ISSN 2347-1921

Bayes Estimators of the Scale Parameter of an Inverse Weibull Distribution under two different Loss Functions

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ABSTRACT

In this paper, we obtain Bayesian estimators of the scale parameter of the inverse Weibull distribution (IWD). We derive those estimators under two different loss functions: the quasi-squared error loss function and the nonlinear exponential loss function (NLINEX). Two priors are considered for finding the estimators: a class of natural–conjugate informative prior, namely; the exponential prior information and inverted-Levy prior information. Based on a Monte Carlo simulation study, the performance of those estimators is compared. The comparison criteria, the mean square errors (MSE) are computed and presented in tables. Comparison results show that MLE was the best followed by Bayes estimators based on the inverse Levy prior under NLINEX loss function which was preferable among the others.

Keywords:

Inverse Weibull distribution; MLE; Bayes' Estimators; exponential prior; inverse-Levy prior; quasi-quadratic loss functions; NLINEX loss function.

Council for Innovative Research

Peer Review Research Publishing System

Journal: Journal of Advances in Mathematics

Vol 8, No 2

editor@cirjam.org

www.cirjam.com, www.cirworld.com

INTRODUCTION

The inverse Weibull distribution (IWD) is a life time probability distribution which is widely used in reliability engineering and plays an important role in many applications. It can be used to model a variety of failure characteristics such as infant mortality, random failures, wear-out, and failure-free periods. The IWD can also be used to determine the cost effectiveness and maintenance periods of reliability-centered maintenance activities [3].

The inverse Weibull distribution may be used to analyze data coming from a distribution that have non-monotone hazard function and is uni-modal. The Bayes estimation for the IW parameters was discussed in [1, 4, 5, and 6]

The present paper describes the classical and the Bayes estimators of the scale parameter of inverse Weibull distribution based on two informative priors, under two different loss functions. The proposed estimators have been compared on the basis of the mean square of the estimates.

The estimators are derived in the following order: Maximum likelihood estimator, Bayes estimators with exponential prior and inverted Levy prior, under quasi-quadratic loss function, and the non- linear exponential loss function (NLINEX). Comparison was made through a Monte Carlo simulation study on the performance of these estimators.

MODEL DESCRIPTION AND MAXIMUM LIKELIHOOD ESTIMATOR

A random variable X is said to follow the two parameter IW distribution if its pdf is given by:

$$f(x;\alpha,\beta)=\alpha\beta x^{-(\beta+1)}e^{-\alpha x^{-\beta}} \qquad x\geq 0;\ \alpha,\beta>0 \tag{1}$$

where α and β are the scale and shape parameters respectively

The cumulative distribution function (cdf) in its simplest form is given by:

$$F(x;\alpha,\beta)=\begin{cases} e^{-\alpha x^{-\beta}}, & x\geq 0;\ \alpha,\beta>0 \\ 0, & otherwise \end{cases}$$

Let $X_1, X_2, \ldots, X_n$ be a random sample each of them has IW distribution having unknown scale parameter α. The likelihood function of the sample observations $x_1, x_2, \ldots, x_n$ is:

$$L=\alpha^n\beta^n\prod_{i=1}^{n}x_i^{-(\beta+1)}e^{-\alpha\left(\sum_{i=1}^{n}x_i^{-\beta}\right)}$$

The log likelihood function is:

$$\ln L=n\ln\alpha+n\ln\beta-(\beta+1)\sum_{i=1}^{n}\ln x_i-\alpha\sum_{i=1}^{n}x_i^{-\beta}$$

Differentiating the log likelihood with respect to α and then equating to zero we have

$$\frac{d\ln L}{d\alpha}=-\frac{n}{\alpha}-\sum_{i=1}^{n}x_i^{-\beta}=0$$

Hence, the MLE of α is:

$$\hat{\alpha}_{MLE}=-\frac{n}{\sum_{i=1}^{n}x_i^{-\beta}} \tag{2}$$

BAYES' ESTIMATORS

To obtain Bayes estimators, we assume that α is a real valued random variable with probability density function g(α). The posterior distribution of α is the conditional probability density function of α given the data. A loss function is used to represent a penalty associated with each estimate. The loss should be zero if and only if $\hat{\alpha}=\alpha$.

