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Numerical Solutions of Volterra Integral Equation of Second kind Using Implicit Trapezoidal

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ABSTRACT

In this paper, we will be find numerical solution of Volterra Integral Equation of Second kind through using Implicit trapezoidal and that by using Maple 17 program, then we found that numerical solution was highly accurate when it was compared with exact solution.

Keywords

Volterra integral equation of second kind; Implicit trapezoidal Method; Nonlinear programing.

Academic Discipline And Sub-Disciplines

Mathematics, Integral Equation and Elasticity.

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INTRODUCTION

In this paper, we consider the Volterra integral equation of the second kind

$$x(t) = f(t) + \int_0^t k\big(t,s,x(s)\big)\,ds, \tag{1}$$

Where x, f and k are vector-valued functions with m components. If f and k are continuous and $k(t,s,x(s))$ satisfies a Lipcshitz condition with respect to x, then a unique solution $x(t)$ of (1) exists[1,4,7].
Volterra integral equations have been found to be effective to describe some application such as potential theory and Dirichlet problemsand electrostatics. Also, Volterra integral equations areapplied in the biology, chemistry, engineering, mathematical problems of radiation equilibrium,theparticle transport problems of astrophysics and reactor theory, and radiation heat transfer problems [1,10].
In this paper, we present the computation of numerical solution of Volterra integral equation of the second kind.

PRELIMINARIES

In this section, we recall the main theorems [7].
Theorem 1. Consider the equation

$$x(t) = f(t) + \int_0^t p(t,s)k(t,s)x(s)\,ds \tag{2}$$

Where

1) $f(t)$is continuous in $0 \le t \le T$.
2) $k(t,s)$ is a continuous function in $0 \le s \le t \le T$,
3) for each continuous function $h$andall $0 \le \tau_1 \le \tau_2 \le t$the integra**ls**

$$\int_{\tau_1}^{\tau_2} p(t,s)k(t,s)h(s)\,ds$$

$$\int_0^t p(t,s)k(t,s)h(s)\,ds$$

are continuous functions of t ,
4) $p(t,s)$ is absolutely integrable with respect to s for all $0 \le t \le T$,
5) there exist points$0 = T_0 < T_1 < T_2 < \cdots < T_N = T$ such that with $t \ge T_1$

$$k \int_{T_i}^{\min(t,T_{i+1})} |p(t,s)|\,ds \le \alpha < 1,$$

Were $k = \max_{0 \le s \le t \le T} |k(t,s)|$,

6) for every $t \ge 0$ such that with $t \ge T_1$

$$\lim_{\delta \to 0^+} \int_t^{t+\delta} |p(t+\delta,s)|\,ds = 0.$$

Then (2) has a unique continuous solution in $0 \le t \le T$**.**

Theorem 2. Consider the equation

$$x(t) = f(t) + \int_0^t p(t,s)k(t,s,x(s))\,ds \tag{3}$$

Where

1) $f(t)$is continuous in $0 \le t \le T$.
2) $k(t,s,u)$ is a continuous function in $0 \le s \le t \le T$, $-\infty < u < \infty$,
3) the Lipschitz condition $|k(t,s,y)$ - $k(t,s,z)| \le L|y - z|$ is satisfies for $0 \le s \le t \le T$ and all y and z,
4) $p(t,s)$ satisfies conditions (3)-(4) of Theroem 1 with k replaced by L and $k(t,s,h(s))$ instead of $k(t,s)h(s)$.
Then (3) has a unique continuous solution in $0 \le t \le T$**.**

THE MATHEMATICS OF THE VOLTERRA PROCEDURE

In this section, we use the technique of the Volterraeqution [2,7] to find an approximates the solution $x(t)$ of (1) at the equally spaced points $t_n = t_0 + nh$ for $n = 1, \cdots, N$ where $t_0 = 0$ and $N$is the total number of steps of size $h. X_n$ denotes the approximation of $x(t)$ at $t = t_n$.

Setting $t = t_n$ in (1), we have

$$x(t_n) = f(t_n) + \int_0^{t_n} k(t_n, t, x(t))\, dt \qquad (4)$$

By the composite trapezoidal rule an approximation of the integral in (4) is

$$\frac{h}{2}\left[k\big(t_n, t_0, x(t_0)\big) + 2\sum_{j=1}^{n-1} k(t_n, t_j, x\big(t_j\big)) + k(t_n, t_n, x(t_n))\right] \qquad (5)$$

Replacing $x(t_n)$ in (4) and (5) by X_n, we obtain the implicit trapezoidal rule

$$X_n = f(t_n) + h\left[\frac{1}{2}k(t_n, t_0, X_0) + \sum_{j=1}^{n-1} k(t_n, t_j, X_j) + \frac{1}{2}k(t_n, t_n, X_n)\right] \qquad (6)$$

