ISSN 2347-1921

Oscillation Criteria for Third-Order Nonlinear Delay Difference Equations

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Abstract

In this paper, we are concerned with oscillation of a class of third-order nonlinear delay difference equation of the form

$$\Delta\big(a(n)\Delta(b(n)\Delta y(n))\big) + q(n)y^{\gamma}(n-k) = 0, \quad n \geq n_0 .$$

We establish some new oscillation criteria by transforming this equation to the first-order delayed and advanced difference equations. Employing suitable comparison theorems we present new results on oscillation of the studied equation. Some examples are provided to illustrate the results.

2010 AMS Subject Classification: 39A10

Keywords and Phrases: Oscillation; third-order; non-linear equation; delay difference equation.

Council for Innovative Research
Peer Review Research Publishing System
Journal: Journal of Advances in Mathematics

Vol 8, No 2
editor@cirjam.org
www.cirworld.org, www.cirjam.com

1 Introduction

In recent years, determination of oscillatory behavior for solutions of first and second order difference equations has occupied a great part of researchers' interest. Compared to the first and second order difference equations, the study of third order difference equations has received considerably less attention in the literature, even though such equations arise in the study of economics, mathematical biology and other areas of mathematics. For contributions, we refer the reader to the papers [1-6] and the references cited therein.

Therefore in this paper, we are concerned with the oscillation of the third-order nonlinear delay difference equation of the form

$$\Delta\big(a(n)\Delta(b(n)\Delta y(n))\big) + q(n)y^{\gamma}(n-k) = 0, \quad n \geq n_0 \tag{1.1}$$

subject to the following conditions:

(i) γ is the quotient of odd positive integers;

(ii) $\{a(n)\}$, $\{b(n)\}$ and $\{q(n)\}$ are sequences of positive real numbers;

(iii) k is a positive integer.

By a solution of equation (1.1) we mean a real sequence $\{y(n)\}$ and satisfying equation (1.1) for all $n \geq n_0$. We consider only those solution $\{y(n)\}$ of equation (1.1) which satisfy $sup\,\{|y(n)| : n \geq N\} > 0$ for all $N \geq n_0$. A solution of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and non oscillatory otherwise.

In [15], the authors considered the equation (1.1) and studied the oscillatory behavior of all solutions when

$$\sum_{n=n_0}^{\infty} \frac{1}{a(n)} = \infty \qquad and \qquad \sum_{n=n_0}^{\infty} \frac{1}{b(n)} = \infty\,.$$

Motivated by this observation, in this paper we discussed the oscillatory behavior of equation (1.1) for the following two cases:

$$\sum_{n=n_0}^{\infty} \frac{1}{a(n)} < \infty \qquad and \qquad \sum_{n=n_0}^{\infty} \frac{1}{b(n)} = \infty \tag{1.2}$$

and

$$\sum_{n=n_0}^{\infty} \frac{1}{a(n)} < \infty \qquad and \qquad \sum_{n=n_0}^{\infty} \frac{1}{b(n)} < \infty \tag{1.3}$$

In Section 2, we establish some sufficient conditions which ensure that all solutions of equation (1.1) are either oscillatory or converges to zero. Examples are provided in Section 3 to illustrate the main results.

2 Oscillation Results

In this section, we will establish some oscillation criteria for the equation (1.1). To simplify our notation, let us denote $z(n) := -w(n) := -a(n)(\Delta b(n)\Delta y(n))$ and $B(n) := \sum_{s=n}^{\infty} \frac{1}{b(s)}$.

We begin with the following Lemmas.

Lemma 2.1.

Let $q(n) > 0$ for all $n \geq n_0$ and $\alpha > 0$ is a ratio of odd positive integers and k is a positive integer.

(i) If $\alpha < 1$ and

$$\sum_{n=n_0}^{\infty} q(n) = \infty\,,$$

then all solutions of the equation

$$\Delta x(n) + q(n)x^{\alpha}(n-k) = 0 \qquad (2.1)$$

are oscillatory.

(ii) If $\alpha = 1$ and

$$\lim_{n\to\infty} inf \sum_{s=n-k}^{n-1} q(s) > \left(\frac{k}{k+1}\right)^{k+1},$$

then all solutions of equation (2.1) are oscillatory.

