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The Randić index of line, subdivision and total graphs of some graphs

[1] Rasoul Mojarad, [2] Jafar Asadpour
[2] Department of Mathematics, Bushehr Branch, Islamic Azad University, Bushehr, Iran
[2] Mojarrad@gmail.com
[1] Department of Mathematics, Miyaneh Branch, Islamic Azad University, Miyaneh, Iran
[1] jafar_asadpour@yahoo.com

ABSTRACT

The Randić index *R*(*G*) of an organic molecule whose molecular graph is *G* is the sum of the weights $(d(u)d(v))^{-\frac{1}{2}}$ of all edges *uv* of *G*, where $d(u)$ (or $d(v)$) denote the degree of vertex *u*(or *v*). In this paper, we compute the Randić index of line, subdivision and total graphs for some graphs.

Indexing terms/Keywords:

Randić index; Line graph; Total graph.

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INTROUDUCTION

In 1975 Randic [8] proposed several numbering schemes for the edges of the associated hydrogen-suppressed graph based on the degrees of the endvertices of an edge in studying the properties of alkane. To preserve rankings of certain molecules, several inequalities involving the weights of edges needed to be satisfied. Randic stated that weighting all edges uv of the associated graph G by $(d(u)d(v))^{-\frac{1}{2}}$ preserved these inequalities, where d(u) denotes the

degree of a vertex u in G. The sum of these latter weights over the edges of G is called the Randic index of G, denoted by R(G). Some researchers often call it connectivity index [2]. Randic index is an important molecular descriptor and has been closely correlated with many chemical properties [6] and found to parallel the boiling point, Kovats constants, and a calculated surface. In addition, the Randić index appears to predict the boiling points of alkanes more closely, and only it takes into account the bonding or adjacency degree among carbons in alkanes (see [7]). It is said in [5] that Randić index “together with its generalizations it is certainly the molecular-graph-based structure-descriptor, that found the most numerous applications in organic chemistry, medicinal chemistry, and pharmacology”. More data and additional references on the index can be found in [3,4].

The vertex set and the edge set of a graph G are denoted by *V(G)* and *E(G)*, respectively. Let *G* = (*V*,*E*) be a simple graph with vertex set {$v_1, v_2,..., v_n$}. The Randić index of a graph *G* is defined as

$$R(G)=\sum_{uv\in E(G)}\frac{1}{\sqrt{d(u)d(v)}},$$

where *d(u)* denote the degree of vertex *u*.

Obviously, If G be a *r*-regular graph, then $R(G)=\frac{|E(G)|}{r}$.

For example, The Randić index of the cycle C_n, complete graph K_n and complete bipartite graph $K_{n,n}$ are as follows:

$$R(K_n)=\frac{n(n-1)}{2(n-1)}=\frac{n}{2},\ R(K_{n,n})=\frac{n^2}{n}=n \text{ and } R(C_n)=\frac{n}{2}.$$

Let x_{ij} denotes the number of edges with degree *i*, *j*, *resp.* The Randić index can be rewritten as

$$R(G)=\sum_{1\le i\le j\le n-1}\frac{x_{ij}}{\sqrt{ij}}.$$

There are many results concerning Randić index. In [1], Bollobás and Erdös gave the sharp lower bound of $R(G)\ge\sqrt{n-1}$, when *G* is a graph of order *n* without isolated vertices. Yu [10] gave the sharp upper bound of $R(T)\le\frac{n+2\sqrt{2}-3}{2}$, when *T* is a tree of order *n*. Caporossi et al [2] gave another description of the Randić index by using linear programming.

In this paper, we derive formulae of the Randić index of line, subdivision and total graphs for some graphs.

Main results

Firstly, we restate three graphs which constructed from a graph *G*.

Subdivision Graph: The subdivision graph *S(G)* is the graph obtained from *G* by replacing each of its edge by a path of length two, or equivalently, by inserting an additional vertex into each edge of *G*. (See [9]):

Line graph: The line graph of *G*, denoted by *L(G)*, is the intersection graph whose vertices correspond to the edges of *G*, and two vertices of *L*(*G*) are joined by an edge if and only if the corresponding edges in *G* are adjacent.

Total graph: A natural extension of line graphs is the total graph. The total graph *T(G)* is the graph whose vertex set is the set of all elements of *G*, and two vertices are adjacent if and only if the corresponding elements are associated in *G*, or equivalently, The vertex set of *T(G)* is $V(G) \cup E(G)$. Two vertices *x*, *y* in the vertex set of *T(G)* are adjacent in *T(G)* in case one of the following holds: (i) *x*, *y* are in *V(G)* and x is adjacent to *y* in *G*. (ii) *x*, *y* are in *E(G)* and *x*, *y* are adjacent in *G* (iii) *x* is in *V(G)*, *y* is in *E(G)*, and *x*, *y* are incident in *G*.

