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Double Stage Shrinkage Estimation of the Reliability Function of the Proportional Hazard Family of Distribution Function under Different Loss Functions Using Progressive Type II Censored Sample

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Abstract

In this paper we propose two double stage shrinkage estimators of the reliability function of the proportional hazard family of distribution functions, using progressive type II censored sample. The risk functions and relative risk functions of the suggested estimators are derived under symmetric and asymmetric loss functions, viz., the squared error loss function and the general entropy loss function. The results show that the proposed estimators have better performance than a classical estimator in terms relative risk.

Keywords and phrases: double shrinkage estimation; reliability function; progressive type II censored sample.

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1. Introduction

One of the important probability models of family of distributions in reliability and life experiments is the proportional hazard family of distribution functions. It was originally proposed by Cox (1972) and later it was referred as Cox's proportional hazard regression model. The model led to the proportional hazard family of distribution functions which includes several well-known life distributions such as exponential distribution, Pareto (type I and type II) distribution, beta distribution, Burr type XII distribution etc as special cases. The proportional hazard family of distribution functions is defined as follows.

Let us consider the continuous random variable X with the cumulative distribution function (cdf) $F(x;\theta)$. The proportional hazard family of distribution functions with underlying distribution F_0 is defined as

$$F(x;\theta) = 1 - [\bar{F}_0(x)]^{\theta} \quad , \quad -\infty < c < x < d < \infty, \quad \theta > 0 \quad , \tag{1}$$

where $\bar{F}_0(.) = 1 - [F_0(.)]$, and F_0 is an arbitrary continuous cumulative distribution function with $F_0(c) = 0$, $F_0(d) = 1$. The family in (1) was studied by many authors such as Tibshirani and Ciampi (1983), Fergusson et al (1984) (applied this family in the study of family breakdown). In recent years Asgharzadeh and Valiollahi (2009, 2010) and Liang Wang and Yimin Shi (2012) studied it under progressive censored sample.

The reliability of a given system (or component) for a given time has been defined as the probability that the system (or component) functions longer than the time of duration t, and is given by

$$R(t) = P(X > t) = 1 - F(t;\theta) \quad , \quad t > 0 \tag{2}$$

From (1) the reliability function (at some time t) and probability density function (pdf) are respectively given by

$$R(t) = [\bar{F}_0(t)]^{\theta} \tag{3}$$

$$f(x;\theta) = \theta\, f_0(x)[\bar{F}_0(x)]^{\theta-1} \quad , \quad -\infty < c < x < d < \infty, \quad \theta > 0 \tag{4}$$

where $f_0(x) = \dfrac{dF_0(x)}{dx}$.

The prior information of about the unknown parameter θ as an initial guess value θ_0, based on the past experience, must be used while estimating θ. Thompson (1968) suggested single stage shrinkage estimation by modifying the usual estimator of the unknown parameter θ by moving it closer to θ_0. The shrinkage estimator performs batter than the usual estimator when the initial value is close to the true value of the parameter θ. The shrinkage estimators have been discussed for different parameters or parametric functions under different life of distributions by number of authors. With respect to reliability function Pandey and Upadhyay (1985) studied the problem of Bayes shrinkage estimation of reliability function from type II censored sample with exponential failure model. Later Chiou(1992,1993) suggested and studied properties some empirical Bayes and ordinary shrinkage estimators of reliability function for exponential distribution. Recently Al Hemyari and Jehel(2011) proposed pooling shrinkage estimator of reliability function for exponential model using the sampling plan (n, C, T) here n stands initial experimental units, C for renewal and T for stopping time of the experiment. Finally, Shanubhogue and Jiheel (2013) suggested and studied properties single stage shrinkage estimators of the reliability function of the proportional hazard family of distribution functions under and asymmetric loss functions using progressive type II censored sample.

It's important in estimation how we get a good estimators using minimum number of sample units and economize the cost of experimentation. Therefore, the double stage pooling estimation is a good procedure to achieve this goal. Katti (1962) suggested two stage pooling estimation for estimating the mean of the normal distribution when the variance is known and the initial value μ_0 of μ is given. He used the initial value only to construct a region for of the mean of the first sample, i.e., if the mean of the first sample belongs to this region the it will be used as estimate of μ otherwise second sample will be drawn and pooled mean will be constructed as estimator of μ. Arnold and Al-Bayyati (1970, 1972) proposed the double stage shrinkage estimator for the mean of the normal distribution and parameters of regression model, using the first stage sample to test the accuracy of initial value and if it is accurate then use shrinkage of the usual estimator towards a initial value, otherwise use pooled estimator as suggested in Katti (1962). On the line of this many authors study double stage shrinkage estimations like Waikar et al (1984) , Adke et al (1987) , Handa et al (1988) and Kambo et al (1991). Recently Srivastava and Tanna (2007), Al-Hemyari (2009) and Prakash and Singh (2009) studied similar type shrinkage estimation for different life distributions under various sampling schemes.

Therefore, the aim of this article is to extend Shanubhogue and Jiheel (2013) work from single stage to double stage shrinkage estimators of reliability function for the proportional hazard family of distribution functions and study the properties under symmetric and asymmetric loss functions using progressive type II censored sample.

In progressive type II censored sample the experimenter put n identical units, with life time distribution having cumulative function $F(x)$ and probability density function $f(x)$, for life testing. After observing the first failure , R_1 surviving units are removed from the test at random; next, immediately following the second failure, R_2 surviving units are removed from the test at random, and so on ; finally ,at the time of the m[th] failure all the remaining $R_m = n - m - \sum_{i=1}^{m-1} R_i$ surviving units are removed from the test. In this progressive censoring scheme $R_1, R2, \dots, Rm$ are assumed to be pre-fixed . The complete sample and usual censored sample are special cases of type II progressive censored scheme. If $R_1 = R_2 = \dots = R_m = 0$ we get complete sample and if $R_1 = R_2 = \dots = R_{m-1} = 0$, $R_m = n-m$ we get type II censored sample. Balakrishnan (2007) gives a complete idea and developments on progressive censored sample.

2. Double Shrinkage Estimators of R(t)

Let $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$ be Type II progressive censored sample from the proportional hazard family (1). Then the joint density of the progressive censored sample (see Balakrishnan and Aggarwala 2000) is

$$f(x_{1:m::n}, x_{2:m:n}, \dots, x_{m:m:n}) = C\prod_{i=1}^{m} f(x_{i:m:n})(1 - F(x_{i:m:n}))^{R_i}, \; 0 \le x_{1:m::n} \le x_{2:m:n} \le \dots \le x_{m:m:n} \quad , \tag{5}$$

where C= n (n-R_1-1)(n-R_1-R_2-2) . . . (n-R_1-R_2- . . . – R_{m-1}-m+1) . Substituting (1) and (4) in (5) we get

$$f(x_{1:m::n}, x_{2:m:n}, \dots, x_{m:m:n}) = C\prod_{i=1}^{m} \frac{f_0(x_{i:m:m})}{\bar{\bar{F}}_0(x_{i:m:m})} \theta^m e^{-\theta S}, \; 0 \le x_{1:m::n} \le x_{2:m:n} \le \dots \le x_{m:m:n} \tag{6}$$

where $S = -\sum_{i=1}^{m} (R_i + 1)\, Ln\bar{F}_0(x_{i:m:m})$. The log likelihood function can be written as

$$L(x_{1:m::n}, x_{2:m:n}, \dots, x_{m:m:n}) = Ln(C) + \sum_{i=1}^{m} Ln\left(\frac{f_0(x_{i:m:m})}{\bar{\bar{F}}_0(x_{i:m:m})}\right) + mLn\,\theta - \theta S$$

Then the MLE of θ is

$$\hat{\theta} = \frac{m}{S}$$

Now, by replacing θ by its MLE $\hat{\theta}$ in (4) we get the MLE of reliability function of proportional hazard family function as

$$\hat{R}(t) = [\bar{F}_0(t)]^{\hat{\theta}} \tag{7}$$

It is seen that (from Asgharzadeh and Valiollahi (2009)) S has Gamma distribution with parameters m and θ. By using the result we can show that the distribution of $\hat{\theta}$ has distribution as

$$f(\hat{\theta};\theta) = \frac{(m\theta)^m \exp\left(-\frac{m\theta}{\hat{\theta}}\right)}{\Gamma(m)\,(\hat{\theta})^{m+1}}, \hat{\theta} > 0 \tag{8}$$