Where $X_0 = f(0)$since $x(0) = f(0)$.
Defining σ_n by

$$\sigma_n = f(t_n) + h\left[\frac{1}{2}k(t_n, t_0, X_0) + \sum_{j=1}^{n-1} k(t_n, t_j, X_j)\right] \qquad (7)$$

We can rewrite (6) as

$$X_n - \frac{1}{2}hk(t_n, t_n, X_n) - \sigma_n = 0, \qquad (8)$$

Where 0denotes the zero vector. From (8), we see that X_n is the solution of the vector equation

$$\phi(u) = 0, \qquad (9)$$

Where ϕ is the vector-valued function

$$\phi(u) = u - \frac{1}{2}hk(t_n, t_n, u) - \sigma_n \qquad (10)$$

We will obtain an approximation to the solution $X_n$of (9) by way of the matrix-valued function Gdefined in (11). If $A(u)$ is an m by m matrix-valued function that is invertible in a neighborhood of X_n, then X_n is a fixed point of

$$G(u) = u - A(u)\phi(u). \qquad (11)$$

Assuming the components of $G(u)$ have continuous first and second order partial derivatives and that the first order partial derivatives and that the first order partial derivatives at
$X_n$are equal to zero, it can be shown that if $A(u)$ is set equal to the Jacobian matrix of the function ϕ, the iterates $X_n^{(p)}$ defined by (13) below will usually converge quadratically to X_n provided the starting value is sufficiently close to X_n. The Jacobian matrix of ϕ is the m by m matrix $J(u)$ with the element

$$J(u)_{ij} = \frac{\partial}{\partial u_j}\phi_i(u) = \delta_{ij} - \frac{1}{2}h\frac{\partial}{\partial u_j}k_i(t_n, t_n, u) \qquad (12)$$

In row i and column, where δ_{ij} is the Kronecker delta. Details of the statements made here follow from the discussion of Newton's method for nonlinear systems in [2]. Linz gives a brief outline of the trapezoidal rule and Newton's method for Volterra integral systems of the second kind in Section of [7].
We obtain $X_n$from $X_{n-1}$by setting $X_n^{(0)} = X_{n-1}$and then generating the iterates $X_n^{(p)}$ from

$$X_n^{(p)} = G\left(X_n^{(p-1)}\right) = X_n^{(p-1)} - J^{-1}\left(X_n^{(p-1)}\right)\phi\left(X_n^{(p-1)}\right) \qquad (13)$$

For $p = 1,2,3,\cdots$. (This is Newton's method for nonlinear systems.) Let y denote the solution of the matrix equation

$J\left(X_n^{(p-1)}\right)y = \phi\left(X_n^{(p-1)}\right)$. (14)
Then the iteration formula (13) becomes

$$X_n^{(p)} = X_n^{(p-1)} - y. \qquad (15)$$

We compute the solution $y = J^{-1}\left(X_n^{(p-1)}\right)\phi\left(X_n^{(p-1)}\right)$ using the command Linear Solve. The iterates$X_n^{(p)}$ are computed until the infinity norm of the vector y is less than a prescribed tolerance Tol. Then $X_n$is assigned the value of the last iterate [2,7].

NUMERICAL EXAMPLES

In this section, we solve some examples, and we can compare the numerical results with the exact solution.

Example1. Consider the Volterra integral equation of second kind

$$\mathrm{X}(t) = 6t - t^3 + \int_0^t (t-s)\,\mathrm{X}_1(s)ds\,,$$

with the exact solution $\mathrm{X}(t) = 6t$.

Table.1 Numerical results and exact solution of Volterra integral equation for example 1.

t	$X_1(t)$		E
0.0	0.00000	0.00000	0.00000
0.1	0.59900	0.60000	0.00100
0.2	1.19799	1.20000	0.00201
0.3	1.79696	1.80000	0.00304
0.4	2.39590	2.40000	0.00410
0.5	2.99480	3.00000	0.00520
0.6	3.59364	3.60000	0.00636
0.7	4.19243	4.20000	0.00757
0.8	4.79113	4.80000	0.00887
0.9	5.38975	5.40000	0.01025
1.0	5.98827	6.00000	0.01173
1.1	6.58667	6.60000	0.01333
1.2	7.18493	7.20000	0.01507
1.3	7.78305	7.80000	0.01695
1.4	8.38099	8.40000	0.01901
1.5	8.97875	9.00000	0.02125

