ISSN 2347-1921

Hermite – Lagrange Interpolation on a Unit Circle

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ABSTRACT:

In this paper , we consider the explicit representation and convergence of Hermite-Lagrange interpolation on two disjoint sets of nodes, which are obtained by projecting vertically the zeros of $\left(1\text{-}x^2\right)P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha,\beta)\prime}(x)$ respectively on the unit circle , where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial.

Mathematics subject classification: 41A05, 30E10.

Keywords: Jacobi polynomial; explicit representation; Convergence.

Council for Innovative Research
Peer Review Research Publishing System
Journal: Journal of Advances in Mathematics
Vol 9, No 1
editor@cirjam.org
www.cirjam.com, www.cirworld.com

ISSN 2347-1921

1. INTRODUCTION:

In a paper, T. N. T. Goodman and A. Sharma [8] considered convergence and divergence behaviour of Hermite interpolation in the circle of radius $\rho^{\frac{3}{2}}$. In addition, T. N. T .Goodman, K.G. Ivanov and A. Sharma [9] considered the behaviour of the Hermite interpolant in the roots of unity. In 1998, S. Bahadur and K. K. Mathur [2] proved the convergence of Quasi - Hermite interpolation on the nodes obtained by projecting vertically the zeros of $(1-x^2)P_n(x)$on the unit circle, where $P_n(x)$ stands for n[th] Legendre polynomial. Later on, the authors [6] considered the convergence of Hermite interpolation.

Earlier, in 1975 L.G. Pál [13] proved that, when function values are prescribed on one set of n points and derivative values on another set of n-1 points, then there exists no unique polynomial of degree ≤ 2n-2, but by prescribing function value at one more point not belonging to the former set of n points, there exists a unique polynomial of degree $\leq 2n-1$. Later on, many authors ([3] - [16]) have dealt with the above method of interpolation on the various sets of nodes.

In 2006, M. Lenard [11] considered the weighted (0, 2) Pál-type interpolation problems on the zeros of Legendre polynomial $P_n(x)$ and gave the explicit formulae. In another paper, first author considered ([3] [4]) (0,1;0) and (0; 0,1) interpolation problem for the vertically projected zeros of the Legendre polynomial on a unit circle. Also, P. Mathur [12] considered (0,1; 0) interpolation on infinite interval. Many other mathematicians also worked in the same direction. This has motivated us to consider Hermite -Lagrange interpolation on the unit circle. In this paper, we consider two pair wise disjoint sets $\{z_k\}_{k=0}^{2n+1}$ and $\{t_k\}_{k=1}^{2n-2}$ which are the vertically projected zeros of $(1-x^2)P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha,\beta)\prime}(x)$ respectively on the unit circle.

Let $\boldsymbol{Z_n} = \{z_k : k = 0(1)2n+1\}$ satisfying:

$$(1.1)\qquad \begin{cases} Z_n = \{z_0 = 1, z_{2n+1} = -1, \\ \qquad\qquad z_k = \cos\theta_k + i\sin\theta_k, z_{n+k} = -z_k, k = 1(1)n\} \end{cases}$$

$\boldsymbol{T_n} = \{t_k : k = 1(1)2n-2\}$ such that

$$(1.2)\qquad \begin{cases} T_n = \{t_0 = 1, t_{2n-1} = -1, \\ \qquad\qquad t_k = \cos\phi_k + i\sin\phi_k, t_{n+k} = -t_k, k = 1(1)n-1 \end{cases}$$

In section 2, we give some preliminaries and in section 3, we describe the problem and obtained the regularity of the same. In section 4, we give the explicit formulae of the interpolatory polynomials. In section 5 and 6, estimation of interpolatory polynomials and convergence are given respectively.

2. PRELIMINARIES:

In this section, we shall give some well-known results, which we shall use.

