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First Order Uniform Solution for General Perturbed Harmonic Oscillator

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ABSTRACT

In this paper, first order uniform solutions with respect to small parameter ε are established analytically for general perturbed harmonic oscillator of the form $\ddot{U}+\omega_0^2 U=\varepsilon U^n \dot{U}^m$, $\varepsilon \ll 1$, n and m are nonnegative integers. Comparison between these analytical solutions and the numerical solutions of the differential equations is also given for different *n*, *m*, and ε, and showed excellent agreement. A result that confirming the validity of our analytical solutions.

Indexing terms/Keywords

Harmonic oscillator; perturbation theory; regularization; universal Solution.

Academic Discipline And Sub-Disciplines

Perturbation Theory; Celestial Mechanics; Dynamical Astronomy.

SUBJECT CLASSIFICATION

Celestial Mechanics

TYPE (METHOD/APPROACH)

Uniform Solution using the method of multiple scales.

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INTRODUCTION

Applications of the theory of non-linear oscillations may not only be found in classical mechanics but also in various branches of science, of these are for example, electronics, communication, biology and quantum mechanics [12, 13, and 14]. Moreover, new problems have raised new questions and on these the subject is still evolved. On the other hand, harmonic oscillators play important roles in both, theoretical astrophysics and space dynamics.

As for example, the disruption of the clusters due to galactic tidal force [8] is governed by a typical equation of a harmonic oscillator form. Many other astrophysical problems which are formulated in terms of harmonic oscillators may be found in references [1, 10].

In fact, the most important applications of harmonic oscillators are the regularized theories of space dynamics .The basic idea of these theories relied on transforming the equations of motion to a harmonic oscillator [11] form which is characterized by stable properties with respect to the numerical as well as the analytical integrations. On the contrary to, the usage of either the analytical or numerical techniques on the conventional equations of space dynamics yield inaccurate predictions of position and velocity .This is because that the conventional equations are nearly singular for the cases of close approach, which are of common occurrence in the mission and the re-entry problems of space travel.

Of these transformations for the perturbed two body problem of space dynamics are KS, Burdet and Euler parameters and was found to be efficient and accurate methods for obtaining numerical solutions to any type of perturbing force [2], [4], [5], [6], [7] and [9].

The above mentioned importance of harmonic oscillators is what motivated our present work to establish first order uniform analytical solutions of the general perturbed harmonic oscillator of the form

$$\ddot{U} + \omega_0^{\,2} U = \varepsilon U^n \dot{U}^m, \quad \varepsilon \ll 1, \tag{1.1}$$

for all possible nonnegative values of the integers *n* and *m* to suit many applications.

Note that a dot over a symbol denotes the derivative with respect to the time *t*.

FIRST ORDER UNIFORM SOLUTION

In this section, an analytical first order uniform solution of Equation (1.1) will be established for any possible nonnegative integer values of n and m . To do so we shall use the method of multiple scales [3] as follows.

Introduce the scales

$$T_0 = t, \; T_1 = \varepsilon t, \tag{2.1}$$

then using the chain rule, Equation (1.1) to the first order could be written as

$$\frac{\partial^2 U}{\partial T_0^{\,2}} + 2\varepsilon \frac{\partial^2 U}{\partial T_0 \partial T_1} + \omega_0^{\,2} U = \varepsilon U^n \left(\frac{\partial U}{\partial T_0} \right)^m . \tag{2.2}$$

Let the uniform expansion for U of the form

$$U = U_0(T_0, T_1) + \varepsilon U_1(T_0, T_1), \tag{2.3}$$

be substituted in Equation (2.2) we get by equating coefficients of like powers of ε

$$\frac{\partial^2 U_0}{\partial T_0^{\,2}} + \omega_0^{\,2} U_0 = 0, \tag{2.4}$$

$$\frac{\partial^2 U_1}{\partial T_0^{\,2}} + \omega_0^{\,2} U_1 = -2 \frac{\partial^2 U_0}{\partial T_0 \partial T_1} + U_0^{\,n} \left(\frac{\partial U_0}{\partial T_0} \right)^m . \tag{2.5}$$