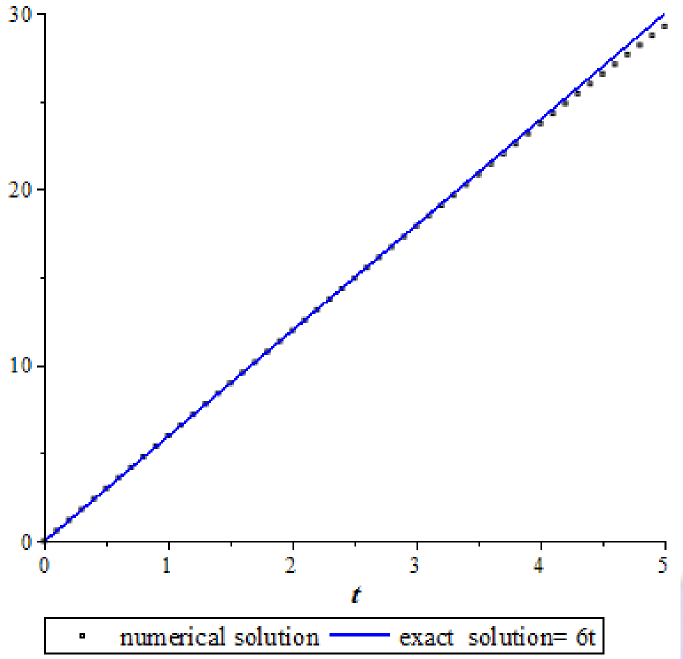

Fig. 1 the exact and approximate solutions result of Volterra integral equation for example 1.

Example 2. Consider the Volterra integral equation of second kind

$$X(t) = t - \frac{2}{3}t^3 - 2\int_0^t X_1(s)\,ds\,,$$

with the exact solution $X(t) = t - t^2$.

Table. 2 Numerical results and exact solution of Volterra integral equation for example 2.

t	$X_1(t)$		
0.0	0.00000	0.00000	0.00000
0.1	0.09030	0.09000	0.00030
0.2	0.16055	0.16000	0.00055
0.3	0.21075	0.21000	0.00075
0.4	0.24092	0.24000	0.00092
0.5	0.25106	0.25000	0.00106
0.6	0.24117	0.24000	0.00117
0.7	0.21126	0.21000	0.00126
0.8	0.16133	0.16000	0.00133
0.9	0.09139	0.09000	0.00139
1.0	0.00144	0.00000	0.00144
1.1	- 0.10852	- 0.11000	0.00148
1.2	- 0.23848	- 0.24000	0.00152
1.3	- 0.38846	- 0.39000	0.00154
1.4	- 0.55843	- 0.56000	0.00157
1.5	- 0.74842	- 0.75000	0.00158

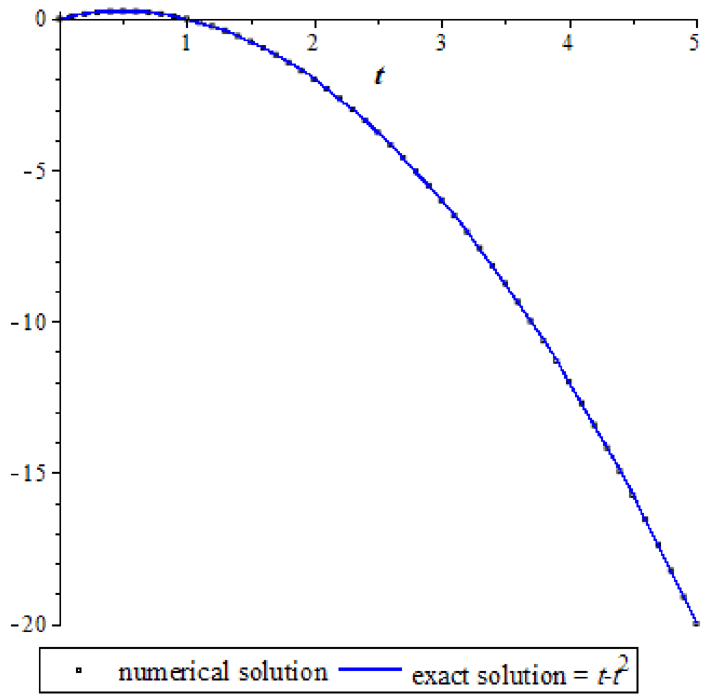

Fig. 2 the exact and approximate solutions result of Volterra integral equation for example 2.

Example 3. Consider the Volterra integral equation of second kind

$$X(t) = 1 - t - \int_0^t (t-s) X_1(s)\, ds ,$$

with the exact solution $X(t) = cost - sint$.