Prior and Posterior Distributions

Under the assumption that the shape parameter β is known, Bayes' estimators for the scale parameter α is considered with informative prior information. We consider two informative priors the exponential prior and the inverted Levy prior distributions.The parameters of the prior distribution are called hyper-parameters [9].

1. Posterior Distribution of the Scale Parameter Based on an Exponential Prior:

The exponential prior is assumed to be

$$g(\alpha) = \lambda e^{-\lambda\alpha} \qquad \lambda > 0 , \alpha > 0$$

where λ is the hyper-parameter.
The posterior distribution of the scale parameter α given the data (x_1, x_2… x_n) is given by:

$$h_1(\alpha|\boldsymbol{x}) = \frac{\prod_{i=1}^{n} f(x_i|\alpha)g(\alpha)}{\int_0^\infty \prod_{i=1}^{n} f(x_i|\alpha)g(\alpha)\,d\alpha}$$

$$= \frac{\alpha^n \lambda e^{-\alpha\left(\lambda+\sum_{i=1}^{n} x_i^{-\beta}\right)}}{\int_0^\infty \alpha^n \lambda e^{-\alpha\left(\lambda+\sum_{i=1}^{n} x_i^{-\beta}\right)} d\alpha}$$

$$= \frac{\left(\lambda + \sum_{i=1}^{n} x_i^{-\beta}\right)^{n+1} \alpha^n e^{-\alpha(\lambda+\sum_{i=1}^{n} x_i^{-\beta})}}{\Gamma(n+1)}$$

This posterior density is recognized as the density of the gama distribution. That is $\alpha \sim Gamma\left((n+1), \left(\lambda + \sum_{i=1}^{n} x_i^{-\beta}\right)\right)$.

And

$$E(\alpha) = \frac{n+1}{\lambda + \sum_{i=1}^{n} x_i^{-\beta}}$$

2. Posterior Distribution of the Scale Parameter α Based on an Inverted Levy Prior:

The inverted Levy prior is assumed to be [8]

$$g(\alpha) = \sqrt{\frac{\theta}{2\pi}}\, \alpha^{-1/2}\, e^{-\frac{\theta\alpha}{2}} \qquad \alpha > 0, \theta > 0$$

where θ is the hyper parameter.

The posterior distribution of the scale parameter α given the data (x_1, x_2… x_n) is given by

$$h_2(\alpha|\boldsymbol{x}) = \frac{\alpha^{n-\frac{1}{2}}\, e^{-\alpha\left(\frac{\theta}{2}+\sum_{i=1}^{n} x_i^{-\beta}\right)}}{\int_0^\infty \alpha^{n-\frac{1}{2}} e^{-\alpha\left(\frac{\theta}{2}+\sum_{i=1}^{n} x_i^{-\beta}\right)} d\alpha}$$

$$= \frac{\left(\frac{\theta}{2} + \sum_{i=1}^{n} x_i^{-\beta}\right)^{n+\frac{1}{2}} \alpha^{n-\frac{1}{2}}\, e^{-\alpha\left(\frac{\theta}{2}+\sum_{i=1}^{n} x_i^{-\beta}\right)}}{\Gamma\left(n+\frac{1}{2}\right)}$$

This posterior density is also recognized as the density of the gama distribution. That is, $\alpha \sim Gamma\left(\left(n+\frac{1}{2}\right), \left(\frac{\theta}{2} + \sum_{i=1}^{n} x_i^{-\beta}\right)\right)$.

with,

$$E(\alpha) = \frac{n + \frac{1}{2}}{\frac{\theta}{2} + \sum_{i=1}^{n} x_i^{-\beta}}$$

Loss Functions

The choice of loss functions is an essential part in the estimation problems. In the present work, we consider both symmetric as well as asymmetric loss functions. The first is the quasi-quadratic loss function which is classified as a symmetric function and associates equal importance to the losses [7]. The second is the non-linear exponential loss function proposed by Islam, A., F.M. Saiful *et al*., which is quite asymmetric in nature [2].