(iii) If $\alpha > 1$ and there exists a $\lambda > \frac{1}{k}$ In α such that

$$\lim_{n\to\infty} inf \left[q(n)\exp\left(-e^{\lambda n}\right)\right] > 0\,,$$

then all solutions of equation (2.1) are oscillatory.

Proof.

The proof can be found in [8] and [16]. ◘

Lemma 2.2.

Let $q(n) > 0$ for all $n \geq n_0$ and $\beta > 1$ is a ratio of odd positive integers and k is a positive integer. If

$$\sum_{n=n_0}^{\infty} q(n) = \infty\,,$$

then every solution of the equation

$$\Delta x(n) - q(n)x^{\beta}(n+k) = 0 \qquad (2.2)$$

is oscillatory.

Proof.

Assume that $\{x(n)\}$ is a nonoscillatory solution of equation (2.2). Without loss of generality, we may assume that $x(n) > 0$ for all $n \geq n_1 \geq n_0$. From the given equation, we have

$$\Delta x(n) = q(n)x^{\beta}(n+k) > 0$$

and therefore $\{x(n)\}$ is increasing and $x^{\beta}(n+1) \leq x^{\beta}(n+k)$ for all $n \geq n_1$. Then

$$\int_{x(n)}^{x(n+1)} \frac{ds}{s^{\beta}} \geq \frac{\Delta x(n)}{x^{\beta}(n+1)} = q(n)\frac{x^{\beta}(n+k)}{x^{\beta}(n+1)} \geq q(n), \qquad n \geq n_1\,.$$

Summing last inequality from n_1 to $N-1$, we have

$$\int_{x(n_1)}^{x(N)} \frac{ds}{s^{\beta}} \geq \sum_{n=n_1}^{N-1} q(n)$$

or

$$\sum_{n=n_1}^{N-1} q(n) < \frac{1}{(\beta-1)x^{\beta-1}(n_1)} < \infty\,.$$

Letting $N \to \infty$ in the last inequality, we obtain a contradiction and the proof is now complete. ◘

Lemma 2.3.

Let $q(n) > 0$ for all $n \geq n_0$ and m is a positive integer. If

$$\lim_{n\to\infty} inf \sum_{s=n}^{n+m} q(n) > \left(\frac{m}{m+1}\right)^{m+1}$$

then every solution of

$$\Delta x(n) - q(n)x(n+m) = 0 \tag{2.3}$$

is oscillatory.

Proof.

The proof can be found in [7]. ◘

Theorem 2.1.

Assume that condition (1.2) holds. Let there exist numbers α, β, γ with $\alpha \leq \gamma \leq \beta$ such that α, β, γ are the ratios of odd positive integers. If there exist two positive integers l and m with $2l < k$ and if for all sufficiently large $n_1 \geq n_0$ and for $n_2 > n_1$, the following difference equations

$$\Delta w(n) + c_1^{\gamma-\alpha} q(n) \left[\sum_{s=n_2}^{n-k-1} \frac{\sum_{u=n_1}^{s-1} \frac{1}{a(u)}}{b(s)} \right]^{\alpha} w^{\alpha}(n-k) = 0 \tag{2.4}$$

$$\Delta v(n) + \left(\frac{1}{b(n)} \sum_{s_2=n}^{n+l-1} \frac{1}{a(s_2)} \sum_{s_1=s_2}^{s_2+l-1} q(s_1) \right) v^{\gamma} (n+2l-k) = 0 \tag{2.5}$$

and

$$\Delta z(n) - c_1^{\gamma-\beta} q(n) \left(\sum_{s=n+m}^{\infty} \frac{1}{a(s)} \right)^{\beta} \left(\sum_{s=n_1}^{n-k-1} \frac{1}{b(s)} \right)^{\gamma} z^{\beta}(n+m) = 0 \tag{2.6}$$

are oscillatory for all constants $c_1 > 0$, $c_2 > 0$, then every solution of equation (1.1) is oscillatory.

Proof.