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Table. 3 Numerical results and exact solution of Volterra integral equation for example 3.

t	$X_1(t)$	c	I
0.0	1.00000	1.00000	0.00000
0.1	0.89500	0.89517	0.00017
0.2	0.78105	0.78140	0.00035
0.3	0.65929	0.65982	0.00053
0.4	0.53094	0.53164	0.00071
0.5	0.39727	0.39816	0.00088
0.6	0.25964	0.26069	0.00106
0.7	0.11941	0.12062	0.00122
0.8	- 0.02202	- 0.02065	0.00137
0.9	- 0.16323	- 0.16172	0.00151
1.0	- 0.30280	- 0.30117	0.00163
1.1	- 0.43934	- 0.43761	0.00173
1.2	- 0.57150	- 0.56968	0.00182
1.3	- 0.69793	- 0.69606	0.00187
1.4	- 0.81739	- 0.81548	0.00191
1.5	- 0.92868	- 0.92676	0.00192

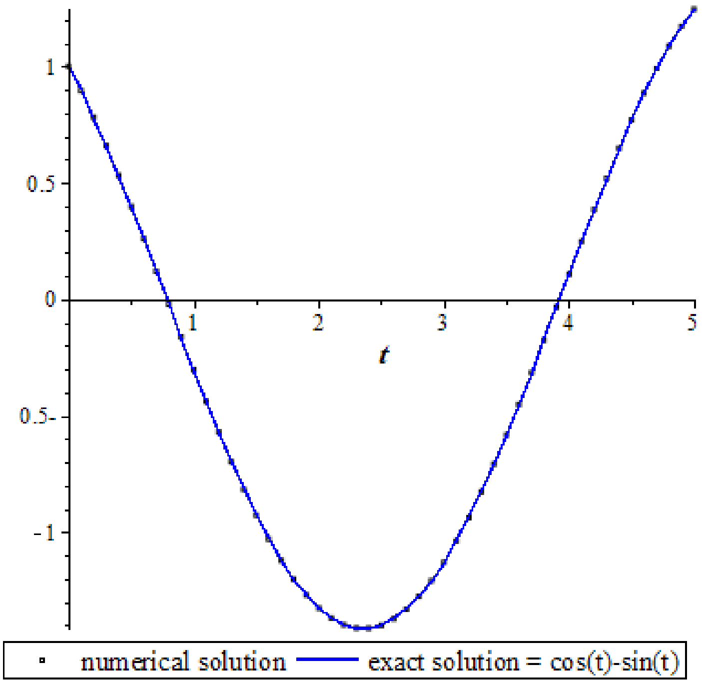

Fig. 3 the exact and approximate solutions result of Volterra integral equation for example 3.

Example 4. Consider theVolterra integral equation of second kind

$$X(t) = 1 + 2\int_0^t (t - s)X_1(s)\,ds\,,$$

with the exact solution $X(t) = e^{t^2}$.

Table. 4 Numerical results and exact solution of Volterra integral equation for example 4.

t	$X_1(t)$		
0.000	1.00000	1.00000	0.00000
0.010	1.00010	0.00010	0.00000
0.020	1.00040	1.00040	0.00000
0.030	1.00090	1.00090	0.00000
0.040	1.00160	1.00160	0.00000
0.050	1.00250	1.00250	0.00000
0.060	1.00360	1.00361	0.00000
0.070	1.00490	1.00491	0.00001
0.080	1.00641	1.00642	0.00001
0.090	1.00811	1.00813	0.00002
0.100	1.01002	1.01005	0.00003
0.110	1.01212	1.01217	0.00005
0.120	1.01443	1.01450	0.00007
0.130	1.01695	1.01704	0.00010
0.140	1.01966	1.01979	0.00013
0.150	1.02258	1.02276	0.00017

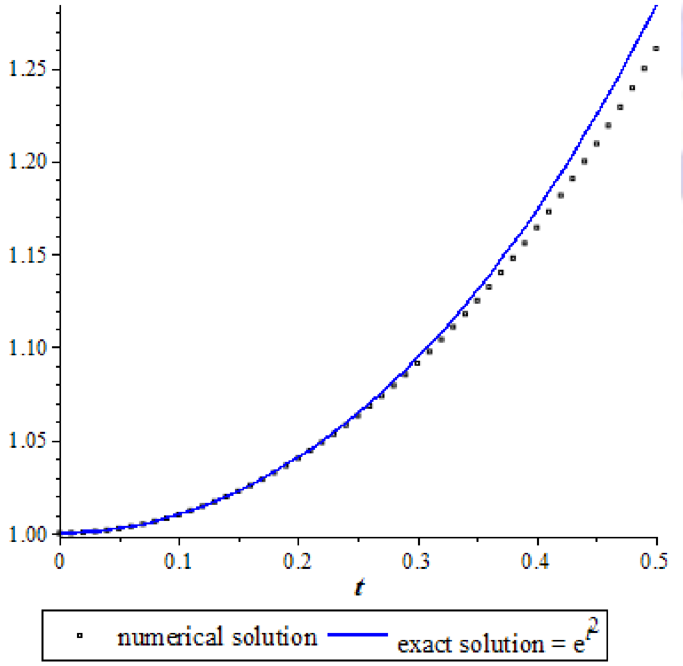

Fig. 4 the exact and approximate solutions result of Volterra integral equation for example 4.

