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The q-Exponential Operator and Generalized Rogers-Szegö Polynomials

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Abstract. This paper is mainly concerned with using q-exponential operator $T(bD_q)$ in proving the identities that involve the generalized Rogers-Szegö polynomials $r_n(x,b)$. We introduce some new roles of the q-exponential operator and prove that the generalized Rogers-Szegö polynomials can be represented by the q-exponential operator, so we use this operator and it's roles in proving the basic identities of $r_n(x,b)$ given in [7, 8] which are: generating function, Mehler's formula and Rogers formula. Then we introduce several extensions of $r_n(x,b)$ identities such that: the extended generating function, extended Mehler's formula, extended Rogers formula and another extended identities. These extended identities of the generalized Rogers-Szegö polynomials can be considered a general form of the corresponding identities for the classical Rogers-Szegö polynomials $h_n(x\,|\,q)$ when $b=1$.

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1 Introduction and Notation

The Rogers-Szegö polynomials play an important role in the theory of the orthogonal polynomials, particularly in the study of the Askey-Wilson polynomials [1, 2, 3, 4, 10, 11, 12, 15]. In 1997, Chen and Liu [5] used a special case of the q-exponential operator $T(bD_q)$ to represent the classical Rogers-Szegö polynomials $h_n(x|q)$ to derive Mehler's formula (1.7) and Rogers formula (1.8) of $h_n(x|q)$. In this paper we represent the generalized Rogers-Szegö polynomials $r_n(x,b)$ by the q-exponential operator to prove the basic and extended identities for $r_n(x,b)$.

Firstly, let us review some common notation and terminology for basic hypergeometric series in [9], where we assume that $|q|<1$, the q-shifted factorial is given by:

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{1.1}$$

Also

$$(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty,$$

$$(a;q)_{n+k} = (a;q)_k(aq^k;q)_n.$$

This notation for multiple q-shifted factorial:

$$(a_1,a_2,\cdots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,$$

$$(a_1,a_2,\cdots,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

The q-binomial coefficient or Gaussian polynomials is given as:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

The basic hypergeometric series ${}_{r+1}\phi_r$ is defined by:

$${}_{r+1}\phi_r\left(\begin{matrix} a_1,\ldots,a_{r+1} \\ b_1,\ldots,b_r \end{matrix} ;q,x\right) = \sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_{r+1};q)_n}{(q,b_1,\ldots,b_r;q)_n}x^n,$$

where a_i, b_j, q and x may be real or complex for all i and j.

The Cauchy identity is defined as:

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x|<1. \tag{1.2}$$

Putting $a=0$, (2) becomes Euler's identity:

$$\sum_{n=0}^{\infty}\frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty}, \quad |x|<1, \tag{1.3}$$

and its inverse relation:

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}} x^k}{(q;q)_k} = (x;q)_\infty. \tag{1.4}$$

The (classical) Rogers-Szegö polynomial is defined by:

$$h_n(x/q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} x^k. \tag{1.5}$$

It has the following generating function:

$$\sum_{n=0}^{\infty} h_n(x|q) \frac{t^n}{(q;q)_n} = \frac{1}{(t, xt; q)_\infty}, \tag{1.6}$$

where $\max\{|t|, |xt|\} < 1$.

Also Mehler's formula of $h_n(x|q)$ is:

$$\sum_{n=0}^{\infty} h_n(x|q) h_n(y|q) \frac{t^n}{(q;q)_n} = \frac{(xyt^2;q)_\infty}{(t, xt, yt, xyt; q)_\infty}, \tag{1.7}$$

where $\max\{|t|, |xt|, |yt|, |xyt|\} < 1$.

and the Rogers formula of $h_n(x|q)$ is:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x|q) \frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$= (xst;q)_\infty \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_n(x|q) h_m(x|q) \frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$= \frac{(xst;q)_\infty}{(t, s, xt, xs; q)_\infty}, \tag{1.8}$$

where $\max\{|s|, |t|, |xs|, |xt|\} < 1$,

where (1.6), (1.7), and (1.8) are the basic identities of $h_n(x|q)$.

The generalized Rogers-Szegö polynomials [7, 8, 13] is defined as:

$$r_n(x,b) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k b^{n-k}.$$

Compared with classical Rogers-Szegö polynomials $h_n(x|q)$, the polynomials $r_n(x,b)$ involve two parameters. Clearly, the polynomials $h_n(x|q)$ can be considered as a special case of the polynomials $r_n(x,b)$ for $b = 1$. Note that

$$r_n(x,b) = r_n(b,x).$$

In 1982, Cigler [7] stated the generating function and Mehler's formula for the generalized Rogers-Szegö polynomials. Also Désarménien [8] states the basic identities for the polynomials $r_n(x,b)$.

Theorem 1.1 *(Cigler [7] and Désarménien [8]).*

1. The generating function for $r_n(x,b)$ is:

$$\sum_{n=0}^{\infty} r_n(x,b) \frac{t^n}{(q;q)_n} = \frac{1}{(xt, bt; q)_\infty}, \tag{1.9}$$

where $\max\{|xt|, |bt|\} < 1$.

2. Mehler's formula for $r_n(x,b)$ is:

$$\sum_{n=0}^{\infty} r_n(x,b) r_n(y,b) \frac{t^n}{(q;q)_n} = \frac{(xyb^2t^2;q)_\infty}{(xyt, xbt, byt, b^2t; q)_\infty}, \tag{1.10}$$

where $\max\{|xyt|, |xbt|, |byt|, |b^2t|\} < 1$.

3. The Rogers formula for $r_n(x,b)$ is:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r_{n+m}(x,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = (xbst;q)_\infty \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r_n(x,b) r_m(x,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}, \tag{1.11}$$

where $\max\{|xs|,|xt|,|bs|,|bt|\} < 1$.