Let $\{y(n)\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may suppose that $\{y(n)\}$ is positive. Then there exist three possible cases:

Case(1) : $y(n) > 0,\ \ \Delta y(n) > 0,\ \ \Delta\big(b(n)\Delta y(n)\big) > 0,$

$$\Delta\Big(a(n)\Delta\big(b(n)\Delta y(n)\big)\Big) < 0;$$

Case(2) : $y(n) > 0,\ \ \Delta y(n) < 0,\ \ \Delta\big(b(n)\Delta y(n)\big) > 0,$

$$\Delta\Big(a(n)\Delta\big(b(n)\Delta y(n)\big)\Big) < 0;$$

and

Case(3) : $y(n) > 0,\ \ \Delta y(n) > 0,\ \ \Delta\big(b(n)\Delta y(n)\big) < 0,$

$$\Delta\Big(a(n)\Delta\big(b(n)\Delta y(n)\big)\Big) < 0$$

for $n \geq n_1$, where $n_1 \geq n_0$ is large enough.

Assume that case(1) holds. Using $\Delta\Big(a(n)\Delta\big(b(n)\Delta y(n)\big)\Big) < 0$, we have

$$b(n)\Delta y(n) \geq \sum_{s=n_1}^{n-1} \frac{a(s)\Delta\big(b(s)\Delta y(s)\big)}{a(s)}$$

$$\geq a(n)\Delta\big(b(n)\Delta y(n)\big) \sum_{s=n_1}^{n-1} \frac{1}{a(s)}.$$

That is,

$$\Delta y(n) \geq \frac{a(n)\Delta\big(b(n)\Delta y(n)\big)}{b(n)} \sum_{s=n_1}^{n-1} \frac{1}{a(s)}.$$

Summing the last inequality from n_2 $(n_2 > n_1)$ to $n-1$, we get by the definition of $w(n)$ that

$$y(n) \geq a(n)\Delta\big(b(n)\Delta y(n)\big) \sum_{s=n_2}^{n-1} \frac{\sum_{u=n_1}^{s-1} \frac{1}{a(u)}}{b(s)}$$

$$= w(n) \sum_{s=n_2}^{n-1} \frac{\sum_{u=n_1}^{s-1} \frac{1}{a(u)}}{b(s)}. \tag{2.7}$$

From equation (1.1) and the fact that $\Delta y(n) > 0$, we see that there exists a constant $c_1 > 0$ such that

$$\Delta\Big(a(n)\Delta\big(b(n)\Delta y(n)\big)\Big) + c_1^{\gamma-\alpha} q(n) y^{\alpha}(n-k) \leq 0.$$

Using (2.7) in the above inequality, we see that $w(n)$ is a positive solution of the inequality

$$\Delta w(n) + c_1^{\gamma-\alpha} q(n) \left[\sum_{s=n_2}^{n-k-1} \frac{\sum_{u=n_1}^{s-1} \frac{1}{a(u)}}{b(s)}\right]^{\alpha} w^{\alpha}(n-k) \leq 0.$$

Therefore, by Lemma 1[16], the associated delay difference equation (2.4) also has a positive solution, which is a contradiction.

Next assume that case(2) holds. Summing equation (1.1) from n to $n+l-1$ implies that

$$a(n)\Delta\big(b(n)\Delta y(n)\big) \geq \sum_{s_1=n}^{n+l-1} q(s_1)\, y^{\gamma}(s_1-k)$$

$$\geq y^{\gamma}(n+l-k) \sum_{s_1=n}^{n+l-1} q(s_1).$$

That is,

$$\Delta\big(b(n)\Delta y(n)\big) \geq \frac{y^{\gamma}(n+l-k)}{a(n)} \sum_{s_1=n}^{n+l-1} q(s_1).$$

Summing the last inequality from n to $n+l-1$, we have

$$-b(n)\Delta y(n) \geq \sum_{s_2=n}^{n+l-1} \frac{y^{\gamma}(s_2+l-k)}{a(s_2)} \sum_{s_1=s_2}^{s_2+l-1} q(s_1)$$

$$\geq\ y^{\gamma}(n+2l-k)\sum_{s_2=n}^{n+l-1}\frac{1}{a(s_2)}\sum_{s_1=s_2}^{s_2+l-1}q(s_1)\,.$$