The differential equation satisfied by $P_n^{(\alpha,\beta)}(x)$ is

$$(2.1)\ (1-x^2)P_n^{(\alpha,\beta)''}(x)+[\beta-\alpha-(\alpha+\beta+2)x]P_n^{(\alpha,\beta)'}(x)+n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)=0$$

$$(2.2)\qquad W(z)=\prod_{k=1}^{2n}(z-z_k)=K_nP_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right)z^n$$

$$(2.3)\qquad H(z)=\prod_{k=1}^{2n-2}(z-t_k)=K_n^*P_n^{(\alpha,\beta)'}\left(\frac{1+z^2}{2z}\right)z^{n-1}$$

We shall require the fundamental polynomials of Lagrange interpolation based on the nodes as zeroes of R(z) and W(z) are given by:

$$(2.4)\qquad L_k(z)=\frac{R(z)}{R'(z_k)(z-z_k)}, k=0(1)2n+1$$

where, $R(z)=(z^2-1)W(z)$

ISSN 2347-1921

(2.5) $$l_k(z) = \frac{H(z)}{H'(t_k)(z - t_k)}, k = 1(1)2n - 2$$

We will also use the following results

(2.6) $$(-1)^n W'(z_{n+k}) = W'(z_k) = -\frac{1}{2} K_n P_n^{(\alpha,\beta)'}(x_k)(1 - z_k^2) z_k^{n-2}, k = 1(1)n$$

We will also use the following well-known inequalities (see [8])

(2.7) $$(1 - x^2)^{\frac{1}{2}} P_n^{(\alpha,\beta)}(x) = o(n^{\alpha-1}) \text{for} \quad x \in [-1,1]$$

(2.8) $$\left(1 - x_k^2\right)^{-1} \sim \left(\frac{k}{n}\right)^{-2}$$

(2.9) $$\left|P_n^{(\alpha,\beta)'}(x_k)\right| \sim k^{-\alpha-\frac{3}{2}} n^{\alpha+2}$$

(2.10) $$\left|P_n^{(\alpha,\beta)}(x)\right| = o(n^{\alpha}),$$

(2.11) $$(1 - x^2)\left|P_n^{(\alpha,\beta)'}(x)\right| \leq cn^{\alpha+1},$$

(2.12) $$\left|P_n^{(\alpha,\beta)}(x_k)\right| \sim k^{-\alpha-\frac{1}{2}} n^{\alpha}$$

(2.13) $$\left|P_n^{(\alpha,\beta)'}(x)\right| = o(n^{\alpha+2})$$

3. THE PROBLEM

Let $\{z_k\}_{k=0}^{2n+1}$ and $\{t_k\}_{k=1}^{2n-2}$ be the two disjoint set of nodes obtained by projecting vertically the zeros of the $(1\text{-}x^2)P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha,\beta)'}(x)$ on the unit circle respectively, we determine the interpolatory polynomials $Q_n(z)$ of degree $\leq 6n+1$ satisfying the conditions:

(3.1) $$\begin{cases} Q_n(z_k) = \alpha_k, \; k = 0(1)2n+1 \\ Q'_n(z_k) = \beta_k, \; k = 0(1)2n+1 \\ Q_n(t_k) = \alpha_k^*, \; k = 1(1)2n-2 \end{cases}$$

where α_k , β_k *and* α_k^* are arbitrary complex numbers. In addition, we are interested in establishing the convergence theorem for the same.

Regularity:

Theorem 1: Hermite - Lagrange interpolation is regular on Z_n and T_n .

Proof: It is sufficient, if we show the unique solution of (3.1) is $Q_n(z) \equiv 0$, when all data $\alpha_k = \beta_k = \alpha_k^* = 0$. Clearly, in this case, we have $Q_n(z) = R(z)H(z)q(z)$, where q(z) is a polynomial of degree $\leq$ 2n +1. Obviously, $Q_n(z_k) = 0$ and $Q_n(t_k) = 0$. Then from $Q_n'(z_k) = 0$, we get $q(z_k) = 0$. Therefore, we have $q(z) = (az+b)W(z)$, where a and b are arbitrary constants. As, $q(\pm 1) = 0$, we get

$(-a+b)W(-1) = 0$

$(a+b)W(1) = 0$ as $W(1) = W(-1) = K_n$. We get $a = b = 0$

It indicates $Q_n(z) \equiv q(z) \equiv 0$. Hence, the theorem follows.