Where (1.9), (1.10), and (1.11) are the basic identities of $r_n(x,b)$.

The q-differential operator D_q is defined by:

$$D_q f(a) = \frac{f(a) - f(aq)}{a}.$$

The q-exponential operator $T(bD_q)$ is defined in [5] as:

$$T(bD_q) = \sum_{n=0}^{\infty}\frac{(bD_q)^n}{(q;q)_n}.$$

The q-exponential operator is the reminiscent of the Euler's identity (1.3), then Chen and Liu [5] obtained the following operator identities:

Proposition 1.2 *Let* D_q *and* $T(bD_q)$ *be defined as above, then*

$$D_q^k\left\{\frac{1}{(at;q)_\infty}\right\} = \frac{t^k}{(at;q)_\infty}, \tag{1.12}$$

$$T(bD_q)\left\{\frac{1}{(at;q)_\infty}\right\} = \frac{1}{(at,bt;q)_\infty}, \qquad |bt| < 1, \tag{1.13}$$

$$T(bD_q)\left\{\frac{1}{(as,at;q)_\infty}\right\} = \frac{(abst;q)_\infty}{(as,at,bs,bt;q)_\infty}, \tag{1.14}$$

where $\max\{|bt|,|bs|\} < 1$.

In 2005, Zhang and Wang [16] used the q-exponential operator $T(bD_q)$ to some terminating summation formulas of basic hypergeometric series and q-integrals to obtain some q-series identities and q-integrals involving ${}_3\phi_2$, they gave the following operator identities:

Proposition 1.3

$$D_q^n\left\{\frac{(av;q)_\infty}{(at;q)_\infty}\right\} = t^n (v/t;q)_n \frac{(avq^n;q)_\infty}{(at;q)_\infty}, \tag{1.15}$$

$$T(dD_q)\left\{\frac{(av;q)_\infty}{(as,at,aw;q)_\infty}\right\} = \frac{(av,dv,adstw/v;q)_\infty}{(as,at,aw,ds,dt,dw;q)_\infty}\, {}_3\phi_2\left(\begin{matrix} v/s, v/t, v/w \\ av, dv \end{matrix}; q, adstw/v\right), \tag{1.16}$$

where $\max\{|ds|,|dt|,|dw|,|adstw/v|\} < 1$.

2 New Roles of the q-Exponential Operator

In this section, we derive some roles of the q-exponential operator $T(bD_q)$, these roles are very important for proving the identities of the generalized Rogers-Szegö polynomials $r_n(x,b)$.

Theorem 2.1 *For* $n \in N$, *we have*

$$T(bD_q)\left\{\frac{a^n}{(as,at;q)_\infty}\right\}=\frac{(abst;q)_\infty}{(as,at,bs,bt;q)_\infty}\sum_{j=0}^{n}nj\frac{(as,at;q)_j}{(abst;q)_j}b^j a^{n-j}, \tag{2.1}$$

where $\max\{|bs|,|bt|\}<1$.

Proof.

$$T(bD_q)\left\{\frac{a^n}{(as,at;q)_\infty}\right\}$$

$$=\sum_{k=0}^{\infty}\frac{b^k}{(q;q)_k}D_q^k\left\{\frac{a^n}{(as,at;q)_\infty}\right\}$$

$$=\sum_{k=0}^{\infty}\frac{b^k}{(q;q)_k}\sum_{j=0}^{k}q^{j(j-k)}kjD_q^j\{a^n\}D_q^{k-j}\left\{\frac{1}{(asq^j,atq^j;q)_\infty}\right\}$$

$$=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{b^{k+j}}{(q;q)_j(q;q)_k}q^{-jk}D_q^j\{a^n\}D_q^k\left\{\frac{1}{(asq^j,atq^j;q)_\infty}\right\}$$

$$=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{b^j}{(q;q)_j}\frac{(q;q)_n}{(q;q)_{n-j}}a^{n-j}\frac{(bq^{-j}D_q)^k}{(q;q)_k}\left\{\frac{1}{(asq^j,atq^j;q)_\infty}\right\}$$

$$=\sum_{j=0}^{\infty}njb^j a^{n-j}T(bq^{-j}D_q)\left\{\frac{1}{(asq^j,atq^j;q)_\infty}\right\}$$

$$=\frac{(abst;q)_\infty}{(as,at,bs,bt;q)_\infty}\sum_{j=0}^{n}nj\frac{(as,at;q)_j}{(abst;q)_j}b^j a^{n-j}.$$

- Setting $n=0$ in (2.1), we get the identity (1.14) of the q-exponential operator.

As a special case of (2.1), we give the following corollary:

Corollary 1 *For* $n\in N$ *, we have*

$$T(bD_q)\left\{\frac{a^n}{(as;q)_\infty}\right\}=\frac{a^n}{(as,bs;q)_\infty}\sum_{j=0}^{n}nj(as;q)_j(b/a)^j, \tag{2.2}$$

where $|bs|<1$.

Proof. Put $t=0$ in (2.1).

- Setting $n=0$ in (2.2), we get the identity (1.13) of the q-exponential operator.

Theorem 2.2 *We have*

$$T(bD_q)\left\{\frac{1}{(as,at,av;q)_\infty}\right\}=\frac{(abtv;q)_\infty}{(as,at,av,bt,bv;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}at,av\\abtv\end{matrix};q,bs\right), \tag{2.3}$$

where $\max\{|bt|,|bv|,|bs|\}<1$.