By the above definitions we have following lemma.

Lemma 2.1. (a) Let $e = uv \in V(L(G) \cap E(G)$. Then $d_{L(G)}(e) = d_G(u) + d_G(v) - 2$.

(b) For every $w \in V(S(G))$, $d_{S(G)}(w) = \begin{cases} d_G(w), & w \in V(G) \\ 2, & w \in E(G) \end{cases}$.

(c) For every $w \in V(T(G))$, $d_{T(G)}(w) = \begin{cases} 2d_G(w), & w \in V(G) \\ d_G(u) + d_G(v), & w = uv \in E(G) \end{cases}$.

Result 2.2. Let *G* be a *r*-regular graph. Then

1. For every $w \in V(T(G))$, $d_{L(G)}(w) = 2(r-1)$ and $|E(L(G))| = (r-1)|E(G)|$.
2. For every $w \in V(S(G))$, $d_{T(G)}(w) = 2$ or r and $|E(S(G))| = 2|E(G)|$.
3. For every $w \in V(T(G))$, $d_{T(G)}(w) = 2r$ and $|E(T(G))| = r(|E(G)| + |V(G)|)$.

Result 2.3. Let *G* be a *r*-regular graph. Then

$R(L(G)) = \frac{|E(G)|}{2}$, $R(S(G)) = \frac{2|E(G)|}{\sqrt{2r}}$ and $R(T(G)) = \frac{|V(G)|+|E(G)|}{2}$.

Example 2.4. $R(L(K_n)) = \frac{n(n-1)}{4}$, $R\left(L(K_{n,n})\right) = \frac{n^2}{2}$ and $R(L(C_n)) = \frac{n}{2}$.

$R(S(K_n)) = \frac{n\sqrt{n-1}}{\sqrt{2}}$, $R\left(S(K_{n,n})\right) = 2n\sqrt{2n}$ and $R(S(C_n)) = n$.

$R(T(K_n)) = \frac{n(n+1)}{4}$, $R\left(T(K_{n,n})\right) = \frac{2n+n^2}{2}$ and $R(T(C_n)) = n$.

Result 2.5. Let *G* be a *r*-regular graph. Then $R(G) + R(L(G)) = R(T(G))$.

Result 2.6. Let *G* be a *r*-regular graph. Then

1. $R(G) < R(L(G)) < R(S(G)) < R(T(G))$, if $r \geq 3$.

2. For $r = 2$, $R(G) = R(L(G))$ and $R\big(T(G)\big) = R\big(S(G)\big) = 2R\big(L(G)\big) = 2R(G)$.

In following we compute Randić index for three graphs, path, star and wheel graph.

The path graph P_n:

Let P_n be a path graph with n vertex. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $E(P_n) = \{e_1, e_2, \dots, e_{n-1}\}$.

Obviously, $R(P_n) = \frac{2}{\sqrt{1\times 2}} + \frac{n-3}{\sqrt{2\times 2}} = \sqrt{2} + \frac{n-3}{2}$.

By the definition of line graph, $V(L(P_n) = E(P_n)$, $E(L(P_n) = \{e_i e_{i+1} | 1 \leq i \leq n-2\}$, so $L(P_n) = P_{n-1}$, thus

$R(L(P_n)) = \sqrt{2} + \frac{n-4}{2}$.

By the definition of subdivision graph, $V(S(P_n)) = \{v_1, v_2, \dots, v_n, e_1, e_2, \dots, e_{n-1}\}$ and

$E(S(P_n) = \{v_i e_i | 1 \leq i \leq n-1\} \cup \{e_i v_{i+1} | 1 \leq i \leq n-1\}$. We have $S(P_n) = P_{2n-1}$, so

$R(S(P_n)) = n - 2 + \sqrt{2}$.

Theorem 2.7. The Randić index of total graph of path, $R(T(P_n)) = \frac{2\sqrt{6}+3\sqrt{2}+4\sqrt{3}}{6} + \frac{4n-13}{4}$.

Proof. By the definition of total graph, $V(T(P_n) = V(P_n) \cup E(P_n)$,

$E(T(P_n) = \{v_i e_i | 1 \leq i \leq n-1\} \cup \{e_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup$

$\{e_i e_{i+1} | 1 \leq i \leq n-2\}$

.