The solution of Equation (2.4) is expressed as

$$U_0 = a \cos\theta, \tag{2.6}$$

where

$$a = a(T_1), \tag{2.7.1}$$

$$\theta = \omega_0 T_0 + \beta(T_1). \tag{2.7.2}$$

Using Equation (2.6) into Equation (2.5) we get

$$\frac{\partial^2 U_1}{\partial T_0{}^2} + \omega_0{}^2 U_1 = 2a\omega_0 \frac{\partial \beta}{\partial T_1}\cos\theta + 2\omega_0 \frac{\partial a}{\partial T_1}\sin\theta + $$
$$+(-1)^m a^{m+n}\omega_0{}^m \sin^m\theta\cos^n\theta. \qquad (2.8)$$

The products $\sin^m\theta\cos^n\theta$ for the possible values of the nonnegative integers n and m are of the forms

$$\sin^m\theta\cos^n\theta = \frac{1}{2}\alpha_0{}^{(m,n)} + \sum_{\ell=1}^{s_1}\alpha_\ell{}^{(m,n)}\cos 2\ell\theta; m \equiv \text{even}; n \equiv \text{even}, \qquad (2.9.1)$$

$$\sin^m\theta\cos^n\theta = \sum_{\ell=1}^{s_1}\gamma_\ell{}^{(m,n)}\sin 2\ell\theta; m \equiv \text{odd}; n \equiv \text{odd}, \qquad (2.9.2)$$

$$\sin^m\theta\cos^n\theta = \sum_{\ell=1}^{s_2}\lambda_\ell{}^{(m,n)}\cos(2s-1)\theta; m \equiv \text{even}; n \equiv \text{odd}, \qquad (2.9.3)$$

$$\sin^m\theta\cos^n\theta = \sum_{\ell=1}^{s_2}\eta_\ell{}^{(m,n)}\sin(2s-1)\theta; m \equiv \text{odd}; n \equiv \text{even}, \qquad (2.9.4)$$

where

$$s_1 = (m+n)/2, \qquad (2.10.1)$$

$$s_2 = (m+n+1)/2 \qquad (2.10.2)$$

and the case of an even integer includes also its zero value.

Consequently, the solution of Equation (2.8) is uniform that is, free from mixed secular terms, according to the conditions listed in Table I.

Table I. Uniformity conditions for the solution of Equation (2.8)

m and *n*	Conditions on *a* and β
1- *m*≡even; n≡even	$a = a_0; \quad \beta = \beta_0$
2- *m*≡odd; n≡odd	$a = a_0; \quad \beta = \beta_0$
3- *m*≡even; n≡odd	$a = a_0,$ $\frac{d\beta}{dT_1} = -\frac{1}{2}a^{m+n-1}\omega_0{}^{m-1}\lambda_1{}^{(m,n)}$
4- *m*≡odd; n≡even	$\beta = \beta_0,$ $\frac{da}{dT_1} = \frac{1}{2}a^{m+n}\omega_0{}^{m-1}\eta_1{}^{(m,n)}$

The third and the fourth conditions yield

$$\beta = -\frac{1}{2}a_0{}^{m+n-1}\omega_0{}^{m-1}\lambda_1{}^{(m,n)}T_1 + \beta_0,$$

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$$a = \frac{a_0}{\left[1 - \frac{1}{2}(m+n-1)a_0^{\;m+n-1}\omega_0^{\;m-1}\eta_1^{(m,n)}T_1\right]^{1/(m+n-1)}},$$

where a_0 and β_0 are constants.

Thus the uniform solution of Equation (1.1) is

$$U = a_0\cos(\omega_0 t+\beta_0)+\varepsilon\left\{\frac{1}{2}A_0^{(m,n)}+\sum_{\ell=1}^{s_1}A_\ell^{(m,n)}\cos 2\ell(\omega_0 t+\beta_0)\right\}, m\equiv \text{even}; n\equiv\text{even}, \qquad (2.11.1)$$

$$U = a_0\cos(\omega_0 t+\beta_0)+\varepsilon\sum_{\ell=1}^{s_1}B_\ell^{(m,n)}\sin 2\ell(\omega_0 t+\beta_0), m\equiv\text{odd}; n\equiv\text{odd}, \qquad (2.11.2)$$