Proof.

$$T(bD_q)\left\{\frac{1}{(as,at,av;q)_\infty}\right\}$$

$$= \sum_{k=0}^{\infty} \frac{b^k}{(q;q)_k} D_q^k \left\{ \frac{1}{(as;q)_\infty} \frac{1}{(at,av;q)_\infty} \right\}$$

$$= \sum_{k=0}^{\infty} \frac{b^k}{(q;q)_k} \sum_{j=0}^{k} q^{j(j-k)} kj D_q^j \left\{ \frac{1}{(as;q)_\infty} \right\} D_q^{k-j} \left\{ \frac{1}{(atq^j, avq^j;q)_\infty} \right\}$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{b^{k+j}}{(q;q)_k (q;q)_j} q^{-jk} \frac{s^j}{(as;q)_\infty} D_q^k \left\{ \frac{1}{(atq^j, avq^j;q)_\infty} \right\}$$

$$= \frac{1}{(as;q)_\infty} \sum_{j=0}^{\infty} \frac{(bs)^j}{(q;q)_j} T(bq^{-j} D_q)^k \left\{ \frac{1}{(atq^j, avq^j;q)_\infty} \right\}$$

$$= \frac{1}{(as;q)_\infty} \sum_{j=0}^{\infty} \frac{(bs)^j}{(q;q)_j} \frac{(abtvq^j;q)_\infty}{(atq^j, avq^j, bt, bv;q)_\infty}, \quad |bt|,|bv|<1$$

$$= \frac{(abtv;q)_\infty}{(as,at,av,bt,bv;q)_\infty} \; {}_2\phi_1 \left(\begin{matrix} at, av \\ abtv \end{matrix} ; q, bs \right), \quad |bs| < 1.$$

- Setting $s = 0$ in (2.3), we get the identity (1.14) of the q -exponential operator.

Theorem 2.3 *We have*

$$T(bD_q)\left\{ \frac{1}{(as,at,av,aw;q)_\infty} \right\}$$

$$= \frac{(abvw;q)_\infty}{(as,at,av,aw,bv,bw;q)_\infty} \sum_{j,n=0}^{\infty} \frac{(av,aw;q)_{j+n}(at;q)_j}{(abvw;q)_{j+n}} \frac{(bs)^j}{(q;q)_j} \frac{(bt)^n}{(q;q)_n}, \tag{2.4}$$

where $\max\{|bt|,|bv|,|bw|\} < 1$.

Proof.

$$T(bD_q)\left\{ \frac{1}{(as,at,av,aw;q)_\infty} \right\}$$

$$= \sum_{k=0}^{\infty} \frac{b^k}{(q;q)_k} D_q^k \left\{ \frac{1}{(as;q)_\infty} \frac{1}{(at,av,aw;q)_\infty} \right\}$$

$$= \sum_{k=0}^{\infty} \frac{b^k}{(q;q)_k} \sum_{j=0}^{k} q^{j(j-k)} kj D_q^j \left\{ \frac{1}{(as;q)_\infty} \right\} D_q^{k-j} \left\{ \frac{1}{(atq^j, avq^j, awq^j;q)_\infty} \right\}$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{b^{k+j}}{(q;q)_k (q;q)_j} q^{-jk} \frac{s^j}{(as;q)_\infty} D_q^k \left\{ \frac{1}{(atq^j, avq^j, awq^j;q)_\infty} \right\}$$

$$= \frac{1}{(as;q)_\infty} \sum_{j=0}^{\infty} \frac{(bs)^j}{(q;q)_j} T(bq^{-j} D_q) \left\{ \frac{1}{(atq^j, avq^j, awq^j;q)_\infty} \right\}$$

$$= \frac{1}{(as;q)_\infty} \sum_{j=0}^{\infty} \frac{(bs)^j}{(q;q)_j} \frac{(abvwq^j;q)_\infty}{(atq^j,avq^j,awq^j,bv,bw;q)_\infty} \, {}_2\phi_1\left(\begin{matrix} avq^j, awq^j \\ abvwq^j \end{matrix} ;q,bt\right)$$

$$= \frac{(abvw,q)_\infty}{(as,at,av,aw,bv,bw;q)_\infty} \sum_{j,n=0}^{\infty} \frac{(av,aw;q)_{j+n}(at;q)_j}{(abvw,q)_{j+n}} \frac{(bs)^j}{(q;q)_j} \frac{(bt)^n}{(q;q)_n}.$$

- Setting $s=0$ in the above theorem, we get the identity (2.3), also setting $s=t=0$, we get the identity (1.14) of the q-exponential operator.

Theorem 2.4 *We have*

$$T(bD_q)\left\{\frac{1}{(as,at,au,av,aw;q)_\infty}\right\} = \frac{(abvw,q)_\infty}{(as,at,au,av,aw,bv,bw;q)_\infty}$$

$$\times \sum_{i,j,n=0}^{\infty} \frac{(at;q)_j(au;q)_{i+j}(av,aw;q)_{i+j+n}}{(abvw,q)_{i+j+n}} \frac{(bt)^i}{(q;q)_i} \frac{(bs)^j}{(q;q)_j} \frac{(bu)^n}{(q;q)_n}, \tag{2.5}$$

where $\max\{|bu|,|bv|,|bw|\}<1.$

Proof.

$$T(bD_q)\left\{\frac{1}{(as,at,au,av,aw;q)_\infty}\right\}$$

$$= \sum_{k=0}^{\infty} \frac{b^k}{(q;q)_k} D_q^k \left\{\frac{1}{(as;q)_\infty} \frac{1}{(at,au,av,aw;q)_\infty}\right\}$$

$$= \sum_{k=0}^{\infty} \frac{b^k}{(q;q)_k} \sum_{j=0}^{k} q^{j(j-k)} kj D_q^j \left\{\frac{1}{(as;q)_\infty}\right\} D_q^{k-j} \left\{\frac{1}{(atq^j,auq^j,avq^j,awq^j;q)_\infty}\right\}$$

$$= \frac{1}{(as;q)_\infty} \sum_{j=0}^{\infty} \frac{(bs)^j}{(q;q)_j} T(bq^{-j}D_q) \left\{\frac{1}{(atq^j,auq^j,avq^j,awq^j;q)_\infty}\right\}$$

$$= \frac{(abvw,q)_\infty}{(as,at,au,av,aw,bv,bw;q)_\infty} \sum_{i,j,n=0}^{\infty} \frac{(at;q)_j(au;q)_{i+j}(av,aw;q)_{i+j+n}}{(abvw,q)_{i+j+n}} \frac{(bt)^i}{(q;q)_i} \frac{(bs)^j}{(q;q)_j} \frac{(bu)^n}{(q;q)_n}.$$

- Setting $u=0$ in the above theorem, we get the identity (2.4), setting $t=u=0$, we get the identity (2.3), and setting $s=t=u=0$, we get the identity (1.14) of the q-exponential operator.