By Lemma 2.1(a), $d_{T(p_n)}(v_1) = d_{T(p_n)}(v_n) = 2$, $d_{T(p_n)}(e_1) = d_{T(p_n)}(e_{n-1}) = 3$ and $d_{T(p_n)}(v_i) = 4$, for

$i \neq 1, n$ and $d_{T(p_n)}(e_i) = 4$, for $i \neq 1, n-1$. Thus

$$R(T(P_n)) = \frac{x_{23}}{\sqrt{2\times 3}} + \frac{x_{24}}{\sqrt{2\times 4}} + \frac{x_{34}}{\sqrt{3\times 4}} + \frac{x_{44}}{\sqrt{4\times 4}} = \frac{2}{\sqrt{6}} + \frac{2}{2\sqrt{2}} + \frac{4}{2\sqrt{3}} + \frac{4n-13}{4}.$$

This proves the result. □

The star graph $K_{1,n}$

Let $V(K_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$ and $E(K_{1,n}) = \{e_1, e_2, \ldots, e_n\}$, where $e_i = vv_i$. Then

$$R(K_{1,n}) = \frac{n}{\sqrt{1 \times n}} = \sqrt{n}.$$

By the definition of line graph, $V(L(K_{1,n}) = E(K_{1,n})$, $E(L(K_{1,n}) = \{e_i e_j | 1 \leq i < j \leq n\}$. So $L(K_{1,n}) = K_n$,

thus $R(L(K_{1,n})) = \frac{n}{2}$.

Theorem 2.8. The Randić index of subdivision graph of star graph, $R(S(K_{1,n})) = \frac{n+\sqrt{n}}{\sqrt{2}}$.

Proof. By the definition of subdivision graph, $V(S(K_{1,n}) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_n\}$,

$E(S(K_{1,n}) = \{v_i e_i | 1 \leq i \leq n\} \cup \{ve_i | 1 \leq i \leq n\}$.

By Lemma 2.1(b), We have $d_{S(K_{1,n})}(v_i) = n$ and $d_{S(K_{1,n})}(e_i) = 2$, for $i = 1,2, \ldots, n$. Thus

$R(S(K_{1,n})) = \frac{x_{21}}{\sqrt{2\times 1}} + \frac{x_{2n}}{\sqrt{2\times n}} = \frac{n}{\sqrt{2}} + \frac{n}{\sqrt{2n}} = \frac{n+\sqrt{n}}{\sqrt{2}}$, as desired. □

Theorem 2.9. The Randić index of total graph of star graph,

$$R(T(K_{1,n})) = \frac{\sqrt{n}}{2} + \frac{n+\sqrt{n}}{\sqrt{2(n+1)}} + \frac{n(n-1)}{2(n+1)}.$$

Proof. By the definition of total graph, $V(T(K_{1,n}) = V(K_{1,n}) \cup E(K_{1,n})$ and

$$E(T(K_{1,n}) = \{v_i e_i | 1 \leq i \leq n\} \cup \{vv_i | 1 \leq i \leq n\} \cup \{ve_i | 1 \leq i \leq n\} \cup \{e_i e_j | 1 \leq i \neq j \leq n\}.$$

By Lemma 2.1(c) We have $d_{T(K_{1,n})}(v) = 2n$, $d_{T(K_{1,n})}(v_i) = 2$ and $d_{T(K_{1,n})}(e_i) = n+1$ for $1 \leq i \leq n$. Also

$x_{2(2n)} = x_{2(n+1)} = x_{(2n)(n+1)} = n$ and $x_{(n+1)(n+1)} = \binom{n}{2}$. Thus

$$R(T(K_{1,n})) = \frac{n}{\sqrt{2\times(n+1)}} + \frac{n}{\sqrt{2\times 2n}} + \frac{n}{\sqrt{2n\times(n+1)}} + \frac{\binom{n}{2}}{\sqrt{(n+1)\times(n+1)}} = \frac{\sqrt{n}}{2} + \frac{n+\sqrt{n}}{\sqrt{2(n+1)}} + \frac{n(n-1)}{2(n+1)}.$$

This proves the result. □

The wheel graph W_n

Let $V(W_n) = \{v, v_1, v_2, \dots, v_n\}$ where $d_{w_n}(v) = n$, $d_{w_n}(v_i) = 3$ for $1 \le i \le n$ and

$E(W_n) = \{e_1, e_2, \dots, e_n, e'_1, e'_2, \dots, e'_n\}$, where $e'_i = vv_i$ and $e_i = v_i v_{i+1}$, for $i = 1,2,\dots,n$ and $v_{n+1} = v_1$.