$$U = a_0\cos(\omega_0 t+\varepsilon\omega_1 t+\beta_0), m\equiv\text{even}; n\equiv\text{odd} \qquad (2.11.3)$$

$$U = a_0\cos(\omega_0 t+\beta_0)\left[1-(m+n-1)\varepsilon\omega_2 t\right]^{-1/(m+n-1)}, m\equiv\text{odd}; n\equiv\text{even} \qquad (2.11.4)$$

where

$$A_\ell^{(m,n)} = \frac{a^{n+m}\omega_0^{\;m-2}}{(1-4\ell^2)}\alpha_\ell^{(m,n)} \quad \forall\ell = 0,1,2,\dots,\ s_1$$

$$B_\ell^{(m,n)} = -\frac{a^{n+m}\omega_0^{\;m-2}}{(1-4\ell^2)}\gamma_\ell^{(m,n)} \quad \forall\ell = 1,2,\dots,\ s_1$$

$$\omega_1 = -\frac{1}{2}a_0^{\;m+n-1}\omega_0^{\;m-1}\lambda_1^{(m,n)}$$

$$\omega_2 = \frac{1}{2}a_0^{\;m+n-1}\omega_0^{\;m-1}\eta_1^{(m,n)}.$$

It remains for us, to find the explicit forms of the *A*'s, B's, ω_1 and ω_2 coefficients as follows.

Since

$$\sin^m\theta\cos^n\theta = \frac{(-J)^m}{2^{n+m}}\left(\Phi-\Phi^{-1}\right)^m\left(\Phi+\Phi^{-1}\right)^n, \qquad (2.12)$$

where

$$\Phi = \exp(J\theta), \quad J = \sqrt{-1}.$$

Using the binomial theorem we get

$$\sin^m\theta\cos^n\theta = \frac{(-J)^m}{2^{n+m}}\sum_{c=0}^{n+m}Q_c^{(m,n)}\Phi^{m+n-2c}, \qquad (2.13)$$

where

$$Q_c^{(m,n)} = \sum_{\ell=\ell_1}^{\ell_2}(-1)^\ell\binom{m}{\ell}\binom{n}{c-\ell}, \qquad (2.14)$$

$$\ell_1 = \max(0, c-n); \quad \ell_2 = \min(c,m),$$

where, for n_1 and n_2 nonnegative integers

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$$\max(n_1,n_2)=\frac{1}{2}\left\{n_1+n_2+|n_1-n_2|\ \right\},$$

$$\min(n_1,n_2)=\frac{1}{2}\left\{n_1+n_2-|n_1-n_2|\ \right\}.$$

From Equation (2.14) we deduce that

$$\sum_{c=0}^{m+n} Q_c^{(m,n)}=0, \tag{2.15}$$

$$Q_{s_1}^{(m,n)}=0,\ \text{if } m \text{ and } n \text{ are positive odd integers,} \tag{2.16.1}$$

$$Q_{2i-1}^{(m,n)}=0;\quad i=1,2,\ldots,m \tag{2.16.2}$$

$$Q_{2j}^{(m,n)}=(-1)^j\binom{m}{j};\ \ j=0,1,2,\ldots,[m/2], \tag{2.16.3}$$

and

$$Q_{2k}^{(m,m)}=(-1)^m Q_{2m-2k}^{(m,m)};\ \ k=[m/2]+1,[m/2]+2,\ldots,m, \tag{2.16.4}$$

where [ℓ] denotes the largest integer $\le \ell$; $\ell \ge 0$, and s_1 is given from Equation (2.10.1).

From Equations (2.12), (2.13) and (2.14) we have

$$\sin^m\theta\cos^n\theta=\frac{(-J)^m}{2^{n+m}}\sum_{c=0}^{n+m}\sum_{\ell=\ell_1}^{\ell_2}(-1)^{\ell}\binom{m}{\ell}\binom{n}{c-\ell}\times \\ \times\left\{\cos(m+n-2c)\theta+J\sin(m+n-2c)\theta\right\}. \tag{2.17}$$