Theorem 2.5 *We have*

$$T(bD_q)\left\{\frac{1}{(ac,as,at,au,av,aw;q)_\infty}\right\} = \frac{(abvw,q)_\infty}{(ac,as,at,au,av,aw,bv,bw;q)_\infty}$$

$$\times \sum_{i,j,m,n=0}^{\infty} \frac{(as;q)_j(at;q)_{j+m}(au;q)_{i+j+m}(av,aw;q)_{i+j+m+n}}{(abvw,q)_{i+j+m+n}} \frac{(bt)^i}{(q;q)_i} \frac{(bc)^j}{(q;q)_j} \frac{(bs)^m}{(q;q)_m} \frac{(bu)^n}{(q;q)_n}, \tag{2.6}$$

where $\max\{|bu|,|bv|,|bw|\}<1.$

Proof.

$$T(bD_q)\left\{\frac{1}{(ac,as,at,au,av,aw;q)_\infty}\right\}$$

$$= \sum_{k=0}^{\infty} \frac{b^k}{(q;q)_k} \sum_{j=0}^{k} q^{j(j-k)} kj D_q^j \left\{ \frac{1}{(ac;q)_\infty} \right\} D_q^{k-j} \left\{ \frac{1}{(asq^j, atq^j, auq^j, avq^j, awq^j; q)_\infty} \right\}$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{b^{k+j} q^{-jk}}{(q;q)_k (q;q)_j} \frac{c^j}{(ac;q)_\infty} D_q^k \left\{ \frac{1}{(asq^j, atq^j, auq^j, avq^j, awq^j; q)_\infty} \right\}$$

$$= \frac{1}{(ac;q)_\infty} \sum_{j=0}^{\infty} \frac{(bc)^j}{(q;q)_j} T(bq^{-j} D_q) \left\{ \frac{1}{(asq^j, atq^j, auq^j, avq^j, awq^j; q)_\infty} \right\}$$

$$= \frac{1}{(ac;q)_\infty} \sum_{j=0}^{\infty} \frac{(bc)^j}{(q;q)_j} \frac{(abvwq^j;q)_\infty}{(asq^j, atq^j, auq^j, avq^j, awq^j, bv, bw; q)_\infty}$$

$$\times \sum_{i,m,n=0}^{\infty} \frac{(atq^j;q)_m (auq^j;q)_{i+m} (avq^j, awq^j;q)_{i+m+n}}{(abvwq^j;q)_{i+m+n}} \frac{(bt)^i}{(q;q)_i} \frac{(bs)^m}{(q;q)_m} \frac{(bu)^n}{(q;q)_n}$$

$$= \frac{(abvw,q)_\infty}{(ac, as, at, au, av, aw, bv, bw; q)_\infty}$$

$$\times \sum_{i,j,m,n=0}^{\infty} \frac{(as;q)_j (at;q)_{j+m} (au;q)_{i+j+m} (av,aw;q)_{i+j+m+n}}{(abvw,q)_{i+j+m+n}} \frac{(bt)^i (bc)^j (bs)^m (bu)^n}{(q;q)_i (q;q)_j (q;q)_m (q;q)_n}.$$

- Setting $c = 0$ in the above theorem, we get the identity (2.5), setting $c = u = 0$, we get the identity (2.4), setting $c = s = u = 0$, we get the identity (2.4), and setting $c = s = t = u = 0$, we get the identity (1.14) of the q-exponential operator.

3 The Basic Identities of $r_n(x,b)$

In this section, we represent the generalized Rogers-Szegö polynomials $r_n(x,b)$ by the q-exponential operator, then by using this operator representation we give a simple proof for the basic identities of the polynomials $r_n(x,b)$.

The generalized Rogers-Szegö polynomials $r_n(x,b)$ can be represented by the q-exponential operator $T(bD_q)$ as follows:

Theorem 3.1 *We have*

$$T(bD_q)\{x^n\} = r_n(x,b). \qquad (3.1)$$

Proof.

$$T(bD_q)\{x^n\} = \sum_{k=0}^{\infty} \frac{b^k}{(q;q)_k} D_q^k \{x^n\}$$

$$= \sum_{k=0}^{\infty} \frac{b^k}{(q;q)_k} \frac{(q;q)_n}{(q;q)_{n-k}} x^{n-k}$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k b^{n-k}$$

$$= r_n(x,b).$$

By using the operator representation (3.1), we derive the basic identities of generalized Rogers-Szegö polynomials.

Theorem 3.2 *(The generating function of* $r_n(x,b)$ *), we have*

$$\sum_{n=0}^{\infty} r_n(x,b)\frac{t^n}{(q;q)_n} = \frac{1}{(xt,bt;q)_\infty}, \tag{3.2}$$

where $\max\{|xt|,|bt|\}<1$.

Proof.

$$\sum_{n=0}^{\infty} r_n(x,b)\frac{t^n}{(q;q)_n} = \sum_{n=0}^{\infty} T(bD_q)\{x^n\}\frac{t^n}{(q;q)_n}$$

$$= T(bD_q)\left\{\sum_{n=0}^{\infty}\frac{(xt)^n}{(q;q)_n}\right\}, \qquad |xt|<1$$

$$= T(bD_q)\left\{\frac{1}{(xt;q)_\infty}\right\}$$

$$= \frac{1}{(xt,bt;q)_\infty}.$$

- Setting $b=1$ in (3.2), we get the generating function of the classical Rogers-Szegö polynomials (1.6).