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Example 5. Consider theVolterra integral equation of second kind

$$X(t) = t^2 + \int_0^t (t-s)X_1(s)\,ds\ ,$$

with the exact solution $X(t) = 2cosht - 2$.

Table.5 Numerical results and exact solution of Volterra integral equation for example 5.

t	$X_1(t)$		
0.000	0.00000	0.00000	0.00000
0.100	0.01000	0.01001	0.00001
0.200	0.04010	0.04013	0.00003
0.300	0.09060	0.09068	0.00008
0.400	0.16201	0.16214	0.00014
0.500	0.25504	0.25525	0.00022
0.600	0.37061	0.37093	0.00032
0.700	0.50990	0.51034	0.00044
0.800	0.67428	0.67487	0.00059
0.900	0.86540	0.86617	0.00077
1.000	1.08518	1.08616	0.00098
1.100	1.33581	1.33704	0.00122
1.200	1.61980	1.62131	0.00151
1.300	1.93999	1.94183	0.00184
1.400	2.29958	2.30180	0.00222
1.500	2.70216	2.70482	0.00266

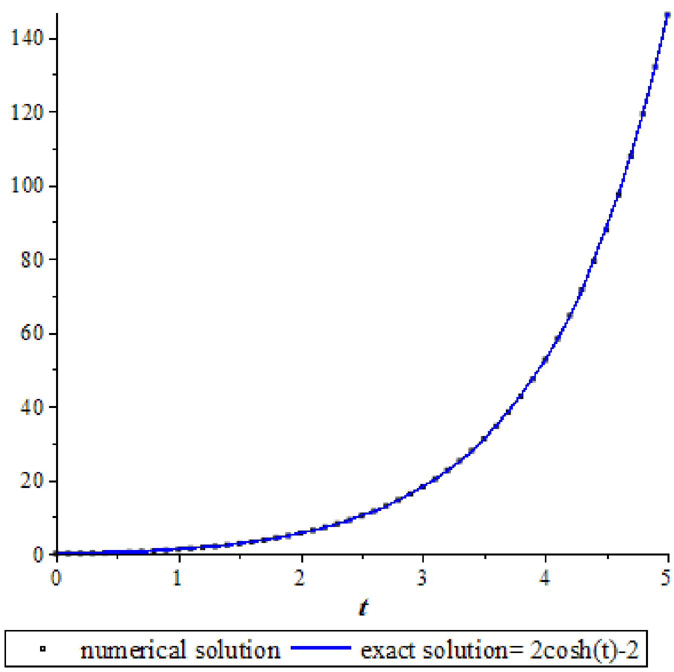

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Fig. 5 the exact and approximate solutions result of Volterra integral equation for example 5.

Example 6. Consider the Volterra integral equation of second kind

$$\mathrm{X}(t) = 1 - \frac{t^2}{2} + \frac{1}{6}\int_0^t (t-s)^3 \mathrm{X}_1(s)ds$$

With the exact solution $\mathrm{X}(t) = \cos t$..

Table.6 Numerical results and exact solution of Volterra integral equation for example 6.

t	$X_1(t)$		$Error = \|X_1(t) - \cos(t)\|$
0.0	1.00000	1.00000	0.00000
0.1	0.99501	0.99500	0.00000
0.2	0.98008	0.98007	0.00002
0.3	0.95537	0.95534	0.00004
0.4	0.92113	0.92106	0.00007
0.5	0.87769	0.87758	0.00010
0.6	0.82549	0.82534	0.00015
0.7	0.76505	0.76484	0.00020
0.8	0.69697	0.69671	0.00027
0.9	0.62195	0.62161	0.00034
1.0	0.54072	0.54030	0.00042
1.1	0.45410	0.45360	0.00051
1.2	0.36296	0.36236	0.00060
1.3	0.26821	0.26750	0.00071
1.4	0.17079	0.16997	0.00083
1.5	0.07169	0.07074	0.00095