Now, the α's, γ's, η's and η's coefficients of Equations (2.9) could be written in unified form as:

$$\sin^{n}\phi\cos^{m}\phi=\begin{cases}\frac{1}{2}g_0^{(n,m)}+\sum\limits_{\ell=1}^{u}g_{\ell}^{(n,m)}\cos 2\ell\phi & : n\equiv \text{even}\ ;\ m\equiv\text{even}\\ \sum\limits_{\ell=1}^{u}g_{\ell}^{(n,m)}\cos(2\ell-1)\phi & : n\equiv\text{even}\ ;\ m\equiv\text{odd}\\ \sum\limits_{\ell=1}^{u}g_{\ell}^{(n,m)}\sin(2\ell-1)\phi & : n\equiv\text{odd}\ ;\ m\equiv\text{even}\\ \sum\limits_{\ell=1}^{u}g_{\ell}^{(n,m)}\sin 2\ell\phi & : n\equiv\text{odd}\ ;\ m\equiv\text{odd}\end{cases}$$

where

$$u=\frac{n+m+\delta}{2}\quad ;\quad \delta=\left(\frac{1-(-1)^{n+m}}{2}\right)$$

$$g_{\ell}^{(n,m)}=(-1)^{\frac{n+\varepsilon}{3}}\,2^{-n-m+1}\sum_{j=q_1}^{q_2}(-1)^j\binom{n}{j}\binom{m}{\frac{m-n-\delta}{2}+\ell+j};$$

$$q_2 = \mathrm{Min}(u-\ell, n) \quad q_1 = \mathrm{Max}\left(0, \frac{n-m+\delta}{2} - \ell\right) \quad ; \quad \varepsilon = \begin{cases} 0 & n \equiv \text{even} \\ 3 & n \equiv \text{odd} \end{cases}$$

By making $(J)^q$ equal to zero when q odd, that is by writing

$$(J)^q = \frac{1}{2}(-1)^{[q/2]}\left\{1+(-1)^q\right\},$$

we deduce by means of Equations (2.16) and (2.17) for the required coefficients the expressions

$$A_\ell^{(m,n)} = (-1)^{\frac{m}{2}} H_\ell^{(m,n)};\ m \equiv \text{even(or zero)}; n \equiv \text{even(or zero)}; \ell = 0,1,2,\ldots,s_1, \tag{2.18}$$

$$B_\ell^{(m,n)} = (-1)^{\frac{m+1}{2}} H_\ell^{(m,n)};\ m \equiv \text{odd}; n \equiv \text{odd}; \ell = 1,2,\ldots,s_1, \tag{2.19}$$

$$\omega_1 = (-1)^{\frac{m}{2}} R^{(m,n)};\ m \equiv \text{even(or zero)}; n \equiv \text{odd}, \tag{2.20}$$

$$\omega_2 = (-1)^{\frac{m+1}{2}} R^{(m,n)};\ m \equiv \text{odd}; n \equiv \text{even(or zero)}, \tag{2.21}$$

where

$$H_\ell^{(m,n)} = 2\left(\frac{a_0}{2}\right)^{n+m} \frac{\omega_0^{m-2}}{1-4\ell^2} \sum_{j=j_1}^{j_2} (-1)^j \binom{m}{j} \binom{n}{\frac{n-m}{2}+\ell+j}, \tag{2.22}$$

$$R^{(m,n)} = -\frac{1}{2}\left(\frac{a_0}{2}\right)^{n+m-1} \omega_0^{m-1} \sum_{j=j_3}^{j_4} (-1)^j \binom{m}{j} \binom{n}{\frac{n-m+1}{2}+j}, \tag{2.23}$$

$$j_1 = \max\left(0, \frac{m-n}{2} - \ell\right);\quad j_2 = \min\left(m, \frac{m+n}{2} - \ell\right), \tag{2.24}$$

$$j_3 = \max\left(0, \frac{m-n-1}{2}\right);\quad j_4 = \min\left(m, \frac{m+n-1}{2}\right). \tag{2.25}$$

Equations (2.11) and Equations (2.18) to (2.25) are what we required to establish for the first order uniform solution of the general perturbed harmonic oscillator of Equation (1.1) for all possible nonnegative integer values of m and n.