Theorem 3.3 *(Mehler's formula of $r_n(x,b)$), we have*

$$\sum_{n=0}^{\infty} r_n(x,b)r_n(y,b)\frac{t^n}{(q;q)_n} = \frac{(xyb^2t^2;q)_\infty}{(xyt,xbt,byt,b^2t;q)_\infty}, \tag{3.3}$$

where $\max\{|xyt|,|xbt|,|byt|,|b^2t|\}<1$.

Proof.

$$\sum_{n=0}^{\infty} r_n(x,b)r_n(y,b)\frac{t^n}{(q;q)_n} = \sum_{n=0}^{\infty} r_n(y,b)T(bD_q)\{x^n\}\frac{t^n}{(q;q)_n}$$

$$= T(bD_q)\left\{\sum_{n=0}^{\infty} r_n(y,b)\frac{(xt)^n}{(q;q)_n}\right\}, |xyt|,|xbt|<1$$

$$= T(bD_q)\left\{\frac{1}{(xyt,xbt;q)_\infty}\right\}$$

$$= \frac{(xyb^2t^2;q)_\infty}{(xyt,xbt,byt,b^2t;q)_\infty}, |byt|,|b^2t|<1$$

- Setting $b=1$ in (3.3), we get Mehler's formula of the classical Rogers-Szegö polynomials (1.7).

Theorem 3.4 *(The Rogers formula of $r_n(x,b)$), we have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r_{n+m}(x,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = (xbst;q)_\infty \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r_n(x,b)r_m(x,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}, \tag{3.4}$$

where $\max\{|xs|,|xt|,|bs|,|bt|\}<1$.

Proof.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r_{n+m}(x,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$

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$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} T(bD_q)\{x^{n+m}\}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$

$$= T(bD_q)\left\{\sum_{n=0}^{\infty}\frac{(xt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(xs)^m}{(q;q)_m}\right\}, \quad |xs|,|xt|<1$$

$$= T(bD_q)\left\{\frac{1}{(xt,xs;q)_\infty}\right\}$$

$$= \frac{(xbst;q)_\infty}{(xs,xt,bs,bt;q)_\infty}$$

$$= (xbst;q)_\infty \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r_n(x,b) r_m(x,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}.$$

- Setting $b=1$ in (3.4), we get Rogers formula (1.8) of the classical Rogers-Szegö polynomials.

4 Some Extensions for the Basic Identities of $r_n(x,b)$

In this section, we extend the basic identities of generalized Rogers-Szegö polynomials by using q-exponential operator $T(bD_q)$, where we introduce the extended generating function, extended Mehler's formula and extended Rogers formula of $r_n(x,b)$.

Theorem 4.1 *(Extended generating function of* $r_n(x,b)$ *),we have*

$$\sum_{n=0}^{\infty} r_{n+k}(x,b)\frac{t^n}{(q;q)_n} = \frac{x^k}{(xt,bt;q)_\infty}\sum_{m=0}^{k}\begin{bmatrix}k\\m\end{bmatrix}(xt;q)_m\,(b/x)^m, \tag{4.1}$$

where $\max\{|xt|,|bt|\}<1$.

Proof.

$$\sum_{n=0}^{\infty} r_{n+k}(x,b)\frac{t^n}{(q;q)_n} = \sum_{n=0}^{\infty} T(bD_q)\{x^{n+k}\}\frac{t^n}{(q;q)_n}$$

$$= T(bD_q)\left\{x^k\sum_{n=0}^{\infty}\frac{(xt)^n}{(q;q)_n}\right\}, |xt|<1$$

$$= T(bD_q)\left\{\frac{x^k}{(xt;q)_\infty}\right\}$$

$$= \frac{x^k}{(xt,bt;q)_\infty}\sum_{m=0}^{k}\begin{bmatrix}k\\m\end{bmatrix}(xt;q)_m\,(b/x)^m, \ |bt|<1.$$

- Setting $k=0$ in (4.1), we get the generating function (1.9) for the generalized Rogers-Szegö polynomials.

Theorem 4.2 *(Extended Mehler's formula of* $r_n(x,b)$ *), we have*

$$\sum_{n=0}^{\infty} r_n(x,b)\,r_{n+k}(y,b)\frac{t^n}{(q;q)_n} = \frac{(xyb^2t^2;q)_\infty}{(xyt,byt,bxt,b^2t;q)_\infty}\sum_{m=0}^{k}\begin{bmatrix}k\\m\end{bmatrix}\frac{(xyt,byt;q)_m}{(xyb^2t^2;q)_m}b^m\,y^{k-m}, \tag{4.2}$$

where $\max\{|xyt|,|byt|,|bxt|,|b^2t|\}<1$.

Proof.

$$\sum_{n=0}^{\infty} r_n(x,b) r_{n+k}(y,b) \frac{t^n}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} r_n(x,b) T_y(bD_q)\{y^{n+k}\} \frac{t^n}{(q;q)_n}$$

$$= T_y(bD_q)\left\{ y^k \sum_{n=0}^{\infty} r_n(x,b) \frac{(yt)^n}{(q;q)_n} \right\}, \qquad |xyt|,|byt|<1$$

$$= T_y(bD_q)\left\{ \frac{y^k}{(xyt,byt;q)_\infty} \right\}$$

$$= \frac{(xyb^2t^2;q)_\infty}{(xyt,byt,bxt,b^2t;q)_\infty} \sum_{m=0}^{k} km \frac{(xyt,byt;q)_m}{(xyb^2t^2;q)_m} b^m y^{k-m}, |bxt|,|b^2t|<1.$$

- Setting $k=0$ in (4.2), we get Mehler's formula (1.10) of the generalized Rogers-Szegö polynomials.