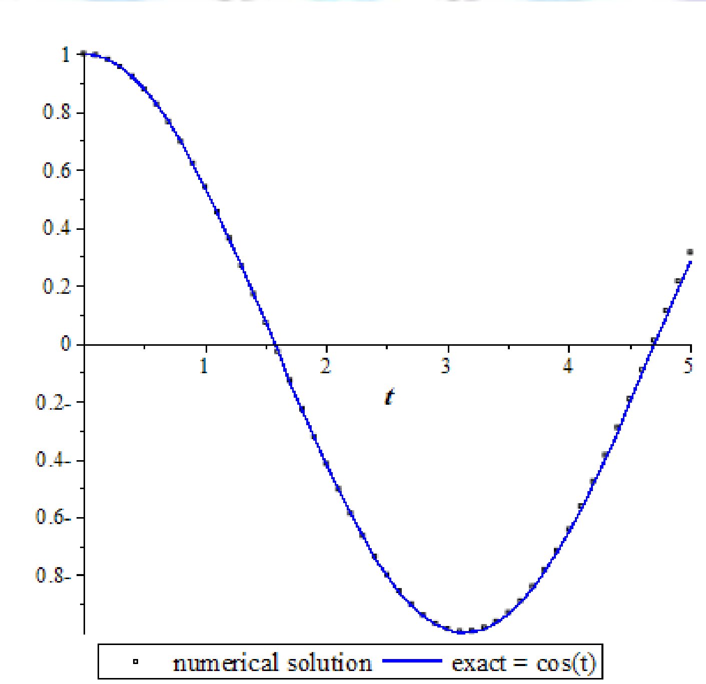

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Fig. 6 The exact and approximate solutions result of Volterra integral equation for example 6.

Example 7. Consider the Volterra integral equation of second kind

$$\mathrm{X}(t) = 1 + t^2 - \int_0^t (t - s + 1)^2 \mathrm{X}_1(s)ds$$

With the exact solution $\mathrm{X}(t) = e^{-t}$.

Table.7 Numerical results and exact solution of Volterra integral equation for example 7.

t	$X_1(t)$	e^{-t}	$Error = \lvert X_1(t) - e^{-t} \rvert$
0.0	1.00000	1.00000	0.00000
0.1	0.90429	0.00055	0.00055
0.2	0.81770	0.81873	0.00103
0.3	0.73937	0.74082	0.00145
0.4	0.66854	0.67032	0.00178
0.5	0.60448	0.60653	0.00205
0.6	0.54657	0.54881	0.00224
0.7	0.49423	0.49659	0.00236
0.8	0.44693	0.44933	0.00240
0.9	0.40419	0.40657	0.00238
1.0	0.36559	0.36788	0.00229
1.1	0.33073	0.33287	0.00214
1.2	0.29926	0.30119	0.00193
1.3	0.27085	0.27253	0.00168
1.4	0.24522	0.24660	0.00138
1.5	0.22208	0.22313	0.00105

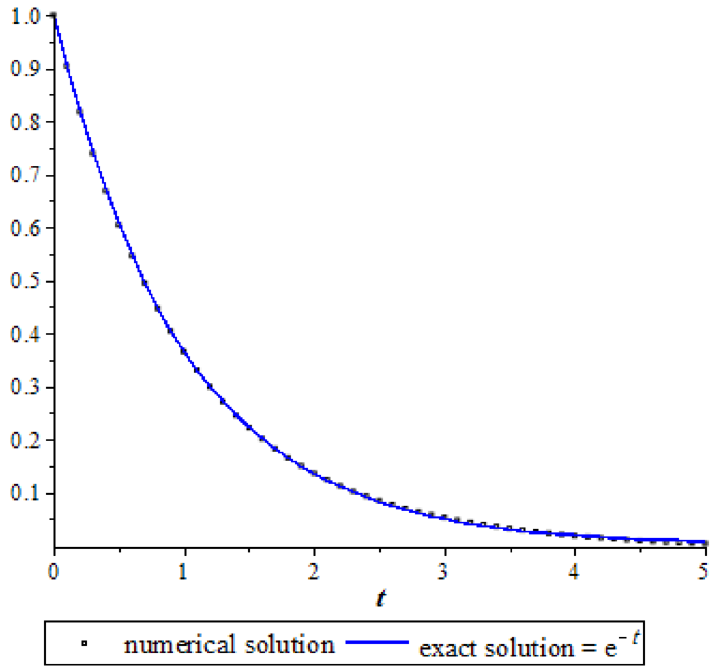

Fig. 7 The exact and approximate solutions result of Volterra integral equation for example 7..