NUMERICAL APPLICATIONS

In this section, numerical examples have been done to check the solutions of the present analytical uniform solution for general perturbed harmonic oscillator. Table II shows comparisons between numerical and analytical solutions ΔU and the first derivatives $\Delta U'$ for different values of the small parameter ε. The case of even values of m and n is considered in Table II, while the case of odd values of m and n is considered in Table III. Initial conditions are: $\omega_0 = 3,\ \beta_0 = 0,$ and $a_0 = 1.$

Table II. Comparison of Numerical and Analytical Solutions with m=14 and n=16

Time	ε	ΔU	$\Delta U'$
4	10^{-2}	-0.06422859	0.07962767
7	10^{-3}	0.00397473	0.00459682
10	10^{-4}	-0.00010846	-0.00104540
12	10^{-5}	4.33762E-06	-0.00011050
15	10^{-6}	-2.27777E-06	1.18447E-05
18	10^{-7}	2.43057E-07	-2.2976E-06
20	10^{-8}	-2.78831E-07	-1.22597E-06
25	10^{-10}	-9.06716E-10	2.64509E-06

Table III. Comparison of Numerical and Analytical Solutions with m=11 and n=13

Time	ε	ΔU	$\Delta U'$
4	10^{-2}	0.01189570	-0.05575550
7	10^{-3}	-0.00121418	0.0023505
10	10^{-4}	0.00012993	-6.1964E-05
12	10^{-5}	1.2771E-05	2.7458E-06
15	10^{-6}	-1.0084E-06	-5.6401E-07
18	10^{-7}	-3.5999E-07	-6.6772E-07
20	10^{-8}	-5.2957E-07	-1.3440E-06
25	10^{-10}	8.3092E-08	2.2661E-06

Tables IV and V include results for the other two cases m-even & n-odd and m-odd & n-even respectively. The accuracy reached to the order of 10^{-10} by decreasing the value of the small parameter ε to 10^{-10}. Figures 1, 2 and 3 confirms this accuracy for arbitrary values of time.

Table IV. Comparison of Numerical and Analytical Solutions with m=8 and n=11

Time	ε	ΔU	$\Delta U'$
4	10^{-2}	-0.01096716	0.04726696
7	10^{-3}	0.00327659	-0.00541146
10	10^{-4}	-0.00057251	0.00017741
12	10^{-5}	-7.0291E-05	-3.6924E-05
15	10^{-6}	8.2221E-06	1.5880E-05
18	10^{-7}	-1.0440E-06	-3.2993E-06
20	10^{-8}	-4.9368E-07	-1.7586E-06
25	10^{-10}	7.4752E-08	2.4530E-06

Table V. Comparison of Numerical and Analytical Solutions with m=7 and n=8

Time	ε	ΔU	$\Delta U'$
4	10^{-2}	0.00101806	-0.00558697
7	10^{-3}	-0.00028733	0.00144005
10	10^{-4}	4.2329E-05	-4.6295E-05
12	10^{-5}	3.8123E-06	2.8918E-06
15	10^{-6}	1.8722E-07	-1.5032E-06
18	10^{-7}	-5.1164E-07	-4.3860E-07
20	10^{-8}	-5.2929E-07	-1.3404E-06
25	10^{-10}	5.6180E-08	2.5825E-06