Now, let ${}_1X_{i:m_1:n_1}, (i = 1,2,\dots,m_1)$ with schemes $R_{1i}, (i = 1,2,\dots,m_1)$ and ${}_2X_{j:m_2:n_2}, \; (j = 1,2,\dots,m_2)$ with schemes $R_{2j}, (j = 1,2,\dots,m_2)$ be the two independent progressive censored samples of size m_1 and m_2 respectively, drawn from the distribution defined in (1). Then by (4) the ML estimators for the scale parameter θ based on the first sample and the second sample are given respectively

$$\hat{\theta}_1 = \frac{m_1}{S_1} \quad , \tag{9}$$

where $S_1 = -\sum_{i=1}^{m_1} ({}_1R_i + 1)\, Ln\bar{F}_0({}_1x_{i:m:m}) \quad ,$

$$\hat{\theta}_2 = \frac{m_2}{S_2} \quad , \tag{10}$$

where $S_2 = -\sum_{j=1}^{m_2} ({}_2R_j + 1)\, \mathrm{Ln}\bar{F}_0({}_2x_{j:m:m})$,

and the pooled estimator of θ is given

$$\hat{\theta}_p = \frac{m_1\hat{\theta}_1 + m_2\hat{\theta}_2}{m_1 + m_2} \quad . \tag{11}$$

Now, by replacing θ by its MLE $\hat{\theta}$ of the first sample, second sample and pooled estimator in (4) we get the MLE of reliability function of proportional hazard family for first, second and pooled estimator respectively as

$$\hat{R}_1(t) = [F_0(t)]^{\hat{\theta}_1} \quad , \tag{12}$$

$$\hat{R}_2(t) = [F_0(t)]^{\hat{\theta}_2} \tag{13}$$

and

$$\hat{R}_p(t) = [F_0(t)]^{\hat{\theta}_p} \quad . \tag{14}$$

Depending on the testing of hypothesis H_0: $\theta = \theta_0$ against H_1:$\theta \neq \theta_0$ at level α, if H_0 accepted we use only first sample and take the shrinkage estimator $[F_0(t)]^{k\,\hat{\theta}_1 + (1-k)\,\theta_0}$ or $k[F_0(t)]^{\hat{\theta}_1} + (1-k)[F_0(t)]^{\theta_0}$. Otherwise we drawn a second sample and take the estimator defined in (14). Therefore, we proposed the following two double shrinkage estimators,

$$\tilde{R}_1(t) = \begin{cases} [F_0(t)]^{k\,\hat{\theta}_1 + (1-k)\,\theta_0} & H_0 : \theta = \theta_0 \text{ Accepted} \\ [F_0(t)]^{\hat{\theta}_p} & \text{otherwise} \end{cases} \tag{15}$$

$$\tilde{R}_2(t) = \begin{cases} k[F_0(t)]^{\hat{\theta}_1} + (1-k)[F_0(t)]^{\theta_0} & H_0 : \theta = \theta_0 \text{ Accepted} \\ [F_0(t)]^{\hat{\theta}_p} & \text{otherwise} \end{cases} \tag{16}$$

In next section, we derive risks of the estimators under symmetric loss function (square error loss function (SELF)) and asymmetric loss function (general entropy loss function (GELF)). As many real life situations over estimation or under estimation are not of equal consequences and hence for such situations, a useful asymmetric loss function is LINEX loss function introduced by Varian (1975). It is given by

$$L(\Delta) = b(e^{a\Delta} - a\,\Delta - 1) \tag{17}$$

where $\Delta = \frac{\hat{\theta}}{\theta} - 1$, b is the scale parameter and a is the shape parameter. The sign and values of a respectively represent the direction and degree of asymmetry. The positive value of a is used when over estimation is more serious than under estimation and negative value is used for the other case. If we replace a by q, $b=1$ and Δ by $\mathrm{Ln}(\Delta)$ then we get another loss called GELF proposed by Calabria and Pulcini(1996) and is

$$L(\Delta) = \Delta^q - q\mathrm{Ln}(\Delta) - 1;\ q \neq 0, \Delta = \frac{\hat{\theta}}{\theta} \tag{18}$$

Here q plays the same role as in LINEX loss function.

3. Risk of the estimators

3.1 The Risk of the MLE estimator $\hat{R}_p(t)$

The risk of the estimator $\hat{R}(t)$ under GELF is defined as follows.

$R_{GELF}(\hat{R}_p(t)) = E(\hat{R}_p(t) | GELF)$

$$= \int_0^\infty \int_0^\infty \left[\left(\frac{[F_0(t)]^{\hat{\theta}_p}}{[F_0(t)]^{\theta_0}} \right)^q - qLn\left(\frac{[F_0(t)]^{\hat{\theta}_p}}{[F_0(t)]^{\theta_0}} \right) - 1 \right] f(\hat{\theta}_1;\theta) f(\hat{\theta}_2;\theta) d\hat{\theta}_1 d\hat{\theta}_2 ,$$

where $f(\hat{\theta}_i;\theta)$; i= 1,2 define in (8).

Now, by using the transformation $x = \frac{\hat{\theta}_1}{m_1 \theta}$, $y = \frac{\hat{\theta}_2}{m_2 \theta}$ and substituting in the integrals above, we get

$$R_{GELF}(\hat{R}_p(t)) = \exp(qp) \int_0^\infty \frac{\exp(-(\frac{qp m_1^2}{m_1+m_2} x + \frac{1}{x}))}{\Gamma(m_1) x^{m_1+1}} dx \int_0^\infty \frac{\exp(-(\frac{qp m_2^2}{m_1+m_2} y + \frac{1}{y}))}{\Gamma(m_2) y^{m_2+1}} dy ,$$

$$\frac{qp m_1^2}{m_1+m_2} \int_0^\infty \frac{\exp(-1/x)}{\Gamma(m_1) x^{m_1}} dx + \frac{qp m_2^2}{m_1+m_2} \int_0^\infty \frac{\exp(-1/y)}{\Gamma(m_2) y^{m_2}} dy - qp - 1$$

where $p = -\theta Ln(\bar{F}_0(t))$. To evaluate the first integrals in the above expression we recall a result due to Watson (1952), viz.

$$\int_0^\infty \frac{\exp(-(a z + \frac{b}{z}))}{z^r} dz = 2 (\frac{a}{b})^{\frac{(r-1)}{2}} K_{r-1}(2\sqrt{a b}) \qquad (19)$$

where $K_r(\ )$ is the modified Bessel function of the second kind of order r for r = 0,1,2, ... Substituting $r = m_i+1$ and $a = \frac{qp m_i^2}{m_1+m_2}$, where i=1, 2 and b=1 we obtain

$$R_{GELF}(R_p(t)) = \frac{4\exp(qp) \left(\frac{qp m_1^2}{m_1+m_2}\right)^{(m_1/2)} \left(\frac{qp m_2^2}{m_1+m_2}\right)^{(m_2/2)}}{\Gamma(m_1)\Gamma(m_2)} K_{m_1}(2\sqrt{\frac{qp m_1^2}{m_1+m_2}}) K_{m_2}(2\sqrt{\frac{qp m_2^2}{m_1+m_2}}) + \frac{qp}{m_1+m_2}\left(\frac{m_1^2}{m_1-1} + \frac{m_2^2}{m_2-1} \right) - qp - 1 . \qquad (20)$$

Also, we can define the risk of estimator $\hat{R}(t)$ under SELF as

$R_{SELF}(\hat{R}_p(t)) = E(\hat{R}_p(t) - R(t))^2$

$$= \int_0^\infty \int_0^\infty ([\bar{F}_0(t)]^{\hat{\theta}_p} - [F_0(t)]^{\theta})^2 f(\hat{\theta}_1;\theta) f(\hat{\theta}_2;\theta) d\hat{\theta}_1 d\hat{\theta}_2 .$$