Example 8. Consider the Volterra integral equation of second kind

$$X(t) = 1 + \frac{t}{2} + \frac{1}{2}\int_0^t (t-s+1)^2 X_1(s)ds$$

With the exact solution $X(t) = e^t$.

Table.8 Numerical results and exact solution of Volterra integral equation for example 8.

t	$X_1(t)$	e^t	$Error = \|X_1(t) - e^t\|$
0.0	1.00000	1.00000	0.00000
0.1	1.10513	1.10517	0.00004
0.2	1.22131	1.22140	0.00009
0.3	1.34972	1.34986	0.00014
0.4	1.49164	1.49182	0.00019
0.5	1.64848	1.64872	0.00024
0.6	1.82182	1.82212	0.00030
0.7	2.01339	2.01375	0.00036
0.8	2.22511	2.22554	0.00043
0.9	2.45910	2.45960	0.00051
1.0	2.71770	2.71828	0.00059
1.1	3.00349	3.00417	0.00067
1.2	3.31935	3.32012	0.00077
1.3	3.66842	3.66930	0.00087
1.4	4.05421	4.05520	0.00099
1.5	4.48058	4.48169	0.00111

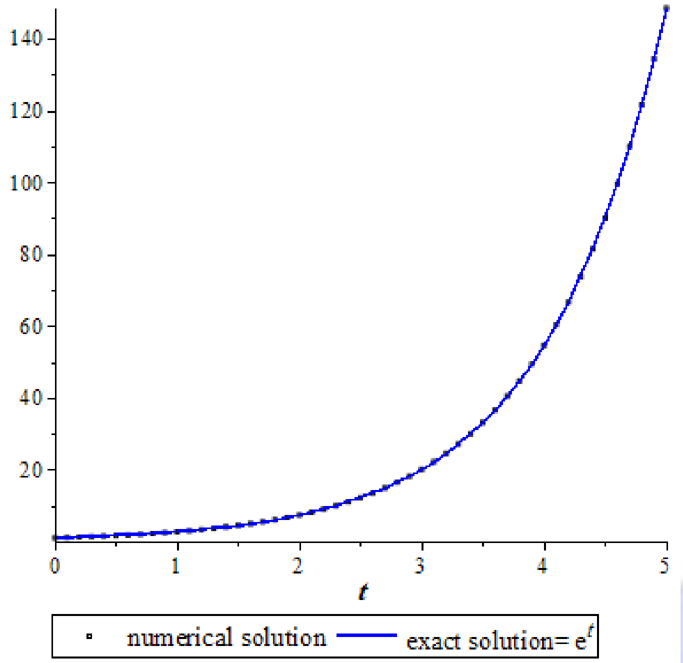

Fig. 8 The exact and approximate solutions result of Volterra integral equation for example 8..

Example 9. Consider the Volterra integral equation of second kind

$$X(t) = 3t^2 + \left(1 - e^{-t^3}\right) - \int_0^t e^{-t^3+s^3} X_1(s)ds$$

With the exact solution $X(t) = 3t^2$.

Table.9 Numerical results and exact solution of Volterra integral equation for example 9.

t	$X_1(t)$	$3t^2$	$Error = \|X_1(t) - 3t^2\|$
0.0	0.00000	0.00000	0.00000
0.1	0.02952	0.03000	0.00048
0.2	0.11908	0.12000	0.00092
0.3	0.26865	0.27000	0.00135
0.4	0.47817	0.48000	0.00183
0.5	0.74758	0.75000	0.00242
0.6	1.07680	1.08000	0.00320
0.7	1.46575	1.47000	0.00425
0.8	1.91429	1.92000	0.00571
0.9	2.42232	2.43000	0.00768
1.0	2.98967	3.00000	0.01033
1.1	3.61621	3.63000	0.01379
1.2	4.30174	4.32000	0.01826
1.3	5.04608	5.07000	0.02392
1.4	5.84902	5.88000	0.03098