Thus $R(W_n) = \frac{n}{\sqrt{3\times n}} + \frac{n}{\sqrt{3\times 3}} = \frac{n+\sqrt{3n}}{3}$.

Theorem 2.10. The Randić index of line graph of wheel graph,

$$R(L(W_n)) = \frac{\sqrt{n}}{4} + \frac{n}{\sqrt{n+1}} + \frac{n(n-1)}{2(n+1)}.$$

Proof. By the definition of line graph, $V(L(W_n) = E(W_n)$ and

$E(L(W_n) =$

$\{e_i e_{i+1} | 1 \le i \le n, e_{n+1} = e_1\} \cup \{e'_i e_i, e'_i e_{i-1} | 1 \le i \le n, e_0 = e_n\} \cup \{e'_i e'_j | 1 \le i \ne j \le n\}$

According to Lemma 2.1(a), We have $d_{S(W_n)}(e_i) = 4$ and $d_{S(W_n)}(e'_i) = n+1$ for $1 \le i \le n$.

On other hand, $x_{44} = n$, $x_{4(n+1)} = 2n$ and $x_{(n+1)(n+1)} = \binom{n}{2}$. Thus

$$R(L(W_n)) = \frac{n}{\sqrt{4\times 4}} + \frac{2n}{\sqrt{4\times(n+1)}} + \frac{\binom{n}{2}}{\sqrt{(n+1)\times(n+1)}} = \frac{\sqrt{n}}{4} + \frac{n}{\sqrt{n+1}} + \frac{n(n-1)}{2(n+1)},$$

as desired. □

Theorem 2.11. The Randić index of subdivision graph of wheel graph, $R(S(W_n)) = \frac{n\sqrt{6}+\sqrt{2n}}{2}$.

Proof. By the definition of subdivision graph,

$$V(W_n) = \{v, v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_n, e'_1, e'_2, \ldots, e'_n\},$$

$$E(S(W_n)) = \{v_i e_i, e_i v_{i+1}, v_i e'_i | 1 \leq i \leq n\} \cup \{v e'_i | 1 \leq i \leq n\}.$$

By Lemma 2.1(b), $d_{S(W_n)}(e_i) = d_{S(W_n)}(e'_i) = 2$, $d_{S(W_n)}(v_i) = 3$ and $d_{S(W_n)}(v) = n$. Thus

$$R(S(W_n)) = \frac{x_{23}}{\sqrt{2\times 3}} + \frac{x_{2n}}{\sqrt{2\times n}} = \frac{3n}{\sqrt{6}} + \frac{n}{\sqrt{2n}}.$$

The result now follows. □

Theorem 2.12. The Randić index of total graph of wheel graph,

$$R(T(W_n)) = \frac{(4n+\sqrt{3n})}{6} + \frac{(3\sqrt{2n}+3n\sqrt{6})}{6\sqrt{(n+3)}} + \frac{n(n-1)}{2(n+3)}.$$

Proof. By the definition of total graph, $V(T(W_n) = V(W_n) \cup E(W_n)$ and

$E(T(W_n) = \{v_i v_{i+1}, v_i e_i, e_i v_{i+1}, e_i e_{i+1} | 1 \leq i \leq n\} \cup \{v v_i | 1 \leq i \leq n\} \cup \{v e'_i | 1 \leq i \leq n\} \cup$

$\{v_i e'_i | 1 \leq i \leq n\} \cup \{e'_i e_i, e'_i e_{i-1} | 1 \leq i \leq n, e_0 = e_n\} \cup \{e'_i e'_j | 1 \leq i \neq j \leq n\}$

By Lemma 2.1(c) $d_{T(W_n)}(v) = 2n$, $d_{T(W_n)}(v_i) = d_{T(W_n)}(e_i) = 6$ and $d_{T(W_n)}(e'_i) = n + 3$ for $1 \leq i \leq n$.

Also $x_{66} = 4n$, $x_{6(2n)} = x_{6(n+3)} = 2n$, $x_{(2n)(n+3)} = n$, and $x_{(n+3)(n+3)} = \binom{n}{2}$.Thus

$$R(T(W_n)) = \frac{4n}{\sqrt{6\times 6}} + \frac{n}{\sqrt{6\times 2n}} + \frac{n}{\sqrt{2n\times(n+3)}} + \frac{3n}{\sqrt{6\times(n+3)}} + \frac{\binom{n}{2}}{\sqrt{(n+3)\times(n+3)}}.$$

This proves the result. □

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