By using same the transformation $x = \frac{\hat{\theta}_1}{m_1\theta}$, $y = \frac{\hat{\theta}_2}{m_2\theta}$ and the other required results we get

$$R_{SELF}(\hat{R}_p(t)) = \frac{4\left(\frac{2pm_1^2}{m_1+m_2}\right)^{(m_1/2)}\left(\frac{2pm_2^2}{m_1+m_2}\right)^{(m_2/2)}}{\Gamma(m_1)\Gamma(m_2)} K_{m_1}\left(2\sqrt{\frac{2pm_1^2}{m_1+m_2}}\right) K_{m_2}\left(2\sqrt{\frac{2pm_2^2}{m_1+m_2}}\right) - \frac{8\exp(-p)\left(\frac{pm_1^2}{m_1+m_2}\right)^{(m_1/2)}\left(\frac{pm_2^2}{m_1+m_2}\right)^{(m_2/2)}}{\Gamma(m_1)\Gamma(m_2)} K_{m_1}\left(2\sqrt{\frac{pm_1^2}{m_1+m_2}}\right) K_{m_2}\left(2\sqrt{\frac{pm_2^2}{m_1+m_2}}\right) + \exp(-2p) \quad (21)$$

3.2 The Risk of the shrinkage estimator $\tilde{R}_1(t)$

The risk of the estimator $\tilde{R}_1(t)$ under GELF is defined as follows:

$R_{GELF}(\tilde{R}_1(t)) = E(\tilde{R}_1(t) \mid GELF)$

$$= \int_{r_1}^{r_2} \left(\left[\frac{[\bar{F}_0(t)]^{k\hat{\theta}_1+(1-k)\theta_0}}{[\bar{F}_0(t)]^{\theta}} \right]^q - q\mathrm{Ln}\left[\frac{[\bar{F}_0(t)]^{k\hat{\theta}_1+(1-k)\theta_0}}{[\bar{F}_0(t)]^{\theta}} \right] - 1 \right) f(\hat{\theta}_1;\theta)\, d\hat{\theta}$$

$$+ \int_0^{\infty}\int_0^{\infty} \left[\left(\frac{[F_0(t)]^{\hat{\theta}_p}}{[F_0(t)]^{\theta_0}} \right)^q - q\mathrm{Ln}\left(\frac{[F_0(t)]^{\hat{\theta}_p}}{[F_0(t)]^{\theta_0}} \right) - 1 \right] f(\hat{\theta}_1;\theta) f(\hat{\theta}_2;\theta)\, d\hat{\theta}_1 d\hat{\theta}_2$$

$$- \int_0^{\infty}\int_{r_1}^{r_2} \left[\left(\frac{[F_0(t)]^{\hat{\theta}_p}}{[F_0(t)]^{\theta_0}} \right)^q - q\mathrm{Ln}\left(\frac{[F_0(t)]^{\hat{\theta}_p}}{[F_0(t)]^{\theta_0}} \right) - 1 \right] f(\hat{\theta}_1;\theta) f(\hat{\theta}_2;\theta)\, d\hat{\theta}_1 d\hat{\theta}_2$$

where r_1 and r_2 are boundaries of the acceptance region of a test of the hypothesis $H_0{:}\theta = \theta_0$ against the alternative $H_1{:}\ \theta \neq \theta_0$. Define $r_1 = \frac{2m_1\theta_0}{\chi_2^2}$, $r_2 = \frac{2m_2\theta_0}{\chi_1^2}$ where χ_1^2 and χ_2^2 are respectively lower and upper αth percentile values of chi-square distribution with 2m degrees of freedom. Now, also by using the transformation $x = \frac{\hat{\theta}_1}{m_1\theta}$, $y = \frac{\hat{\theta}_2}{m_2\theta}$ and evaluating the integrals in the expression for GELF we get

$$R_{GELF}(\tilde{R}_1(t)) = \exp(qp(1-(1-k)\lambda))\int_{r_1'}^{r_2'} \frac{\exp(-(qpm_1kx+\frac{1}{x}))}{\Gamma(m_1)\,x^{m_1+1}}dx$$

$$+\frac{qpm_1k}{m_1-1}\left[I(1/r_1',m_1-1)-I(1/r_2',m_1-1)\right]+qp(1-k)\lambda\left[I(1/r_1',m_1)-I(1/r_2',m_1)\right]$$

$$+\frac{2\exp(qp)(\frac{qpm_2^2}{m_1+m_2})^{\frac{m_2}{2}}k_m(2\sqrt{\frac{qpm_2^2}{m_1+m_2}})}{\Gamma(m_1)\Gamma(m_2)}\left[\begin{array}{l}2\exp(qp)(\frac{qpm_1^2}{m_1+m_2})^{\frac{m_1}{2}}k_m(2\sqrt{\frac{qpm_1^2}{m_1+m_2}})\\ -\int_{r_1'}^{r_2'}\frac{\exp(-(\frac{qpm_2^2}{m_1+m_2}x+\frac{1}{x}))}{x^{m_1+1}}dx\end{array}\right] \quad (22)$$

$$+\frac{qpm_1^2}{(m_1+m_2)(m_1-1)}\left[1-\left[I(1/r_1',m_1-1)-I(1/r_2',m_1-1)\right]\right]$$

$$+\frac{qpm_2^2}{(m_1+m_2)(m_2-1)}\left[1-\left[I(1/r_1',m_1)-I(1/r_2',m_1)\right]\right]-qp-1$$

where $r_1' = \frac{2\lambda}{\chi_2^2}$, $r_2' = \frac{2\lambda}{\chi_1^2}$, $\lambda = \frac{\theta_0}{\theta}$ and $I(x, n)$ is the cdf of the gamma distribution given by $I(x,n) = \frac{\int_0^x t^{n-1}e^{-t}dt}{\Gamma(n)}$.

Now, we can define the risk of estimator $\tilde{R}_1(t)$ under SELF as

$R_{SELF}(\tilde{R}_1(t)) = E(\tilde{R}_1(t) - R(t))^2$

$$=\int_{r_1}^{r_2}([\bar{F}_0(t)]^{k\hat{\theta}_1+(1-k)\theta_0}-[F_0(t)]^{\theta})^2 f(\hat{\theta}_1;\theta)d\hat{\theta}_1+\int_0^{\infty}\int_0^{\infty}([\bar{F}_0(t)]^{\hat{\theta}_p}-[F_0(t)]^{\theta})^2 f(\hat{\theta}_1;\theta)f(\hat{\theta}_2;\theta)d\hat{\theta}_1 d\hat{\theta}_2$$

$$=\int_0^{\infty}\int_{r_1}^{r_2}([\bar{F}_0(t)]^{\hat{\theta}_p}-[F_0(t)]^{\theta})^2 f(\hat{\theta}_1;\theta)f(\hat{\theta}_2;\theta)d\hat{\theta}_1 d\hat{\theta}_2$$

Also, by using the transformation $x = \frac{\hat{\theta}_1}{m_1\theta}$, $y = \frac{\hat{\theta}_2}{m_2\theta}$ and evaluating the integrals above, we obtain

$$R_{SELF}\,(\tilde{R}_1(t)) = \exp(-2p((1-k)\lambda))\int_{r_1'}^{r_2'} \frac{\exp(-(2pm_1kx+\frac{1}{x}))}{\Gamma(m_1)\,x^{m_1+1}}dx$$

$$-2\exp(-p(1+(1-k)\lambda))\int_{r_1'}^{r_2'} \frac{\exp(-(pm_1kx+\frac{1}{x}))}{\Gamma(m_1)\,x^{m_1+1}}dx$$

$$+\frac{2(\frac{2pm_2^2}{m_1+m_2})^{\frac{m_2}{2}}k_{m_2}(2\sqrt{\frac{2pm_2^2}{m_1+m_2}})}{\Gamma(m_1)\,\Gamma(m_2)}\left[\begin{array}{l} 2(\frac{2pm_1^2}{m_1+m_2})^{\frac{m_1}{2}}k_{m_1}(2\sqrt{\frac{2pm_1^2}{m_1+m_2}}) \\ -\int_{r_1'}^{r_2'}\frac{\exp(-(\frac{2pm_1^2}{m_1+m_2}x+\frac{1}{x}))}{x^{m_1+1}}dx \end{array}\right]+\exp(-2p)$$

$$-\frac{4\exp(-p)(\frac{pm_2^2}{m_1+m_2})^{\frac{m_2}{2}}k_{m_2}(2\sqrt{\frac{pm_2^2}{m_1+m_2}})}{\Gamma(m_1)\,\Gamma(m_2)}\left[\begin{array}{l} 2(\frac{pm_1^2}{m_1+m_2})^{\frac{m_1}{2}}k_{m_1}(2\sqrt{\frac{pm_1^2}{m_1+m_2}}) \\ -\int_{r_1'}^{r_2'}\frac{\exp(-(\frac{pm_1^2}{m_1+m_2}x+\frac{1}{x}))}{x^{m_1+1}}dx \end{array}\right] \quad (23)$$