Theorem 4.3 *(Extended Rogers formula of* $r_n(x,b)$*), we have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r_{n+m+k}(x,b) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} = \frac{(xbst;q)_\infty}{(xt,xs,bt,bs;q)_\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(xt,xs;q)_j}{(xbst;q)_j} b^j x^{k-j}, \tag{4.3}$$

where $\max\{|xs|,|xt|,|bs|,|bt|\}<1$.

Proof.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r_{n+m+k}(x,b) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} T(bD_q)\{x^{n+m+k}\} \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m}$$

$$= T(bD_q)\left\{ x^k \sum_{n=0}^{\infty} \frac{(xt)^n}{(q;q)_n} \sum_{m=0}^{\infty} \frac{(xs)^m}{(q;q)_m} \right\}, \qquad |xs|,|xt|<1$$

$$= T(bD_q)\left\{ \frac{x^k}{(xt,xs;q)_\infty} \right\}$$

$$= \frac{(xbst;q)_\infty}{(xt,xs,bt,bs;q)_\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(xt,xs;q)_j}{(xbst;q)_j} b^j x^{k-j}, \qquad |bs|,|bt|<1.$$

- Setting $k=0$ in (4.3), we get the Rogers formula (1.11) for the generalized Rogers-Szegö polynomials.

Now we extend some other identities for $r_n(x,b)$ also by using the q-exponential operator.

Theorem 4.4 *We have*

$$\sum_{k=0}^{\infty} r_{m+k}(x,b) r_{n+k}(y,b) \frac{t^k}{(q;q)_k}$$

$$= \frac{(xyb^2t^2;q)_\infty}{(b^2t,bxt,byt,xyt;q)_\infty} \sum_{s=0}^{m}\sum_{j=0}^{n} \begin{bmatrix} m \\ s \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} \frac{(bxt;q)_s (byt;q)_j (xyt;q)_{s+j}}{(xyb^2t^2;q)_{s+j}} b^{s+j} x^{m-s} y^{n-j}, \tag{4.4}$$

where $\max\{|xyt|,|byt|,|bxt|,|b^2t|\}<1$.

Proof.

$$\sum_{k=0}^{\infty} r_{m+k}(x,b) r_{n+k}(y,b) \frac{t^k}{(q;q)_k}$$

$$= \sum_{k=0}^{\infty} r_{m+k}(x,b) T_y(bD_q)\{y^{n+k}\} \frac{t^k}{(q;q)_k}$$

$$= T_y(bD_q)\left\{ y^n \sum_{k=0}^{\infty} r_{m+k}(x,b) \frac{(yt)^k}{(q;q)_k} \right\}$$

$$= T_y(bD_q)\left\{ y^n \frac{x^m}{(xyt,byt;q)_\infty} \sum_{s=0}^{m} ms(xyt;q)_s (b/x)^s \right\}$$

$$= \sum_{s=0}^{m} msb^s x^{m-s} T_y(bD_q)\left\{ \frac{y^n}{(xytq^s,byt;q)_\infty} \right\}$$

$$= \sum_{s=0}^{m} ms \frac{(xyb^2t^2q^s;q)_\infty}{(xytq^s,byt,bxtq^s,b^2t;q)_\infty} \sum_{j=0}^{n} nj \frac{(xytq^s,byt;q)_j}{(xyb^2t^2q^s;q)_j} b^{s+j} x^{m-s} y^{n-j}$$

$$= \frac{(xyb^2t^2;q)_\infty}{(b^2t,bxt,byt,xyt;q)_\infty} \sum_{s=0}^{m}\sum_{j=0}^{n} msnj \frac{(bxt;q)_s (byt;q)_j (xyt;q)_{s+j}}{(xyb^2t^2;q)_{s+j}} b^{s+j} x^{m-s} y^{n-j}.$$

- Setting $n=m=0$ in (4.4), we get Mehler's formula (1.10), and setting $m=0$, we get the extension of Mehler's formula (4.2) for $r_n(x,b)$.

By using (2.3), we give the following identity of the generalized Rogers-Szegö polynomials:

Theorem 4.5 *We have*

$$\sum_{n,m,k=0}^{\infty} r_{n+m+k}(x,b) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} \frac{v^k}{(q;q)_k} = \frac{(xbsv;q)_\infty}{(xt,xs,xv,bs,bv;q)_\infty} {}_2\phi_1\left(\begin{matrix} xs, xv \\ xbsv \end{matrix} ;q,bt \right), \quad (4.5)$$

where $\max\{|xt|,|xs|,|xv|,|bt|,|bs|,|bv|\}<1$.

Proof.

$$\sum_{n,m,k=0}^{\infty} r_{n+m+k}(x,b) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} \frac{v^k}{(q;q)_k}$$

$$= T(bD_q)\left\{ \sum_{n=0}^{\infty} \frac{(xt)^n}{(q;q)_n} \sum_{m=0}^{\infty} \frac{(xs)^m}{(q;q)_m} \sum_{k=0}^{\infty} \frac{(xv)^k}{(q;q)_k} \right\}, \qquad |xt|,|xs|,|xv|<1$$

$$= T(bD_q)\left\{ \frac{1}{(xt,xs,xv;q)_\infty} \right\}$$

$$= \frac{(xbsv;q)_\infty}{(xt,xs,xv,bs,bv;q)_\infty} {}_2\phi_1\left(\begin{matrix} xs, xv \\ xbsv \end{matrix} ;q,bt \right), \qquad |bt|,|bs|,|bv|<1.$$

- Setting $t=0$ in (4.5), we get the Rogers formula (1.11) for $r_n(x,b)$.