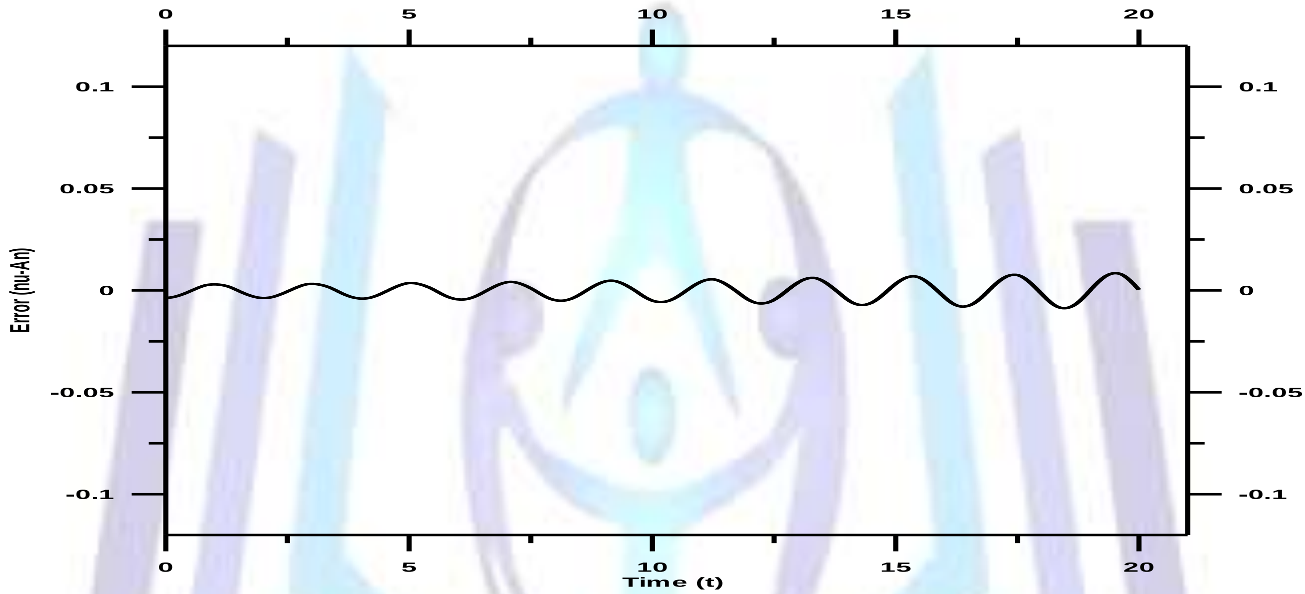

Fig. 1: Difference between numerical and analytical solutions of the general perturbed harmonic oscillator. The error is of order 10^{-3}, for small parameter $\varepsilon = 10^{-3}$.

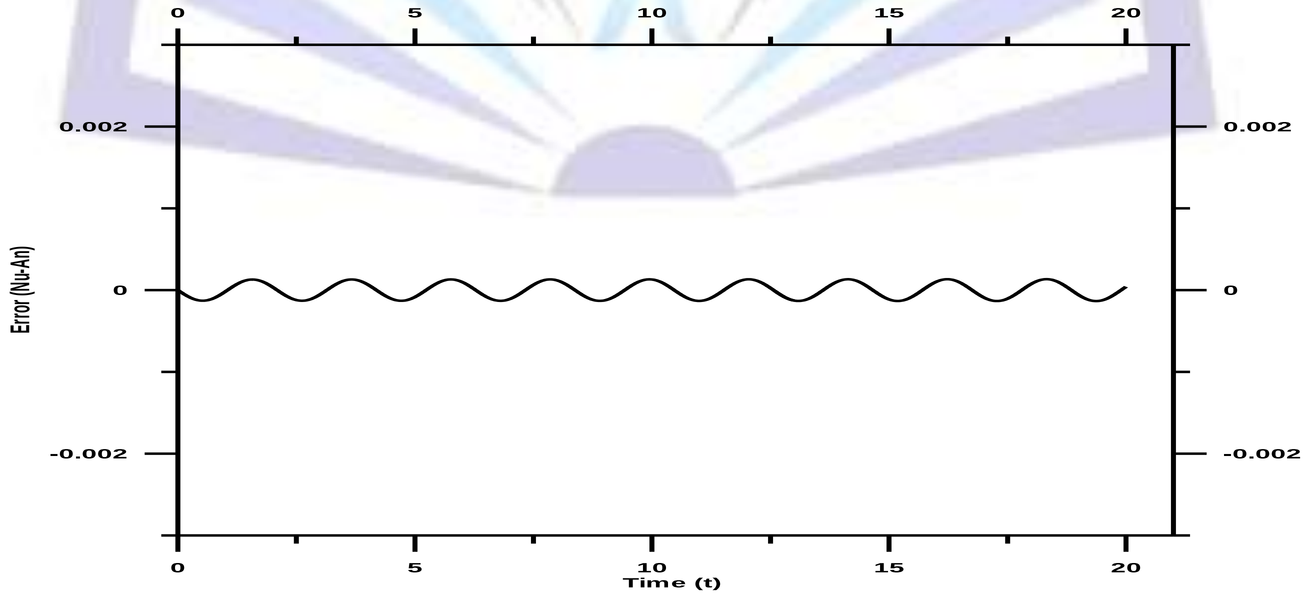

Fig. 2: Difference between numerical and analytical solutions of the general perturbed harmonic oscillator. The error is of order 10^{-4}, for small parameter $\varepsilon = 10^{-4}$.