1.The Quasi-Quadratic Loss Function

By using the quasi-quadratic loss function:

$$L_1(\hat{\alpha}, \alpha) = \left(e^{-c\hat{\alpha}} - e^{-c\alpha}\right)^2$$

where c ≠ 0, is the scale parameter of the loss function. Bayes' estimator will be the estimator that minimizes the posterior risk given by

$$R_1(\hat{\alpha} - \alpha) = E[L_1(\hat{\alpha}, \alpha)] = \int_0^{\infty} \left(e^{-c\hat{\alpha}} - e^{-c\alpha}\right)^2 h(\alpha|\boldsymbol{x})\, d\alpha$$

which is minimized when

$$\hat{\alpha} = -\frac{1}{c}\left[lnE\left(e^{-c\alpha}|\boldsymbol{x}\right)\right]$$

Where $E\left(e^{-c\alpha}\right) = \int_0^{\infty} e^{-c\alpha} h(\alpha|\boldsymbol{x}) d\alpha$

Now based on exponential prior, we have:

$$E\left(e^{-c\alpha}\right) = \int_0^{\infty} e^{-c\alpha}\, h_1(\alpha|\boldsymbol{x})\, d\alpha$$

$$= \int_0^{\infty} e^{-c\alpha} \frac{\left(\lambda + \sum_{i=1}^{n} x_i^{-\beta}\right)^{n+1} \alpha^n e^{-\alpha(\lambda + \sum_{i=1}^{n} x_i^{-\beta})}}{\Gamma(n+1)}\, d\alpha$$

$$= \int_0^{\infty} \frac{\left(\lambda + \sum_{i=1}^{n} x_i^{-\beta}\right)^{n+1} \alpha^n e^{-\alpha(c+\lambda + \sum_{i=1}^{n} x_i^{-\beta})}}{\Gamma(n+1)}\, d\alpha$$

$$E\left(e^{-c\alpha}|\boldsymbol{x}\right) = \left[\frac{\lambda + \sum_{i=1}^{n} x_i^{-\beta}}{c + \lambda + \sum_{i=1}^{n} x_i^{-\beta}}\right]^{n+1}$$

Hence, Bayes estimator is:

$$\hat{\alpha}_1 = -\frac{1}{c}\, ln\left[\frac{\lambda + \sum_{i=1}^{n} x_i^{-\beta}}{c + \lambda + \sum_{i=1}^{n} x_i^{-\beta}}\right]^{n+1} \qquad (3)$$

And based on inverse Levy prior, we have:

$$E(e^{-c\alpha}) = \int_0^\infty e^{-c\alpha}\, h_2(\alpha|\boldsymbol{x})\, d\alpha$$

$$E(e^{-c\alpha}) = \int_0^\infty e^{-c\alpha} \frac{\left(\frac{\theta}{2} + \sum_{i=1}^n x_i^{-\beta}\right)^{n+\frac{1}{2}} \alpha^{n-\frac{1}{2}} e^{-\alpha(\frac{\theta}{2}+\sum_{i=1}^n x_i^{-\beta})}}{\Gamma\left(n+\frac{1}{2}\right)}\, d\alpha$$

$$= \int_0^\infty \frac{\left(\frac{\theta}{2} + \sum_{i=1}^n x_i^{-\beta}\right)^{n+\frac{1}{2}} \alpha^{n-\frac{1}{2}} e^{-\alpha(c+\frac{\theta}{2}+\sum_{i=1}^n x_i^{-\beta})}}{\Gamma\left(n+\frac{1}{2}\right)}\, d\alpha$$

$$E(e^{-c\alpha}|\boldsymbol{x}) = \left[\frac{\frac{\theta}{2} + \sum_{i=1}^n x_i^{-\beta}}{c + \frac{\theta}{2} + \sum_{i=1}^n x_i^{-\beta}}\right]^{n+\frac{1}{2}}$$