1.5	6.71034	6.75000	0.03966

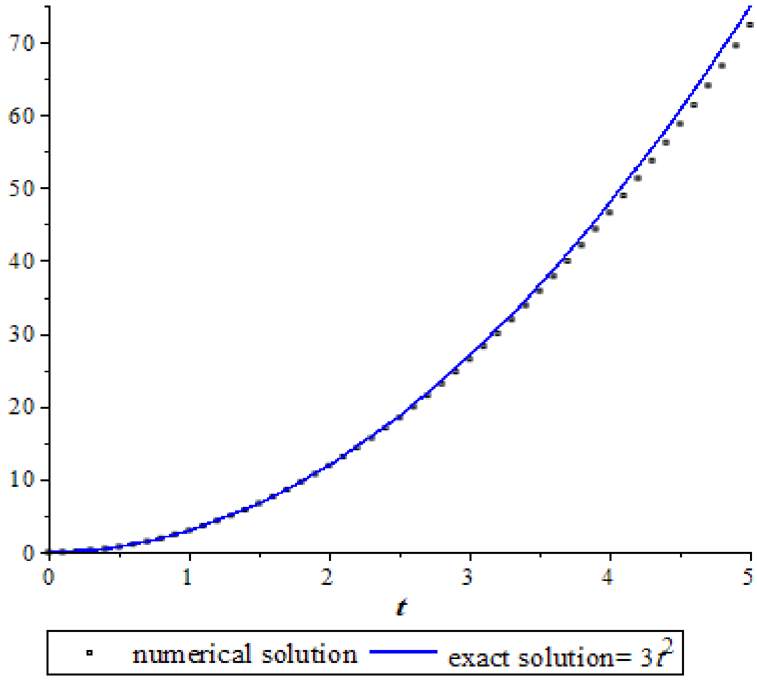

Fig. 9 The exact and approximate solutions result of Volterra integral equation for example 9..

Example 10. Consider the Volterra integral equation of second kind

$$X(t) = \sinh(t) + \frac{1}{10}e - \frac{1}{10}e^{\cosh(t)} + \int_0^t \frac{1}{10}e^{\cosh(s)} X_1(s)ds$$

With the exact solution $X(t) = \sinh(t)$..

Table.10 Numerical results and exact solution Volterra integral equation for example 10.

t	$X_1(t)$	$\sinh(t)$	$Error = \lvert X_1(t) - \sinh(t)\rvert$
0.0	0.00000	0.00000	0.00000
0.1	0.10017	0.10017	0.00000
0.2	0.20135	0.20134	0.00002
0.3	0.30456	0.30452	0.00004
0.4	0.41084	0.41075	0.00008
0.5	0.52124	0.52110	0.00014
0.6	0.63687	0.63665	0.00022
0.7	0.75891	0.75858	0.00033
0.8	0.88859	0.88811	0.00048
0.9	1.02722	1.02652	0.00070
1.0	1.17621	1.17520	0.00101
1.1	1.33711	1.33565	0.00146
1.2	1.51158	1.50946	0.00212
1.3	1.70149	1.69838	0.00311
1.4	1.90895	1.90430	0.00465

1.5	2.13639	2.12928	0.00711

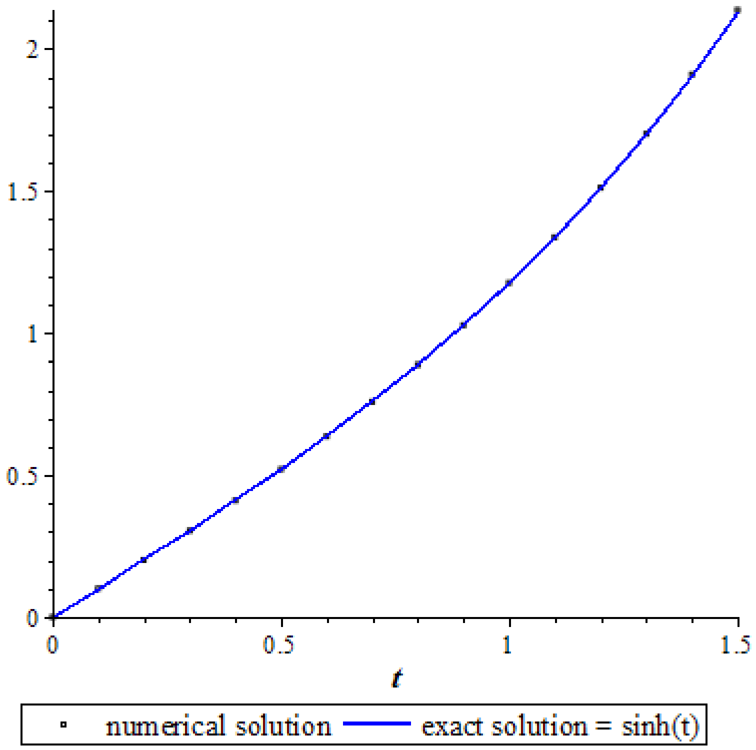

Fig. 10 The exact and approximate solutions result of Volterra integral equation for example 10.

Conclusion

In this paper,we have creatednumerical solutionof a number ofexamples ofthe equation ofVolterraintegral equation of thesecond kindusing thetrapezoidalimplicitand we have foundwhen comparing thenumerical solutionand the exact solutionthatthe results wereveryclosеandthe percentage oferrorbetweenthe two solutionsis very smallwhich indicatesto accuracyimplicit trapezoidalMethod.

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