That is,

$$-\Delta y(n)\ \geq\ \frac{y^{\gamma}(n+2l-k)}{b(n)}\sum_{s_2=n}^{n+l-1}\frac{1}{a(s_2)}\sum_{s_1=s_2}^{s_2+l-1}q(s_1)\,.$$

Summing the last inequality from n to ∞, we get

$$y(n)\ \geq\ \sum_{s_3=n}^{\infty}\frac{y^{\gamma}(n+2l-k)}{b(s_3)}\sum_{s_2=s_3}^{s_3+l-1}\frac{1}{a(s_2)}\sum_{s_1=s_2}^{s_2+l-1}q(s_1)\,.$$

Let us denote the right hand side of the last inequality by $v(n)$. Then $v(n)>0$, and one can easily verify that $v(n)$ is a solution of the difference inequality

$$\Delta v(n)+\left(\frac{1}{b(n)}\sum_{s_2=n}^{n+l-1}\frac{1}{a(s_2)}\sum_{s_1=s_2}^{s_2+l-1}q(s_1)\right)v^{\gamma}(n+2l-k)\leq 0.$$

Therefore, by Lemma 1[16], the associated delay difference equation (2.5) also has a positive solution, which is a contradiction.

Assume that case(3) holds. As $\Delta\Big(a(n)\Delta\big(b(n)\Delta y(n)\big)\Big)<0$, we see that $a(n)\Delta\big(b(n)\Delta y(n)\big)$ is decreasing. Thus, we get

$$a(s)\Delta\big(b(s)\Delta y(s)\big)\ \leq\ a(n)\Delta\big(b(n)\Delta y(n)\big)\quad \text{for}\quad s\geq n\geq n_1.$$

Dividing the above inequality by $a(s)$ and summing the resulting inequality from n to $N-1$, we obtain

$$b(N)\Delta y(N)\ \leq\ b(n)\Delta y(n)+a(n)\Delta\big(b(n)\Delta y(n)\big)\sum_{s=n}^{N-1}\frac{1}{a(s)}\,.$$

Letting $N\to\infty$, we have

$$b(n)\Delta y(n)\ \geq\ -a(n)\Delta\big(b(n)\Delta y(n)\big)\sum_{s=n}^{\infty}\frac{1}{a(s)}\,. \qquad (2.8)$$

Using conditions $y(n)>0$ and $\Delta\big(b(n)\Delta y(n)\big)<0$, we have

$$y(n)\geq\ b(n)\Delta y(n)\sum_{s=n_1}^{n-1}\frac{1}{b(s)}\,. \qquad (2.9)$$

Thus,

$$\Delta\left(\frac{y(n)}{\sum_{s=n_1}^{n-1}\frac{1}{b(s)}}\right)\leq 0\,. \qquad (2.10)$$

Combining (2.8) and (2.9), we get

$$y(n) \geq -a(n)\Delta\big(b(n)\Delta y(n)\big) \sum_{s=n}^{\infty} \frac{1}{a(s)} \sum_{s=n_1}^{n-1} \frac{1}{b(s)}. \qquad (2.11)$$

On the other hand, we have by (2.10) and $(n+m) \geq (n-k)$ that

$$y^{\gamma}(n-k) \geq \left(\frac{\sum_{s=n_1}^{n-k-1} \frac{1}{b(s)}}{\sum_{s=n_1}^{n+m-1} \frac{1}{b(s)}} \right)^{\gamma} y^{\gamma}(n+m)$$

$$= \left(\frac{\sum_{s=n_1}^{n-k-1} \frac{1}{b(s)}}{\sum_{s=n_1}^{n+m-1} \frac{1}{b(s)}} \right)^{\gamma} y^{\beta}(n+m) y^{\gamma-\beta}(n+m). \qquad (2.12)$$

By virtue of (2.10), we have that there exists a constant c_2 such that $y(n) \leq c_2 \sum_{s=n_1}^{n-1} \frac{1}{b(s)}$.