3.3 The Risk of the shrinkage estimator $\tilde{R}_2(t)$:

The risk of estimator $\tilde{R}_2(t)$ under SELF can be obtained as

$$R_{SELF}\,(\tilde{R}_2(t)) = E\,(\tilde{R}_2(t)-R(t))^2$$

$$=\int_{r_1}^{r_2}(k\,[\bar{F}(t)]^{\hat{\theta}}+(1-k)[\bar{F}(t)]^{\theta_0}-[\bar{F}(t)]^{\theta})^2\, f(\hat{\theta};\theta)\,d\hat{\theta}$$

$$+\int_{0}^{\infty}([\bar{F}_0(t)]^{\hat{\theta}}-[F_0(t)]^{\theta})^2 f(\hat{\theta};\theta)\,d\hat{\theta}\;-\int_{r_1}^{r_2}([\bar{F}_0(t)]^{\hat{\theta}}-[F_0(t)]^{\theta})^2 f(\hat{\theta};\theta)\,d\hat{\theta}$$

Now, by using the transformation $x=\frac{\hat{\theta}_1}{m_1\,\theta}$, $y=\frac{\hat{\theta}_2}{m_2\,\theta}$ and evaluating the above integrals, we obtain

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$$
\begin{aligned}
R_{SELF}(\tilde{R}_2(t)) &= k^2\left[\begin{array}{l}\int_{r_1'}^{r_2'}\frac{\exp(-(2pm_1x+\frac{1}{x}))}{\Gamma(m_1)\,x^{m_1+1}}dx - 2\exp(-p\lambda)\int_{r_1'}^{r_2'}\frac{\exp(-(pm_1x+\frac{1}{x}))}{\Gamma(m_1)\,x^{m_1+1}}dx\\ +\exp(-2p\lambda)\int_{r_1'}^{r_2'}\frac{\exp(-\frac{1}{x})}{\Gamma(m_1)\,x^{m_1+1}}dx\end{array}\right]\\
&-2k(\exp(-p)-\exp(-p\lambda))\left[\int_{r_1'}^{r_2'}\frac{\exp(-(pm_1x+\frac{1}{x}))}{\Gamma(m_1)\,x^{m_1+1}}dx - \exp(-p\lambda)\int_{r_1'}^{r_2'}\frac{\exp(-\frac{1}{x})}{\Gamma(m_1)\,x^{m_1+1}}dx\right]\\
&+\left[(\exp(-p)-\exp(-p\lambda))^2-1\right]\left[I(1/r_1', m_1) - I(1/r_2', m_1)\right]\\
&+\frac{2(\frac{2pm_2^2}{m_1+m_2})^{\frac{m_2}{2}}k_{m_2}(2\sqrt{\frac{2pm_2^2}{m_1+m_2}})}{\Gamma(m_1)\,\Gamma(m_2)}\left[\begin{array}{l}2(\frac{2pm_1^2}{m_1+m_2})^{\frac{m_1}{2}}k_{m_1}(2\sqrt{\frac{2pm_1^2}{m_1+m_2}})\\ -\int_{r_1'}^{r_2'}\frac{\exp(-(\frac{2pm_1^2}{m_1+m_2}x+\frac{1}{x}))}{x^{m_1+1}}dx\end{array}\right]+\exp(-2p)\\
&-\frac{4\exp(-p)(\frac{pm_2^2}{m_1+m_2})^{\frac{m_2}{2}}k_{m_2}(2\sqrt{\frac{pm_2^2}{m_1+m_2}})}{\Gamma(m_1)\,\Gamma(m_2)}\left[\begin{array}{l}2(\frac{pm_1^2}{m_1+m_2})^{\frac{m_1}{2}}k_{m_1}(2\sqrt{\frac{pm_1^2}{m_1+m_2}})\\ -\int_{r_1'}^{r_2'}\frac{\exp(-(\frac{pm_1^2}{m_1+m_2}x+\frac{1}{x}))}{x^{m_1+1}}dx\end{array}\right]
\end{aligned}
\tag{24}
$$

We find the shrinkage factor k by minimize (16) with respect to k as

$$
k^* = \frac{\left[\begin{array}{l}\int_{r_1'}^{r_2'}\frac{\exp(-(2pm_1x+\frac{1}{x}))}{\Gamma(m_1)\,x^{m_1+1}}dx - 2\exp(-p\lambda)\int_{r_1'}^{r_2'}\frac{\exp(-(pm_1x+\frac{1}{x}))}{\Gamma(m_1)\,x^{m_1+1}}dx\\ +\exp(-2p\lambda)\int_{r_1'}^{r_2'}\frac{\exp(-\frac{1}{x})}{\Gamma(m_1)\,x^{m_1+1}}dx\end{array}\right]}{(\exp(-p)-\exp(-p\lambda))\left[\int_{r_1'}^{r_2'}\frac{\exp(-(pm_1x+\frac{1}{x}))}{\Gamma(m_1)\,x^{m_1+1}}dx - \exp(-p\lambda)\int_{r_1'}^{r_2'}\frac{\exp(-\frac{1}{x})}{\Gamma(m_1)\,x^{m_1+1}}dx\right]}
\tag{25}
$$

Theoretically, we could not show that k^* lies between 0 and 1. Therefore, we restrict the value of ‘k’ as

$$
k = \begin{cases} 0 & k^* < 0 \\ k^* & 0 \le k^* \le 1 \\ 1 & k^* > 1 \end{cases}
\tag{26}
$$

4. Relative Risk:

To study the properties of estimators $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ under SELF and GELF we compare the relative risks of the estimators given above. The relative risk of $\tilde{R}_1(t)$ with respect to $\hat{R}(t)$ under GELF is

$$RR_{GELF}(\tilde{R}_1(t)) = \frac{R_{GELF}(\hat{R}_p(t))}{R_{GELF}(\tilde{R}_1(t))} \quad (27)$$

Also, the relative risk of $\tilde{R}_1(t)$ with respect to $\hat{R}(t)$ under SELF is

$$RR_{SELF}(\tilde{R}_1(t)) = \frac{R_{SELF}(\hat{R}_p(t))}{R_{SELF}(\tilde{R}_1(t))} \quad (28)$$

Finally, the relative risk of $\tilde{R}_2(t)$ with respect to $\hat{R}(t)$ under SELF is

$$RR_{SELF}(\tilde{R}_2(t)) = \frac{R_{SELF}(\hat{R}_p(t))}{R_{SELF}(\tilde{R}_2(t))} \quad (29)$$

We observe that the equations $RR_{GELF}(\tilde{R}_1(t))$, $RR_{SELF}(\tilde{R}_1(t))$ and $RR_{SELF}(\tilde{R}_2(t))$ depend on m_1, m_2, k, q, p, α and λ. To demonstrate the performance of the proposed estimators under GELF and SELF, we have considered few values of the constants as.

m_1= 6, 9, 12, m_2 = 3, 9, 12, k = 0.1, 0.2, q= 1, 2, 3, p = 0.5, 1.0, 1.5, α = 0.01, 0.05, λ = 0.25(0.25)1.75

The Tables 1 to 3 give the values of the relative risk for the above given values of constants. Based on these tables we have the following conclusions.

5. Conclusion:

i. The relative risk of the both proposed estimators under SELF and GELF are high in around λ=1. i.e. if true value of θ is close to θ_0. Further, the range of values of λ for relative risk larger than one become narrower as m_2 and p are increased.

ii. Compared with the results in Shanubhogue and Jiheel (2013) we see the relative risk of the proposed estimators under GELF and SELF are better.

iii. The relative risk of the estimator $\tilde{R}_1(t)$ under SELF is higher than the relative risk of the estimator $\tilde{R}_1(t)$ under GELF.

iv. The relative risk of the estimator $\tilde{R}_2(t)$ under SELF is higher than the relative risk of the estimator $\tilde{R}_1(t)$ under SELF.

v. The relative risk of the estimator $\tilde{R}_1(t)$ under GELF is decreasing function with p and q for all λ except λ=1, also decreasing with m_2 but we can't make any conclusion for m_1.

vi. The relative risk of the estimator $\tilde{R}_1(t)$ under SELF is decreasing function of m_1, m_2 and p for all λ while the relative risk of the estimator $\tilde{R}_2(t)$ under SELF is decreasing function of m_1 and m_2. Further, it is decreasing with p for all λ except λ=1.

vii. The relative risk of the estimator $\tilde{R}_1(t)$ under SELF and GELF and the relative risk of the estimator $\tilde{R}_2(t)$ under SELF are decreasing functions of α.

viii. The relative risk of the estimator $\tilde{R}_1(t)$ under SELF and GELF are decreasing functions of k i.e if we strongly believe in the guess value θ_0 .

