such that $1 + p(n) = 2 + (-1)^n$ *and* $P(n) = \left(\prod_{i=0}^{n-1}(2 + (-1)^i)\right)^{-1}$. *Indeed,*

$$\frac{1}{P(n)}\sum_{m=n}^{\infty} P(m+1) = \frac{1}{(2+(-1)^n)}\left[1 + \frac{1}{(2+(-1)^n)} + \frac{1}{(2+(-1)^n)(2-(-1)^n)} + ...\right]$$
$$\leq 2\left[1 + \frac{1}{1.3} + \frac{1}{1^2.3^2} + ...\right] = 3$$

implies that $\alpha_p \leq 3$. *Hence by Theorem 2.1,* T_p *has the Hyers-Ulam stability with HUS constant* $\alpha_p \leq 3$.

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