Hence by (2.12), we get

$$y^{\gamma}(n-k) \geq c_2^{\gamma-\beta} \left(\frac{\sum_{s=n_1}^{n-k-1} \frac{1}{b(s)}}{\sum_{s=n_1}^{n+m-1} \frac{1}{b(s)}} \right)^{\gamma} y^{\beta}(n+m) \left(\sum_{s=n_1}^{n+m-1} \frac{1}{b(s)} \right)^{\gamma-\beta}$$

$$= c_2^{\gamma-\beta} \left(\sum_{s=n_1}^{n+m-1} \frac{1}{b(s)} \right)^{-\beta} \left(\sum_{s=n_1}^{n-k-1} \frac{1}{b(s)} \right)^{\gamma} y^{\beta}(n+m). \qquad (2.13)$$

Using (2.13) in equation (1.1), we have

$$\Delta w(n) + c_2^{\gamma-\beta} q(n) \left(\sum_{s=n+m}^{\infty} \frac{1}{a(s)} \right)^{\beta} \left(\sum_{s=n_1}^{n-k-1} \frac{1}{b(s)} \right)^{\gamma} \big(-w(n+m)\big)^{\beta} \leq 0.$$

Writing the last inequality in the form

$$\Delta z(n) - c_2^{\gamma-\beta} q(n) \left(\sum_{s=n+m}^{\infty} \frac{1}{a(s)} \right)^{\beta} \left(\sum_{s=n_1}^{n-k-1} \frac{1}{b(s)} \right)^{\gamma} z^{\beta}(n+m) \geq 0.$$

From the above inequality and Lemma 1[16], we deduce that the associated advanced difference equation (2.6) also has a positive solution, which is a contradiction. This completes the proof. ◻

Corollary 2.1.

Let (1.2) holds and let $\alpha = \beta = \gamma = 1.$ Assume that there exist two positive integers l and m with $2l < k$ and if for all sufficiently large $n_1 \geq n_0$ and for $n_2 > n_1$,

$$\lim_{n\to\infty} inf \sum_{s=n-k}^{n-1} q(s) \sum_{\mu=n_2}^{s-k-1} \frac{\sum_{u=n_1}^{\mu-1} \frac{1}{a(u)}}{b(\mu)} > \left(\frac{k}{k+1} \right)^{k+1}, \qquad (2.14)$$

$$\lim_{n\to\infty} inf \sum_{s=n+2l-k}^{n-1} \frac{1}{b(s)} \sum_{s_2=s}^{s+l-1} \frac{1}{a(s_2)} \sum_{s_1=s_2}^{s_2+l-1} q(s_1) > \left(\frac{k-2l}{k-2l+1} \right)^{k-2l+1} \qquad (2.15)$$

and

$$\lim_{n\to\infty} inf \sum_{s=n+1}^{n+m-1} q(s) \sum_{u=s+m}^{\infty} \frac{1}{a(u)} \sum_{u=n_1}^{s-k-1} \frac{1}{b(u)} > \left(\frac{m-1}{m}\right)^m, \tag{2.16}$$

then every solution of equation (1.1) is oscillatory.

Proof.

The proof follows by using Lemma 2.1(ii) and Lemma 2.3 in Theorem 2.1 and therefore the details are omitted. ◻

Corollary 2.2.

Let (1.2) holds and let $\alpha > 1$, $\beta > 1$ and $\gamma > 1$. Assume that there exist two positive integers l and m with $2l < k$ and if for all sufficiently large $n_1 \geq n_0$ and for $n_2 > n_1$, we have if $k \geq 1$ and there exists a $\lambda_1 > \frac{1}{k} \ln \alpha$ such that

$$\lim_{n\to\infty} inf \left[q(n) \left(\sum_{s=n_2}^{n-k-1} \frac{\sum_{u=n_1}^{s-1} \frac{1}{a(u)}}{b(s)} \right)^{\alpha} \exp\left(-e^{\lambda_1 n}\right) \right] > 0, \tag{2.17}$$

also there exists a $\lambda_2 > \frac{1}{k-2l} \ln \gamma$ such that

$$\lim_{n\to\infty} inf \left[\left(\frac{1}{b(n)} \sum_{s_2=n}^{n+l-1} \frac{1}{a(s_2)} \sum_{s_1=s_2}^{s_2+l-1} q(s_1) \right) \exp\left(-e^{\lambda_2 n}\right) \right] > 0 \tag{2.18}$$

and

$$\sum_{n=n_0}^{\infty} q(n) \left(\sum_{s=n+m}^{\infty} \frac{1}{a(s)} \right)^{\beta} \left(\sum_{s=n_1}^{n-k-1} \frac{1}{b(s)} \right)^{\gamma} = \infty, \tag{2.19}$$

then every solution of equation (1.1) is oscillatory.