The following identity of the generalized Rogers-Szegö polynomials will be derived by using the identity (2.3) of the q -exponential operator as follows:

Theorem 4.6 *We have*

$$\sum_{m,k=0}^{\infty} r_{m+k}(x,b)r_{n+k}(y,b)\frac{t^m}{(q;q)_m}\frac{s^k}{(q;q)_k}$$
$$=\frac{(xyb^2s^2;q)_\infty}{(xt,xbs,xys,b^2s,bys;q)_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}\frac{(xbs;q)_l(xys;q)_{j+l}(bys;q)_j}{(xyb^2s^2;q)_{j+l}(q;q)_l}b^{j+l}y^{n-j}t^l, \quad (4.6)$$

where $\max\{|xt|,|xys|,|bxs|,|bt|,|b^2s|,|bys|\}<1$.

Proof.

$$\sum_{m,k=0}^{\infty} r_{m+k}(x,b)r_{n+k}(y,b)\frac{t^m}{(q;q)_m}\frac{s^k}{(q;q)_k}$$
$$=\sum_{m,k=0}^{\infty} T_x(bD_q)\{x^{m+k}\}r_{n+k}(y,b)\frac{t^m}{(q;q)_m}\frac{s^k}{(q;q)_k}$$
$$=T_x(bD_q)\left\{\sum_{m=0}^{\infty}\frac{(xt)^m}{(q;q)_m}\sum_{k=0}^{\infty}r_{n+k}(y,b)\frac{(xs)^k}{(q;q)_k}\right\}, \qquad |xt|,|xys|,|bxs|<1$$
$$=T_x(bD_q)\left\{\frac{1}{(xt,xbs;q)_\infty}\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}\frac{1}{(xysq^j;q)_\infty}b^j y^{n-j}\right\}$$
$$=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}b^j y^{n-j}T_x(bD_q)\left\{\frac{1}{(xt,xbs,xysq^j;q)_\infty}\right\}$$
$$=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}b^j y^{n-j}\frac{(xyb^2s^2q^j;q)_\infty}{(xt,xbs,xysq^j,b^2s,bysq^j;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}xbs,xysq^j\\xyb^2s^2q^j\end{matrix};q,bt\right)$$
$$=\frac{(xyb^2s^2;q)_\infty}{(xt,xbs,xys,b^2s,bys;q)_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}\frac{(xbs;q)_l(xys;q)_{j+l}(bys;q)_j}{(xyb^2s^2;q)_{j+l}(q;q)_l}b^{j+l}y^{n-j}t^l.$$

- Setting $n=t=0$ in (4.6), we get Mehler's formula (1.10), and setting $t=0$ in (4.6), we get the extension of Mehler's formula (4.2) of the generalized Rogers-Szegö polynomials.

5 Another Extensions for the Identities of $r_n(x,b)$

identities of the generalized Rogers-Szegö polynomials $r_n(x,b)$ which are more general than the previous extensions.

Theorem 5.1 *We have*

By using the q -exponential operator $T(bD_q)$, we give some other extensions for the basic

$$\sum_{n,m=0}^{\infty} r_{n+m}(x,b)r_n(y,b)r_m(z,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$=\frac{(xzb^2s^2;q)_\infty}{(xyt,xbt,xzs,xbs,bzs,b^2s;q)_\infty}\sum_{i,j=0}^{\infty}\frac{(xzs,xbs;q)_{i+j}(xbt;q)_j}{(xzb^2s^2;q)_{i+j}}\frac{(byt)^j}{(q;q)_j}\frac{(b^2t)^i}{(q;q)_i}, \quad (5.)$$

where $\max\{|xyt|,|xbt|,|xzs|,|xbs|,|b^2t|,|bzs|,|b^2s|\}<1$.

Proof.

$$\sum_{n,m=0}^{\infty} r_{n+m}(x,b)r_n(y,b)r_m(z,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$

$$=\sum_{n,m=0}^{\infty} T_x(bD_q)\{x^{n+m}\}r_n(y,b)r_m(z,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$

$$=T_x(bD_q)\left\{\sum_{n=0}^{\infty} r_n(y,b)\frac{(xt)^n}{(q;q)_n}\sum_{m=0}^{\infty} r_m(z,b)\frac{(xs)^m}{(q;q)_m}\right\},|xyt|,|xbt|,|xzs|,|xbs|<1$$

$$=T_x(bD_q)\left\{\frac{1}{(xyt,xbt,xzs,xbs;q)_\infty}\right\}$$

$$=\frac{(xzb^2s^2;q)_\infty}{(xyt,xbt,xzs,xbs,bzs,b^2s;q)_\infty}\sum_{i,j=0}^{\infty}\frac{(xzs,xbs;q)_{i+j}(xbt;q)_j}{(xzb^2s^2;q)_{i+j}}\frac{(byt)^j}{(q;q)_j}\frac{(b^2t)^i}{(q;q)_i},|b^2t|,|bzs|,|b^2s|<1.$$

- Setting $t=0$ in (5.1), we get Mehler's formula (1.10) for $r_n(x,b)$.

In the following theorem, we derive new identity for the generalized Rogers-Szegö polynomials depending on (5.1).

Theorem 5.2 *We have*

$$\sum_{n,m,k,j=0}^{\infty} r_{n+m}(x,b)r_{k+j}(y,b)r_n(z,b)r_m(w,b)r_k(f,b)r_j(g,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{u^k}{(q;q)_k}\frac{v^j}{(q;q)_j}$$

$$=\frac{(xwb^2s^2,ygb^2v^2;q)_\infty}{(xzt,xbt,xws,xbs,bws,b^2s,yfu,ybu,ygv,ybv,bgv,b^2v;q)_\infty}$$

$$\times\sum_{i,p,l,h=0}^{\infty}\frac{(xbt;q)_i(xws,xbs;q)_{i+p}(ybu;q)_l(ygv,ybv;q)_{l+h}}{(xwb^2s^2;q)_{i+p}(ygb^2v^2;q)_{l+h}}\frac{(bzt)^i}{(q;q)_i}\frac{(b^2t)^p}{(q;q)_p}\frac{(bfu)^l}{(q;q)_l}\frac{(b^2u)^h}{(q;q)_h},\qquad(5.2)$$

where $\max\{|xzt|,|xbt|,|xws|,|xbs|,|bws|,|b^2s|,|b^2t|,|yfu|,|ybu|,|ygv|,|ybv|,|bgv|,|b^2v|,|b^2u|\}<1$.