Thus, in general, we observe that the double stage shrinkage estimators of the reliability function proposed by us perform better than the classical estimator and single stage shrinkage estimator both under SELF and GELF.

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Appendix

Table (1) show the relative risk of the estimator $\tilde{R}_1(t)$ under GELF when α=0.01

k=0.1	λ									
m1	m2	q	p	0.25	0.50	0.75	1.00	1.25	1.50	1.75
6	3	1	0.5	0.6834	1.0663	3.3386	11.4468	4.4196	1.5989	0.8611
			1	0.5805	0.8983	3.0829	12.4848	3.9519	1.4184	0.7835
			1.5	0.4926	0.7674	2.8457	13.2255	3.6121	1.3066	0.7403
		2	0.5	0.5805	0.8983	3.0829	12.4848	3.9519	1.4184	0.7835
			1	0.4158	0.66	2.6239	13.7203	3.3551	1.2314	0.7145
			1.5	0.2903	0.4944	2.2367	14.2128	3.0007	1.1422	0.691
		3	0.5	0.4926	0.7674	2.8457	13.2255	3.6121	1.3066	0.7403
			1	0.2903	0.4944	2.2367	14.2128	3.0007	1.1422	0.691
			1.5	0.1615	0.328	1.7992	14.2576	2.6988	1.088	0.6913
	6	1	0.5	0.4858	0.6878	2.2896	9.4206	2.6217	0.8979	0.4799
			1	0.4132	0.5934	2.1379	10.3397	2.4563	0.8486	0.4664
			1.5	0.3495	0.5139	1.9848	11.0127	2.3239	0.8157	0.4603
		2	0.5	0.4132	0.5934	2.1379	10.3397	2.4563	0.8486	0.4664
			1	0.294	0.4464	1.8391	11.5038	2.2196	0.7941	0.459
			1.5	0.2044	0.3397	1.583	12.1259	2.0762	0.7733	0.466
		3	0.5	0.3495	0.5139	1.9848	11.0127	2.3239	0.8157	0.4603
			1	0.2044	0.3397	1.583	12.1259	2.0762	0.7733	0.466
			1.5	0.1138	0.2296	1.2905	12.5829	1.9728	0.7783	0.4917
	9	1	0.5	0.3855	0.5269	1.843	8.9345	1.9165	0.6359	0.3381
			1	0.3272	0.457	1.7111	9.6424	1.8273	0.6169	0.3378
			1.5	0.2764	0.3976	1.5848	10.1754	1.7565	0.6054	0.3406
		2	0.5	0.3272	0.457	1.7111	9.6424	1.8273	0.6169	0.3378
			1	0.2323	0.3469	1.4678	10.5804	1.7016	0.5994	0.3456
			1.5	0.1614	0.266	1.2658	11.14	1.6299	0.5994	0.3603
		3	0.5	0.2764	0.3976	1.5848	10.1754	1.7565	0.6054	0.3406
			1	0.1614	0.266	1.2658	11.14	1.6299	0.5994	0.3603
			1.5	0.0901	0.1814	1.036	11.6642	1.5937	0.6213	0.3915
9	3	1	0.5	0.8008	0.7464	2.4354	13.0718	3.3252	1.105	0.6156
			1	0.7393	0.6375	2.2106	13.817	2.9814	1.0087	0.5798
			1.5	0.6773	0.5517	2.0264	14.4245	2.7424	0.9487	0.5612
		2	0.5	0.7393	0.6375	2.2106	13.817	2.9814	1.0087	0.5798
			1	0.6141	0.4804	1.8659	14.8637	2.5642	0.9086	0.5516
			1.5	0.4872	0.3677	1.598	15.307	2.3184	0.8621	0.5474
		3	0.5	0.6773	0.5517	2.0264	14.4245	2.7424	0.9487	0.5612

			1	0.4872	0.3677	1.598	15.307	2.3184	0.8621	0.5474
			1.5	0.3135	0.2499	1.2993	15.1891	2.106	0.8387	0.5613
			0.5	0.6791	0.5142	1.714	11.1041	2.0915	0.6774	0.377
		1	1	0.6213	0.4506	1.591	11.9029	1.9706	0.6536	0.3753
			1.5	0.5626	0.3957	1.4767	12.5322	1.8764	0.6383	0.3771
			0.5	0.6213	0.4506	1.591	11.9029	1.9706	0.6536	0.3753
	6	2	1	0.5039	0.3482	1.3721	13.0167	1.8028	0.6289	0.3813
			1.5	0.3906	0.2705	1.1914	13.6469	1.7008	0.6231	0.3946
			0.5	0.5626	0.3957	1.4767	12.5322	1.8764	0.6383	0.3771
		3	1	0.3906	0.2705	1.1914	13.6469	1.7008	0.6231	0.3946
			1.5	0.2448	0.1869	0.9842	14.0131	1.625	0.6359	0.4234
			0.5	0.6033	0.4148	1.4171	10.8292	1.6029	0.5106	0.2835
		1	1	0.5475	0.3651	1.3151	11.4895	1.5367	0.503	0.2882
			1.5	0.4918	0.3218	1.2213	12.0213	1.485	0.4993	0.2944
			0.5	0.5475	0.3651	1.3151	11.4895	1.5367	0.503	0.2882
	9	2	1	0.437	0.2841	1.1359	12.446	1.4451	0.4989	0.3017
			1.5	0.3341	0.2219	0.9891	13.0482	1.3929	0.5053	0.3192
			0.5	0.4918	0.3218	1.2213	12.0213	1.485	0.4993	0.2944
		3	1	0.3341	0.2219	0.9891	13.0482	1.3929	0.5053	0.3192
			1.5	0.2062	0.1542	0.8201	13.5413	1.3662	0.5289	0.3509
			0.5	0.9362	0.6029	1.8597	13.6712	2.5554	0.8382	0.4952
		1	1	0.915	0.5229	1.6881	14.2266	2.3207	0.7825	0.4778
			1.5	0.8908	0.4582	1.5502	14.7132	2.1573	0.7482	0.4706
			0.5	0.915	0.5229	1.6881	14.2266	2.3207	0.7825	0.4778
	3	2	1	0.8627	0.4032	1.4318	15.0784	2.0349	0.7257	0.4689
			1.5	0.7922	0.314	1.2356	15.4437	1.8649	0.7018	0.4747
			0.5	0.8908	0.4582	1.5502	14.7132	2.1573	0.7482	0.4706
		3	1	0.7922	0.314	1.2356	15.4437	1.8649	0.7018	0.4747
			1.5	0.6482	0.2173	1.0155	15.2416	1.7175	0.6955	0.4963
			0.5	0.8935	0.4375	1.3488	11.8047	1.692	0.5474	0.324
12		1	1	0.8688	0.3882	1.2553	12.4967	1.6093	0.5362	0.3278
			1.5	0.8402	0.3447	1.1697	13.0579	1.5446	0.5298	0.3335
			0.5	0.8688	0.3882	1.2553	12.4967	1.6093	0.5362	0.3278
	6	2	1	0.8073	0.3062	1.0916	13.4991	1.4938	0.527	0.3405
			1.5	0.7275	0.2418	0.9564	14.0738	1.4232	0.5293	0.3576
			0.5	0.8402	0.3447	1.1697	13.0579	1.5446	0.5298	0.3335
		3	1	0.7275	0.2418	0.9564	14.0738	1.4232	0.5293	0.3576
			1.5	0.5751	0.1697	0.7989	14.3412	1.3712	0.547	0.3889
			0.5	0.8627	0.3661	1.1437	11.6624	1.3478	0.4314	0.2549
	9	1	1	0.8349	0.326	1.0658	12.2634	1.3013	0.4295	0.2622
			1.5	0.8031	0.2902	0.9943	12.7603	1.265	0.4301	0.2702