Proof.

The proof follows by using the Lemma 2.1(iii) and Lemma 2.2 in Theorem 2.1 and therefore the details are omitted. ◻

Corollary 2.3.

Let (1.2) holds and let $\alpha < 1$, $\gamma < 1$ and $\beta = 1$. Assume that there exist two positive integers l and m with $2l < k$ and if for all sufficiently large $n_1 \geq n_0$ and for $n_2 > n_1$,

we have

$$\sum_{n=n_0}^{\infty} q(n) \left(\sum_{s=n_2}^{n-k-1} \frac{\sum_{u=n_1}^{s-1} \frac{1}{a(u)}}{b(s)} \right)^{\alpha} = \infty, \tag{2.20}$$

$$\sum_{n=n_0}^{\infty} \frac{1}{b(n)} \sum_{s_2=n}^{n+l-1} \frac{1}{a(s_2)} \sum_{s_1=s_2}^{s_2+l-1} q(s_1) = \infty \tag{2.21}$$

and

$$\lim_{n\to\infty} inf \sum_{s=n+1}^{n+m-1} q(s) \sum_{u=s+m}^{\infty} \frac{1}{a(u)} \sum_{u=n_1}^{s-k-1} \frac{1}{b(u)} > \left(\frac{m-1}{m}\right)^m, \tag{2.22}$$

then every solution of equation (1.1) is oscillatory.

Proof.

The proof follows by using the Lemma 2.1(i) and Lemma 2.3 in Theorem 2.1 and therefore the details are omitted. ◻

Corollary 2.4.

Let (1.2) holds and let $\alpha < 1$, $\gamma < 1$ and $\beta > 1$. Assume that there exist two positive integers l and m with $2l < k$ and if for all sufficiently large $n_1 \geq n_0$ and for $n_2 > n_1$,

$$\sum_{n=n_0}^{\infty} q(n) \left(\sum_{s=n_2}^{n-k-1} \frac{\sum_{u=n_1}^{s-1} \frac{1}{a(u)}}{b(s)} \right)^{\alpha} = \infty, \tag{2.23}$$

$$\sum_{n=n_0}^{\infty} \frac{1}{b(n)} \sum_{s_2=n}^{n+l-1} \frac{1}{a(s_2)} \sum_{s_1=s_2}^{s_2+l-1} q(s_1) = \infty \tag{2.24}$$

and

$$\sum_{n=n_0}^{\infty} q(n) \left(\sum_{s=n+m}^{\infty} \frac{1}{a(s)} \right)^{\beta} \left(\sum_{s=n_1}^{n-k-1} \frac{1}{b(s)} \right)^{\gamma} = \infty, \tag{2.25}$$

then every solution of equation (1.1) is oscillatory.

Proof.

The proof follows by using the Lemma 2.1(i) and Lemma 2.2 in Theorem 2.1 and therefore the details are omitted. ◻

Theorem 2.2.

Let all conditions of Theorem 2.1 hold with (1.2) replaced by (1.3). If

$$\sum_{v=n_0}^{\infty} \frac{1}{b(v)} \sum_{u=n_0}^{v-1} \frac{1}{a(u)} \sum_{s=n_0}^{u-1} q(s)\, B^{\gamma}(s-k) = \infty, \tag{2.26}$$

then every solution of equation (1.1) is oscillatory.

Proof.