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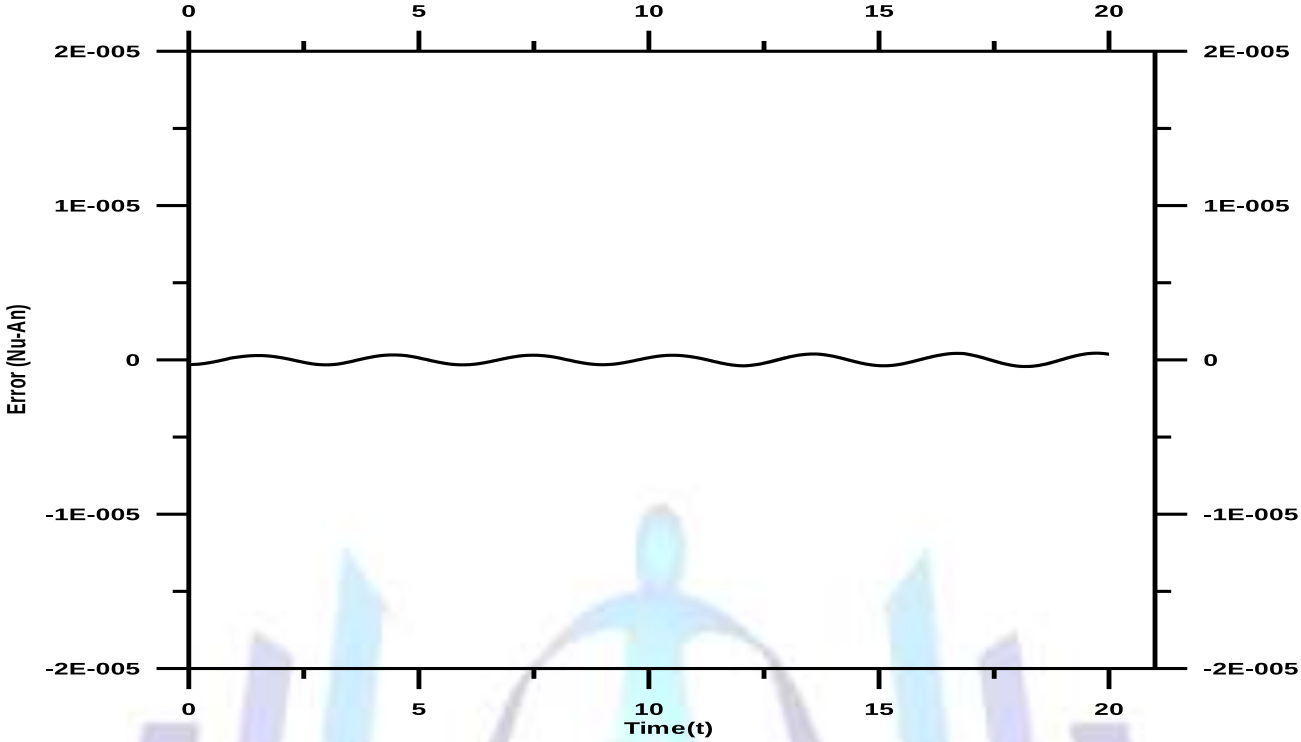

Fig. 3: Difference between numerical and analytical solutions of the general perturbed harmonic oscillator. The error is of order 10^{-7}, for small parameter $\varepsilon = 10^{-5}$.

CONCLUSION

In this paper, first order uniform solutions with respect to small parameter ε are established analytically for the general perturbed harmonic oscillator of the form $\ddot{U} + \omega_0^2 U = \varepsilon U^n \dot{U}^m$, $\varepsilon \ll 1$, n and m are nonnegative integers. The analytic expressions for the solutions are general and suit many applications. Comparison between these analytical solutions and the numerical solutions of the differential equations is also given for different *n*, *m*, and ε, and showed excellent agreement. A result that confirming the validity of our analytical solutions.

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