			0.5	0.8349	0.326	1.0658	12.2634	1.3013	0.4295	0.2622
		2	1	0.7671	0.2584	0.9291	13.1643	1.237	0.4327	0.279
			1.5	0.6818	0.2047	0.8162	13.7375	1.2009	0.4427	0.2983
			0.5	0.8031	0.2902	0.9943	12.7603	1.265	0.4301	0.2702
		3	1	0.6818	0.2047	0.8162	13.7375	1.2009	0.4427	0.2983
			1.5	0.5261	0.1442	0.6839	14.151	1.184	0.4674	0.331
k=0.2										
			0.5	0.7322	1.2284	3.6411	8.8232	4.1012	1.6862	0.9417
		1	1	0.6334	1.0589	3.4189	9.0351	3.6892	1.501	0.8559
			1.5	0.5476	0.9232	3.1994	9.0972	3.3964	1.3856	0.807
			0.5	0.6334	1.0589	3.4189	9.0351	3.6892	1.501	0.8559
	3	2	1	0.4711	0.8087	2.9834	9.0666	3.1778	1.3075	0.777
			1.5	0.3424	0.626	2.5864	8.893	2.8801	1.2135	0.7475
			0.5	0.5476	0.9232	3.1994	9.0972	3.3964	1.3856	0.807
		3	1	0.3424	0.626	2.5864	8.893	2.8801	1.2135	0.7475
			1.5	0.2031	0.4339	2.1132	8.6138	2.6317	1.1533	0.742
			0.5	0.5331	0.8183	2.5632	6.4587	2.4134	0.9497	0.527
		1	1	0.4609	0.719	2.426	6.6661	2.2804	0.9002	0.5112
			1.5	0.3966	0.6329	2.277	6.7607	2.1759	0.8671	0.5034
			0.5	0.4609	0.719	2.426	6.6661	2.2804	0.9002	0.5112
6	6	2	1	0.3394	0.5582	2.1286	6.7951	2.0951	0.8451	0.5007
			1.5	0.2447	0.4367	1.8562	6.7937	1.9877	0.8237	0.5057
			0.5	0.3966	0.6329	2.277	6.7607	2.1759	0.8671	0.5034
		3	1	0.2447	0.4367	1.8562	6.7937	1.9877	0.8237	0.5057
			1.5	0.1445	0.3067	1.5309	6.7957	1.9201	0.8276	0.5296
			0.5	0.429	0.6392	2.1029	5.4957	1.7571	0.6732	0.3718
		1	1	0.3697	0.5629	1.9734	5.6098	1.6916	0.6552	0.3708
			1.5	0.3172	0.4966	1.8435	5.6656	1.641	0.6442	0.3729
			0.5	0.3697	0.5629	1.9734	5.6098	1.6916	0.6552	0.3708
	9	2	1	0.2708	0.439	1.7193	5.6919	1.6031	0.6386	0.3774
			1.5	0.1947	0.3451	1.4977	5.7165	1.5582	0.6392	0.3915
			0.5	0.3172	0.4966	1.8435	5.6656	1.641	0.6442	0.3729
		3	1	0.1947	0.3451	1.4977	5.7165	1.5582	0.6392	0.3915
			1.5	0.1149	0.2437	1.2365	5.7863	1.5495	0.6619	0.4224
			0.5	0.824	0.849	2.7392	10.1524	3.3433	1.2161	0.6885
		1	1	0.7678	0.738	2.5239	10.2375	3.0115	1.1092	0.6457
			1.5	0.7109	0.6493	2.3432	10.2686	2.7818	1.0417	0.6223
9	3		0.5	0.7678	0.738	2.5239	10.2375	3.0115	1.1092	0.6457
		2	1	0.6523	0.5743	2.1811	10.2366	2.6106	0.9958	0.6091
			1.5	0.5326	0.4531	1.9004	10.0543	2.3743	0.9408	0.5997
		3	0.5	0.7109	0.6493	2.3432	10.2686	2.7818	1.0417	0.6223

			1	0.5326	0.4531	1.9004	10.0543	2.3743	0.9408	0.5997
			1.5	0.3613	0.3216	1.5735	9.6727	2.169	0.9085	0.6085
	6	1	0.5	0.7078	0.5987	1.9755	7.8517	2.1036	0.7485	0.4236
			1	0.6538	0.5324	1.8556	8.0382	1.9917	0.7212	0.4196
			1.5	0.5986	0.4742	1.7398	8.1394	1.9047	0.7031	0.4196
		2	0.5	0.6538	0.5324	1.8556	8.0382	1.9917	0.7212	0.4196
			1	0.5428	0.4229	1.6305	8.1846	1.837	0.6915	0.4223
			1.5	0.4329	0.3375	1.4353	8.1858	1.744	0.6821	0.4336
		3	0.5	0.5986	0.4742	1.7398	8.1394	1.9047	0.7031	0.4196
			1	0.4329	0.3375	1.4353	8.1858	1.744	0.6821	0.4336
			1.5	0.2855	0.2426	1.203	8.1238	1.6768	0.6914	0.4605
	9	1	0.5	0.6341	0.4897	1.6617	7.0169	1.6124	0.565	0.319
			1	0.5811	0.4367	1.5569	7.1302	1.5536	0.5557	0.3226
			1.5	0.5278	0.3899	1.4577	7.1932	1.508	0.5508	0.328
		2	0.5	0.5811	0.4367	1.5569	7.1302	1.5536	0.5557	0.3226
			1	0.4748	0.3484	1.3652	7.2251	1.4732	0.5493	0.3347
			1.5	0.3733	0.2789	1.2021	7.2447	1.4293	0.5541	0.3513
		3	0.5	0.5278	0.3899	1.4577	7.1932	1.508	0.5508	0.328
			1	0.3733	0.2789	1.2021	7.2447	1.4293	0.5541	0.3513
			1.5	0.242	0.2012	1.0086	7.2654	1.4114	0.5764	0.3824
12	3	1	0.5	0.9432	0.6773	2.1154	10.651	2.683	0.9387	0.5576
			1	0.9244	0.5961	1.9453	10.6713	2.4428	0.874	0.5352
			1.5	0.903	0.5298	1.8068	10.68	2.2752	0.8332	0.5244
		2	0.5	0.9244	0.5961	1.9453	10.6713	2.4428	0.874	0.5352
			1	0.8783	0.4727	1.6856	10.6463	2.1494	0.8057	0.52
			1.5	0.8163	0.3785	1.4795	10.4726	1.9735	0.7742	0.5218
		3	0.5	0.903	0.5298	1.8068	10.68	2.2752	0.8332	0.5244
			1	0.8163	0.3785	1.4795	10.4726	1.9735	0.7742	0.5218
			1.5	0.6881	0.2729	1.2398	10.0654	1.8182	0.7601	0.5395
	6	1	0.5	0.9037	0.5006	1.5662	8.5075	1.7808	0.6155	0.3665
			1	0.8816	0.4498	1.4729	8.6793	1.6977	0.601	0.3686
			1.5	0.8561	0.4045	1.3853	8.7825	1.6326	0.5919	0.3729
		2	0.5	0.8816	0.4498	1.4729	8.6793	1.6977	0.601	0.3686
			1	0.8268	0.3638	1.3035	8.8354	1.5813	0.5869	0.3788
			1.5	0.7554	0.2945	1.1582	8.8463	1.5098	0.5858	0.3942
		3	0.5	0.8561	0.4045	1.3853	8.7825	1.6326	0.5919	0.3729
			1	0.7554	0.2945	1.1582	8.8463	1.5098	0.5858	0.3942
			1.5	0.617	0.2148	0.9835	8.7545	1.4561	0.5998	0.424
	9	1	0.5	0.8749	0.4235	1.3473	7.8193	1.4201	0.4857	0.2887
			1	0.8499	0.3814	1.2665	7.936	1.3744	0.4821	0.2951
			1.5	0.8213	0.3435	1.1908	8.0081	1.3387	0.4813	0.3025