Let $\{y(n)\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may suppose that $\{y(n)\}$ is positive. Then there exist four possible cases (1), (2), (3) (as those of Theorem 2.1) and

Case(4) : $y(n) > 0,\ \Delta y(n) < 0,\ \Delta\big(b(n)\Delta y(n)\big) < 0,$

$$\Delta\Big(a(n)\Delta\big(b(n)\Delta y(n)\big)\Big) < 0$$

for $n \geq n_1$, where $n_1 \geq n_0$ is large enough. From the proof of Theorem 2.1, we can eliminate cases (1), (2) and (3). Consider now the case (4). Since $\Delta\big(b(n)\Delta y(n)\big) < 0$, we get,

$$\Delta y(s) \leq \frac{b(n)\Delta y(n)}{b(s)} \qquad \text{for} \quad s \geq n.$$

Summing this inequality from n to $N-1$ and letting $N \to \infty$ implies that

$$y(n) \geq -B(n)L(n)\Delta y(n) \geq LB(n) \tag{2.27}$$

for some constant $L > 0$. From equation (1.1), we have,

$$\Delta\left(a(n)\Delta\big(b(n)\Delta y(n)\big)\right) + L^{\gamma} q(n) B^{\gamma}(s-k) \leq 0.$$

Summing the above inequality from n_1 to $n-1$, we get

$$a(n)\Delta\big(b(n)\Delta y(n)\big) + L^{\gamma} \sum_{s=n_1}^{n-1} q(s)\, B^{\gamma}(s-k) \leq 0\,.$$

Summing again, we obtain

$$y(n_1) \geq L^{\gamma} \sum_{v=n_1}^{n-1} \frac{1}{b(v)} \sum_{u=n_1}^{v-1} \frac{1}{a(u)} \sum_{s=n_1}^{u-1} q(s)\, B^{\gamma}(s-k)$$

which contradicts (2.26). This completes the proof. □

Remark 2.1.

Based on Theorem 2.2, similar to Corollaries 2.1 – 2.4, one can obtain some oscillation criteria for equation (1.1). The details are left to the reader.

3 Examples

Example 3.1.

Consider the third order difference equation

$$\Delta\left(n^2\Delta\big(n\Delta y(n)\big)\right) + n^3 y(n-3) = 0\,, \qquad n \geq 1. \tag{3.1}$$

Here, $a(n) = n^2$, $b(n) = n$, $q(n) = n^3$, $k = 3$ and $\gamma = 1$. Choose $\alpha = \beta = 1$ and $l = 1$ and $m = 2$, then all conditions of Corollary 2.1 are satisfied and hence every solution of the equation (3.1) is oscillatory.

Example 3.2.

Consider the third order difference equation

$$\Delta\left(n^2\Delta\big(n\Delta y(n)\big)\right) + n^4 y^3(n-3) = 0\,, \qquad n \geq 1. \tag{3.2}$$

Here, $a(n) = n^2$, $b(n) = n$, $q(n) = n^4$, $k = 3$ and $\gamma = 3$. Choose $\alpha = \frac{5}{3}$, $\beta = 5$ and $l = 1$, then it is easy to see all conditions of Corollary 2.2 are satisfied and hence every solution of the equation (3.2) is oscillatory.

Example 3.3.

Consider the third order difference equation

$$\Delta\left(n^3\Delta\big(n\Delta y(n)\big)\right) + n^6 y^{\frac{1}{3}}(n-4) = 0\,, \qquad n \geq 1. \tag{3.3}$$

Here, $a(n) = n^3$, $b(n) = n$, $q(n) = n^6$, $k = 4$ and $\gamma = \frac{1}{3}$. Choose $\alpha = \frac{1}{5}$, $\beta = 1$, $l = 1$ and $m = 2$, then it is easy to see all conditions of Corollary 2.3 are satisfied. Therefore all solutions of the equation (3.3) are oscillatory.

Example 3.4.

Consider the third order difference equation

$$\Delta\left(n^3\Delta\big(n\Delta y(n)\big)\right) + n^5 y^{\frac{3}{5}}(n-5) = 0\,, \qquad n \geq 1. \tag{3.4}$$

Here, $a(n) = n^3$, $b(n) = n$, $q(n) = n^5$, $k = 5$ *and* $\gamma = \frac{3}{5}$. Choose $\alpha = \frac{1}{3}$, $\beta = \frac{5}{3}$, $l = 2$ and $m = 2$, then it is easy to see all conditions of Corollary 2.4 are satisfied. Hence, every solution of the equation (3.4) is oscillatory.

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