Hence, Bayes estimator is:

$$\hat{\alpha}_2 = -\frac{1}{c} ln\left[\frac{\frac{\theta}{2}+\sum_{i=1}^n x_i^{-\beta}}{c+\frac{\theta}{2}+\sum_{i=1}^n x_i^{-\beta}}\right]^{n+\frac{1}{2}} \qquad (4)$$

2. *The Non-Linear Exponential Loss Function: (NLINEX)*

It is given by

$$L_2(\hat{\alpha},\alpha) = k[e^{cD} - cD^2 - cD - 1], \qquad k > 0, c > 0$$

Where D represents the estimator error i.e., $D = \hat{\alpha} - \alpha$. Without loss of generality, we take k to be 1. And Bayes' estimator will be the estimator that minimizes the posterior risk given by

$$R_2(\hat{\alpha} - \alpha) = E[L_2(\hat{\alpha},\alpha)] = \int_0^\infty L_2(\hat{\alpha},\alpha)\, h(\alpha|\boldsymbol{x})\, d\alpha$$

which is minimized when

$$\hat{\alpha} = -[lnE(e^{-c\alpha}) - 2E(\alpha)]/(c+2)$$

Where

$$E(e^{-c\alpha}) = \int_0^\infty e^{-c\alpha} h(\alpha|\boldsymbol{x}) d\alpha$$

Now Bayes estimator of α based on exponential prior is given by:

$$\hat{\alpha}_3 = -\frac{1}{c+2}\left[ln\left(\frac{\lambda + \sum_{i=1}^n x_i^{-\beta}}{c + \lambda + \sum_{i=1}^n x_i^{-\beta}}\right)^{n+1} - 2\frac{n+1}{\lambda + \sum_{i=1}^n x_i^{-\beta}}\right] \qquad (5)$$

Also bayes estimator of α based on inverted Levy prior is given by

$$\hat{\alpha}_4 = -\frac{1}{c+2}\left[ln\left(\frac{\frac{\theta}{2} + \sum_{i=1}^n x_i^{-\beta}}{c + \frac{\theta}{2} + \sum_{i=1}^n x_i^{-\beta}}\right)^{n+\frac{1}{2}} - 2\frac{n+\frac{1}{2}}{\frac{\theta}{2} + \sum_{i=1}^n x_i^{-\beta}}\right] \qquad (6)$$

SIMULATION AND RESULTS

ISSN 2347-1921

In our simulation study, we generated samples of size n = 20, 50, and100 from IWD distribution with α =1.5, and 3. The values of the hyper parameters are chosen as λ = 2, 4, θ = 2, 4 and c = 1, 3. The process was repeated 3000 times and the expected values for the maximum likelihood estimates and Bayes estimates of the parameter α are obtained along with their mean square error (MSE), where

$$MSE\ (\hat{\alpha}) = \frac{\sum_{i=1}^{R}(\hat{\alpha}-\alpha)^2}{R}$$

The results are summarized and tabulated in the following tables for each estimator and for all sample sizes.

Table 1. Expected values of the parameter *α* and MSE with *α* = 1.5 and c = 1

n	*Criteria*	MLE	Quasi-quadratic				NLINEX			
			Exponential prior		Inverse Levy prior		Exponential prior		Inverse Levy prior	
			λ =2	λ =4	θ=2	θ=4	λ =2	λ =4	θ=2	θ=4
20	$\hat{\alpha}$	1.57505	1.37206	1.20889	1.43710	1.33939	1.40380	1.23324	1.47304	1.37037
	MSE	0.14378	0.08641	0.12648	0.09325	0.09254	0.08622	0.11642	0.09967	0.09014
50	$\hat{\alpha}$	1.53213	1.44955	1.37025	1.47818	1.43534	1.46368	1.38285	1.49305	1.44933
	MSE	0.05324	0.04254	0.04867	0.04468	0.04340	0.04293	0.04676	0.04609	0.04336
100	$\hat{\alpha}$	1.51528	1.47407	1.43189	1.48871	1.46677	1.48135	1.43875	1.49617	1.47401
	MSE	0.02368	0.02124	0.02294	0.02175	0.02147	0.02133	0.02241	0.02207	0.02145