		2	0.5	0.8499	0.3814	1.2665	7.936	1.3744	0.4821	0.2951
			1	0.7888	0.3094	1.1206	8.0489	1.3113	0.4826	0.3108
			1.5	0.7118	0.251	0.9963	8.076	1.276	0.4908	0.3293
		3	0.5	0.8213	0.3435	1.1908	8.0081	1.3387	0.4813	0.3025
			1	0.7118	0.251	0.9963	8.076	1.276	0.4908	0.3293
			1.5	0.5683	0.1834	0.8469	8.0695	1.26	0.5137	0.3615

Table (2) show the relative risk of the estimator $\tilde{R}_1(t)$ under SELF

k=0.1, α=0.01	λ								
m1	m2	p	0.25	0.50	0.75	1	1.25	1.5	1.75
6	3	0.5	0.4676	0.7173	2.7461	14.6062	3.3793	1.2013	0.6848
		1	0.2661	0.4325	2.0258	17.0697	2.582	0.9762	0.6082
		1.5	0.1521	0.2765	1.5176	17.0271	2.1826	0.8934	0.6021
	6	0.5	0.3442	0.5013	1.9637	11.8939	2.2649	0.7904	0.4495
		1	0.2015	0.3201	1.5044	14.0157	1.9245	0.7204	0.4478
		1.5	0.1183	0.2131	1.1688	14.7336	1.7497	0.7088	0.4763
	9	0.5	0.2751	0.3917	1.57	10.7075	1.7264	0.5947	0.3375
		1	0.1635	0.2572	1.2171	12.3168	1.5472	0.5756	0.3575
		1.5	0.0975	0.1749	0.9614	13.0929	1.4639	0.5901	0.3963
9	3	0.5	0.6654	0.5264	1.95	15.0592	2.5933	0.8957	0.5347
		1	0.4752	0.337	1.4702	17.1008	2.0653	0.7756	0.5089
		1.5	0.3178	0.2251	1.14	17.3204	1.7919	0.7347	0.523
	6	0.5	0.5612	0.39	1.4594	13.0846	1.8373	0.626	0.3733
		1	0.3956	0.2619	1.144	15.059	1.6047	0.5963	0.3902
		1.5	0.2637	0.1806	0.9136	15.8678	1.4825	0.6005	0.4256
	9	0.5	0.4928	0.3193	1.2107	12.3958	1.4642	0.4941	0.2941
		1	0.3434	0.2185	0.9592	14.0253	1.3372	0.4935	0.3222
		1.5	0.2279	0.1529	0.775	14.896	1.2779	0.5141	0.3636
12	3	0.5	0.8878	0.4431	1.5001	15.0577	2.0633	0.7177	0.456
		1	0.7911	0.296	1.1547	16.6547	1.7032	0.6511	0.4556
		1.5	0.6655	0.2038	0.9159	16.886	1.5124	0.6341	0.4819
	6	0.5	0.8408	0.3417	1.1583	13.4485	1.5195	0.5233	0.3325
		1	0.7362	0.2377	0.9266	15.1591	1.3604	0.5149	0.3596
		1.5	0.6078	0.1681	0.7545	15.8879	1.276	0.5285	0.4001
	9	0.5	0.8047	0.289	0.9873	13.0473	1.2511	0.4274	0.2711
		1	0.6941	0.2037	0.7969	14.5471	1.1627	0.4369	0.3044
		1.5	0.5643	0.1456	0.6546	15.3583	1.1225	0.4613	0.3482
k=0.2, α=0.01									
6	3	0.5	0.5208	0.8673	3.1146	9.3966	3.178	1.2765	0.7468
		1	0.3124	0.5507	2.3817	9.1298	2.4927	1.0413	0.657

		1.5	0.1872	0.3655	1.816	8.472	2.1548	0.9517	0.6426
		0.5	0.3901	0.6185	2.2638	6.9657	2.1237	0.8415	0.4915
	6	1	0.2391	0.4117	1.7809	7.011	1.8551	0.7701	0.4849
		1.5	0.1465	0.283	1.4023	6.8523	1.7262	0.7572	0.5097
		0.5	0.315	0.4895	1.8318	5.76	1.6159	0.6338	0.3695
	9	1	0.1952	0.3329	1.4489	5.8153	1.4902	0.6161	0.3876
		1.5	0.1212	0.2331	1.1564	5.7984	1.4437	0.6315	0.4249
		0.5	0.6988	0.6206	2.266	10.3632	2.6334	0.9839	0.5924
	3	1	0.5179	0.4153	1.7668	10.2349	2.1226	0.8464	0.5547
		1.5	0.3607	0.2878	1.3984	9.6709	1.8573	0.7946	0.561
		0.5	0.5967	0.4674	1.7241	8.2853	1.867	0.6896	0.4147
9	6	1	0.4356	0.3258	1.3866	8.4064	1.6512	0.6525	0.4263
		1.5	0.3015	0.2322	1.1255	8.2598	1.5395	0.6516	0.4576
		0.5	0.5282	0.3865	1.4475	7.2709	1.4885	0.545	0.3272
	9	1	0.3807	0.2735	1.1703	7.3575	1.3769	0.5408	0.3526
		1.5	0.2619	0.1974	0.9581	7.3209	1.3286	0.559	0.3916
		0.5	0.9001	0.5127	1.7537	10.6991	2.1779	0.7992	0.5075
	3	1	0.814	0.3562	1.3919	10.607	1.8064	0.7171	0.4977
		1.5	0.7005	0.2538	1.1279	10.1341	1.607	0.6901	0.5173
		0.5	0.8564	0.4008	1.3742	8.8961	1.6071	0.5844	0.3711
12	6	1	0.762	0.2885	1.1266	9.0579	1.446	0.5687	0.3937
		1.5	0.6442	0.2105	0.9336	8.9417	1.3597	0.577	0.4304
		0.5	0.8224	0.3418	1.1837	8.0766	1.3247	0.478	0.3029
	9	1	0.7218	0.2486	0.9752	8.1998	1.2377	0.4833	0.3337
		1.5	0.6013	0.183	0.8132	8.1715	1.1985	0.5046	0.3751
k=0.1,α=0.05									
		0.5	0.7398	0.6776	1.8087	5.2478	2.6235	1.1884	0.7603
	3	1	0.5383	0.4439	1.5064	6.0122	2.1331	0.9817	0.6799
		1.5	0.3624	0.2981	1.246	6.2635	1.8143	0.8907	0.6667
		0.5	0.6256	0.5071	1.4416	4.802	1.9088	0.8114	0.5125
6	6	1	0.4434	0.3419	1.2062	5.5578	1.6745	0.7418	0.5093
		1.5	0.295	0.2349	1.005	5.9978	1.5231	0.7218	0.5349
		0.5	0.5449	0.4113	1.2357	4.728	1.5297	0.6224	0.3896
	9	1	0.3795	0.2808	1.0257	5.3961	1.3952	0.6018	0.4107
		1.5	0.2505	0.1953	0.8539	5.8565	1.3173	0.6105	0.4499
		0.5	0.9375	0.5802	1.4045	5.2258	2.1656	0.9461	0.6417
	3	1	0.8704	0.4013	1.1607	5.7079	1.7792	0.8246	0.6118
9		1.5	0.7723	0.2808	0.9698	5.8508	1.5405	0.7748	0.6229
		0.5	0.9032	0.4551	1.1411	4.9035	1.6255	0.679	0.4584
	6	1	0.8249	0.3223	0.9588	5.4588	1.4372	0.6447	0.4763
		1.5	0.7175	0.2297	0.8092	5.7601	1.318	0.6418	0.5127