- Setting $t=u=v=0$ or $t=s=u=0$ in (5.2), we get Mehler's formula (1.10) of the generalized Rogers-Szegö polynomials.

Theorem 5.3 *We have*

$$\sum_{n,m,k=0}^{\infty} r_{n+m+k}(x,b)r_n(y,b)r_m(z,b)r_k(w,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{u^k}{(q;q)_k}$$

$$=\frac{(xb^2wu^2;q)_\infty}{(xyt,xbt,xzs,xbs,xwu,xbu,bwu,b^2u;q)_\infty}$$

$$\times\sum_{i,j,l,p=0}^{\infty}\frac{(xbt;q)_j(xzs;q)_{j+l}(xbs;q)_{i+j+l}(xwu,xbu;q)_{i+j+l+p}}{(xb^2wu^2;q)_{i+j+l+p}}\frac{(bzs)^i}{(q;q)_i}\frac{(byt)^j}{(q;q)_j}\frac{(b^2t)^l}{(q;q)_l}\frac{(b^2s)^p}{(q;q)_p},$$

(5.3)

where $\max\{|xyt|,|xbt|,|xzs|,|xbs|,|xwu|,|xbu|,|bwu|,|b^2s|,|b^2u|\}<1$.

Proof.

$$\sum_{n,m,k=0}^{\infty} r_{n+m+k}(x,b)r_n(y,b)r_m(z,b)r_k(w,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{u^k}{(q;q)_k}$$

$$=T_x(bD_q)\left\{\sum_{n=0}^{\infty} r_n(y,b)\frac{(xt)^n}{(q;q)_n}\sum_{m=0}^{\infty} r_m(z,b)\frac{(xs)^m}{(q;q)_m}\sum_{k=0}^{\infty} r_k(w,b)\frac{(xu)^k}{(q;q)_k}\right\}$$

$$=T_x(bD_q)\left\{\frac{1}{(xyt,xbt,xzs,xbs,xwu,xbu;q)_\infty}\right\}$$

$$=\frac{(xb^2wu^2;q)_\infty}{(xyt,xbt,xzs,xbs,xwu,xbu,bwu,b^2u;q)_\infty}$$

$$\times\sum_{i,j,l,p=0}^{\infty}\frac{(xbt;q)_j(xzs;q)_{j+l}(xbs;q)_{i+j+l}(xwu,xbu;q)_{i+j+l+p}}{(xb^2wu^2;q)_{i+j+l+p}}\frac{(bzs)^i}{(q;q)_i}\frac{(byt)^j}{(q;q)_j}\frac{(b^2t)^l}{(q;q)_l}\frac{(b^2s)^p}{(q;q)_p}.$$

- Setting $s=0$ in (5.3), we get the identity (5.2), and setting $t=s=0$ in (5.1), we get Mehler's formula (1.10) of the generalized Rogers-Szegö polynomials.

Finally, we derive the following identity of the generalized Rogers-Szegö polynomials by using the results (1.14) and (1.16) of the q-exponential operator.

Theorem 5.4 *We have*

$$\sum_{n,m,k=0}^{\infty} r_{n+k}(x,b)r_{m+k}(y,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{u^k}{(q;q)_k}$$

$$=\frac{(xybsu,yb^2su,xbtu/s;q)_\infty}{(xt,ys,xyu,bxu,byu,bt,bs,b^2u;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}ybsu/t,bs,ys\\ xybsu,yb^2su\end{matrix};q,xbtu/s\right), \tag{5.4}$$

where $\max\{|xt|,|ys|,|xyu|,|bs|,|bxu|,|bt|,|byu|,|b^2u|,|xbtu/s|\}<1$.

Proof.

$$\sum_{n,m,k=0}^{\infty} r_{n+k}(x,b)r_{m+k}(y,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{u^k}{(q;q)_k}$$

$$=\sum_{n,m,k=0}^{\infty} T_x(bD_q)\{x^{n+k}\}T_y(bD_q)\{y^{m+k}\}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{u^k}{(q;q)_k}$$

$$=T_x(bD_q)T_y(bD_q)\left\{\sum_{n=0}^{\infty}\frac{(xt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(ys)^m}{(q;q)_m}\sum_{k=0}^{\infty}\frac{(xyu)^k}{(q;q)_k}\right\}, |xt|,|ys|,|xyu|<1$$

$$=T_x(bD_q)T_y(bD_q)\left\{\frac{1}{(xt,ys,xyu;q)_\infty}\right\}$$

$$=T_x(bD_q)\left\{\frac{(ybxsu;q)_\infty}{(xt,ys,xyu,bs,bxu;q)_\infty}\right\},\quad |bs|,|bxu|<1$$

$$=\frac{(xybsu,yb^2su,xbtu/s;q)_\infty}{(xt,ys,xyu,bxu,byu,bt,bs,b^2u;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}ybsu/t,bs,ys\\ xybsu,yb^2su\end{matrix};q,xbtu/s\right).$$

- Setting $u=0$ in (5.4), we get a result of multiplication of two generating functions (1.9) for the generalized Rogers-Szegö polynomials, where

$$\sum_{n=0}^{\infty} r_n(x,b)\frac{t^n}{(q;q)_n}\sum_{m=0}^{\infty} r_m(y,b)\frac{s^m}{(q;q)_m} = \frac{1}{(xt,bt,ys,bs;q)_\infty}.$$

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