Table 2. Expected values of the parameter *α* and MSE with *α* =1.5, and c = 3

n	*Criteria*	MLE	Quasi-quadratic				NLINEX			
			Exponential prior		Inverse Levy prior		Exponential prior		Inverse Levy prior	
			λ =2	λ =4	θ=2	θ=4	λ =2	λ =4	θ=2	θ=4
20	$\hat{\alpha}$	1.57505	1.28870	1.14383	1.34368	1.25802	1.34108	1.18446	1.40261	1.30915
	MSE	0.14378	0.09914	0.16035	0.09255	0.11048	0.08955	0.13821	0.09101	0.09769
50	$\hat{\alpha}$	1.53213	1.33446	1.33446	1.39570	1.39570	1.35634	1.35634	1.41995	1.41995
	MSE	0.05324	0.05604	0.05604	0.04592	0.04592	0.05123	0.05123	0.04400	0.04400
100	$\hat{\alpha}$	1.51528	1.45288	1.41189	1.46699	1.44569	1.46572	1.42401	1.48015	1.45846
	MSE	0.02368	0.02163	0.02506	0.02147	0.02217	0.02128	0.02368	0.02152	0.02164

Table 3. Expected values of the parameter *α* and MSE with *α* =3, and c = 1

n	*Criteria*	MLE	Quasi-quadratic				NLINEX			
			Exponential prior		Inverse Levy prior		Exponential prior		Inverse Levy prior	
			λ =2	λ =4	θ=2	θ=4	λ =2	λ =4	θ=2	θ=4
20	$\hat{\alpha}$	3.15010	2.35028	1.91244	2.59394	2.29433	2.44398	1.97343	2.71246	2.38579
	MSE	0.57511	0.57119	1.24762	0.39938	0.64002	0.48387	1.12739	0.36441	0.54375
50	$\hat{\alpha}$	3.06425	2.70408	2.44129	2.83029	2.67756	2.75356	2.48142	2.88521	2.72656
	MSE	0.21298	0.20826	0.39202	0.17697	0.22229	0.19061	0.35421	0.17337	0.20211
100	$\hat{\alpha}$	3.03056	2.84359	2.69080	2.91226	2.82951	2.87077	2.71510	2.94093	2.85656
	MSE	0.09473	0.09561	0.15258	0.08681	0.09951	0.09062	0.14024	0.08578	0.09377

Table 4. Expected values of the parameter *α* and MSE with *α* =3, and c = 3

n	*Criteria*	MLE	Quasi-quadratic				NLINEX			
			Exponential prior		Inverse Levy prior		Exponential prior		Inverse Levy prior	
			λ =2	λ =4	θ=2	θ=4	λ =2	λ =4	θ=2	θ=4
20	$\hat{\alpha}$	3.15010	2.12369	1.75890	2.31547	2.07313	2.27055	1.85691	2.49797	2.21649
	MSE	0.57511	0.86827	1.58713	0.61894	0.95473	0.66431	1.36502	0.45832	0.73988
50	$\hat{\alpha}$	3.06425	2.57041	2.33175	2.68285	2.54521	2.65357	2.39964	2.77478	2.62755
	MSE	0.21298	0.28323	0.51314	0.22036	0.30359	0.23239	0.43524	0.18822	0.24890
100	$\hat{\alpha}$	3.03056	2.76649	2.62165	2.83105	2.75279	2.81364	2.66390	2.88073	2.79971
	MSE	0.09473	0.11830	0.19452	0.09922	0.12425	0.10301	0.16776	0.09006	0.10772

DISCUSSION

It is observed from simulation results that that the classical MLE was superior over the Bayes estimators. And for Baysian estimation, there is apparently general underestimation particularly in the case of large hyper parameter values. However the use of inverted Levy prior can be preferred especially for small values of hyper parameters. Further it is observed that the asymmetric NLINEX loss function was better in performance than the quasi-quadratic loss function. Finally for all parameter values, an obvious reduction in MSE is observed with the increase in sample size.

ACKNOWLEDGMENTS

Our thanks to Dr. Zeana Z. Jamil who have contributed towards development of the template.

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