	9	0.5	0.8745	0.384	0.9984	4.9328	1.3409	0.5425	0.3644
		1	0.7875	0.2739	0.8365	5.4566	1.2304	0.539	0.3964
		1.5	0.6737	0.1966	0.7059	5.7963	1.1658	0.555	0.4414
12	3	0.5	0.9923	0.5587	1.1543	5.1312	1.8178	0.8001	0.5841
		1	0.9837	0.405	0.9572	5.4598	1.5247	0.7278	0.5824
		1.5	0.9688	0.2935	0.8066	5.5421	1.346	0.7035	0.6088
	6	0.5	0.988	0.4549	0.9521	4.8518	1.3971	0.5958	0.4359
		1	0.9772	0.3355	0.8066	5.2817	1.2543	0.5832	0.4672
		1.5	0.9591	0.2464	0.6879	5.4989	1.1642	0.5921	0.5121
	9	0.5	0.9843	0.3956	0.8466	4.9195	1.179	0.4907	0.3583
		1	0.9716	0.2925	0.7167	5.3414	1.094	0.4983	0.3984
		1.5	0.9506	0.2157	0.6113	5.5978	1.0444	0.5202	0.4488
k=0.2,α=0.05									
6	3	0.5	0.766	0.7595	1.9609	4.5933	2.5143	1.2299	0.8034
		1	0.5756	0.5214	1.6932	4.8993	2.071	1.0202	0.7145
		1.5	0.4021	0.3639	1.4383	4.8777	1.7864	0.9254	0.6951
	6	0.5	0.6552	0.5806	1.5964	3.9874	1.8187	0.8419	0.5433
		1	0.4792	0.4071	1.3764	4.2957	1.621	0.7726	0.5365
		1.5	0.3299	0.2892	1.1712	4.4163	1.4972	0.7519	0.559
	9	0.5	0.5755	0.4773	1.3917	3.7191	1.452	0.6465	0.4136
		1	0.4131	0.3372	1.1843	3.9583	1.3481	0.6276	0.4333
		1.5	0.2816	0.2417	1.0027	4.0944	1.2935	0.637	0.4709
9	3	0.5	0.9436	0.6382	1.5374	4.6828	2.1742	1.0051	0.687
		1	0.8833	0.4581	1.3115	4.8844	1.8006	0.8728	0.648
		1.5	0.795	0.3315	1.1242	4.8556	1.5681	0.8151	0.6522
	6	0.5	0.9116	0.5084	1.2726	4.227	1.6327	0.7239	0.4926
		1	0.8405	0.372	1.0985	4.4892	1.4559	0.6843	0.5057
		1.5	0.7424	0.2731	0.9472	4.5843	1.3437	0.677	0.5381
	9	0.5	0.8845	0.4333	1.1296	4.0875	1.3471	0.5793	0.3922
		1	0.805	0.3183	0.9687	4.3119	1.2473	0.5731	0.4215
		1.5	0.7002	0.2349	0.8325	4.4286	1.1898	0.5867	0.4638
12	3	0.5	0.993	0.6057	1.268	4.6478	1.8732	0.8596	0.6275
		1	0.9853	0.4532	1.0815	4.7805	1.5772	0.7765	0.6177
		1.5	0.9721	0.3384	0.9332	4.7432	1.3944	0.7442	0.6377
	6	0.5	0.989	0.4993	1.0626	4.2623	1.4432	0.6425	0.4701
		1	0.9792	0.3789	0.9225	4.482	1.3005	0.6239	0.4968
		1.5	0.9631	0.2859	0.8031	4.5554	1.2092	0.6279	0.5375
	9	0.5	0.9855	0.4376	0.9563	4.191	1.2198	0.53	0.387
		1	0.974	0.3322	0.8274	4.3923	1.1362	0.5339	0.4242
		1.5	0.9553	0.2512	0.7186	4.4892	1.0868	0.5526	0.4717

Table (3) show the relative risk of the estimator $\tilde{R}_2(t)$ under SELF

α=0.01	λ								
m1	m2	p	0.25	0.50	0.75	1	1.25	1.5	1.75
6	3	0.5	1.0398	1.4874	3.267	18.2537	3.542	1.5122	1.1456
		1	0.9374	1.1975	2.5564	25.7041	2.8556	1.3757	1.1161
		1.5	0.8271	0.9618	1.9315	29.0795	2.5414	1.3265	1.1071
	6	0.5	0.9424	1.1579	2.3909	15.9649	2.3798	1.0031	0.7667
		1	0.8356	0.9461	1.918	22.8238	2.1381	1.0289	0.8431
		1.5	0.722	0.7689	1.4931	27.9082	2.056	1.0755	0.9071
	9	0.5	0.8705	0.9734	1.9446	15.5088	1.8165	0.7577	0.5809
		1	0.7572	0.7956	1.5648	21.6784	1.7236	0.8283	0.6828
		1.5	0.6425	0.6487	1.2324	27.07	1.7292	0.907	0.7716
9	3	0.5	0.9535	1.2761	2.5678	17.8954	2.7745	1.3811	1.1859
		1	0.8994	1.107	2.0864	22.6515	2.3553	1.2843	1.148
		1.5	0.8366	0.9608	1.6769	24.7589	2.1463	1.237	1.1242
	6	0.5	0.898	1.0845	1.9854	16.4088	1.9716	0.9803	0.8559
		1	0.8347	0.9495	1.6529	21.1825	1.8407	1.0072	0.9115
		1.5	0.7626	0.8248	1.3558	24.4067	1.794	1.0388	0.9525
	9	0.5	0.8538	0.9774	1.6868	16.4512	1.574	0.7798	0.6847
		1	0.7828	0.8494	1.4052	20.9662	1.5393	0.8437	0.7688
		1.5	0.7048	0.7331	1.1586	24.5411	1.5566	0.9056	0.8367
12	3	0.5	0.9787	1.1714	2.1518	17.5211	2.342	1.3253	1.2111
		1	0.9599	1.0653	1.8038	20.8996	2.0531	1.2432	1.1616
		1.5	0.935	0.9661	1.5097	22.3433	1.8989	1.196	1.1298
	6	0.5	0.9595	1.0499	1.7261	16.333	1.7332	0.9875	0.922
		1	0.9347	0.9613	1.4822	19.945	1.6523	1.0074	0.9529
		1.5	0.9022	0.8695	1.2606	22.2144	1.621	1.027	0.9759
	9	0.5	0.9432	0.9866	1.5132	16.5658	1.4311	0.8156	0.7662
		1	0.913	0.8951	1.298	20.0859	1.4193	0.8682	0.8254
		1.5	0.8745	0.8017	1.1058	22.6666	1.4377	0.9156	0.8734
α=0.05									
6	3	0.5	0.9541	1.2169	2.0514	5.5249	2.6922	1.3565	1.057
		1	0.8837	1.0563	1.7987	6.5643	2.256	1.2283	1.0434
		1.5	0.7998	0.9023	1.5192	7.0387	1.985	1.1751	1.046
	6	0.5	0.8906	1.056	1.6916	5.1742	1.966	0.9355	0.728
		1	0.8071	0.9111	1.4748	6.2363	1.781	0.9417	0.8022
		1.5	0.7125	0.7694	1.2422	6.9645	1.683	0.9731	0.8668
	9	0.5	0.8378	0.9529	1.4903	5.2273	1.5794	0.7213	0.55
		1	0.7456	0.8099	1.2776	6.2347	1.4898	0.7708	0.6571
		1.5	0.6462	0.6755	1.0669	7.0292	1.4656	0.8348	0.7454

9	3	0.5	0.983	1.0998	1.7138	5.4436	2.2415	1.2552	1.0974
		1	0.9648	1.0028	1.5139	6.0729	1.9193	1.1744	1.0688
		1.5	0.9391	0.9046	1.3204	6.327	1.7219	1.135	1.0523
	6	0.5	0.9666	1.0004	1.455	5.1903	1.6887	0.9176	0.8134
		1	0.9419	0.909	1.2925	5.9142	1.5605	0.9371	0.8594
		1.5	0.9079	0.811	1.1265	6.359	1.4885	0.9639	0.8948
	9	0.5	0.952	0.9386	1.3179	5.312	1.3962	0.74	0.6574
		1	0.9219	0.8412	1.157	6.027	1.342	0.7934	0.7292
		1.5	0.8815	0.74	0.9999	6.5378	1.3266	0.8485	0.787
12	3	0.5	0.9975	1.0143	1.5022	5.3204	1.9534	1.2258	1.1033
		1	0.9951	0.9465	1.3452	5.7457	1.7109	1.1674	1.0676
		1.5	0.9914	0.878	1.2004	5.8978	1.5598	1.134	1.0462
	6	0.5	0.9953	0.9371	1.3018	5.0927	1.5105	0.9365	0.8615
		1	0.9919	0.8721	1.1783	5.6337	1.4192	0.9588	0.8871
		1.5	0.9867	0.8021	1.053	5.9393	1.3646	0.9802	0.9077
	9	0.5	0.9934	0.8883	1.2029	5.2305	1.2794	0.7806	0.7215
		1	0.989	0.8179	1.0789	5.7787	1.2451	0.8316	0.7711
		1.5	0.9825	0.7436	0.9565	6.1407	1.2349	0.8778	0.8122