4. EXPLICIT REPRESENTATION OF INTERPOLATORY POLYNOMIALS

We shall write $Q_n(z)$ satisfying (3.1) as:

(4.1) $$Q_n(z) = \sum_{k=0}^{2n+1} \alpha_k A_k(z) + \sum_{k=0}^{2n+1} \beta_k B_k(z) + \sum_{k=1}^{2n-2} \alpha_k^* C_k(z)$$

where ,$A_k(z)$, $B_k(z)$ and $C_k(z)$ are unique fundamental polynomial each of degree atmost 6n+1 determined by the following conditions:

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For *k= 0 (1) 2n+1*

(4.2) $$\begin{cases} A_k(z_j) = \delta_{jk}\ ,\ j = 0(1)2n+1 \\ A_k'(z_j) = 0\ ,\ j = 0(1)2n+1 \\ A_k(t_j) = 0\ ,\ j = 1(1)2n-2 \end{cases}$$

For k= 0(1)2n+1

(4.3) $$\begin{cases} B_k(z_j) = 0\ ,\ j = 0(1)2n+1 \\ B_k'(z_j) = \delta_{jk}\ ,\ j = 0(1)2n+1 \\ B_k(t_j) = 0\ ,\ j = 1(1)2n-2 \end{cases}$$

For k= 1(1)2n-2

(4.4) $$\begin{cases} C_k(z_j) = 0\ ,\ j = 0(1)2n+1 \\ C_k'(z_j) = 0\ ,\ j = 0(1)2n+1 \\ C(t_j) = \delta_{jk}\ ,\ j = 1(1)2n-2 \end{cases}$$

Theorem 2: For k=1(1)2n-2, we have*,*

(4.5**)** $$C_k(z) = \frac{R^2(z)l_k(z)}{R^2(t_k)}$$

Theorem 3: For k=0(1)2n+1*,* we have,

(4.6) $$B_k(z) = \frac{(z^2-1)R(z)H(z)L_k(z)}{(z_k^2-1)R'(z_k)H(z_k)}$$

Theorem 4: For k=0(1)2n+1, we have,

(4.7) $$A_k(z) = \frac{H(z)L_k^2(z)}{H(z_k)} - \left[\frac{H'(z_k)}{H(z_k)} + 2L'_k(z_k)\right]B_k(z)$$

One can prove theorem 2, 3 and 4 owing to (4.4), (4.3) and (4.2) respectively.

5. ESTIMATION OF FUNDAMENTAL POLYNOMIALS:

Lemma 1: [6] Let $L_k(z)$ be given by (2.4). Then

(5.1) $$\max_{|z|=1} \sum_{k=0}^{2n+1} |L_k(z)| \le c \log n$$

where c is a constant independent of n and z.

***Lemma 2:* Let** $l_k(z)$ be given by (2.5). Then

(5.2) $$\max_{|z|=1} \sum_{k=1}^{2n-2} |l_k(z)| \le \sum_{k=1}^{2n-2} k^{\alpha+\frac{1}{2}}$$

where c is a constant independent of n and z.

Proof: From maximal principal, we know

$$\lambda_n = \max_{|z|=1} \lambda_n(z)$$

$$\lambda_n = \sum_{k=1}^{2n-2} |l_k(z)|$$

Let $z = x + iy$ and $|z| = 1$, then we know, for $0 \le \arg z < \pi$ and $k = 1,2,\ldots\ldots\ldots,n-1$.

$$|l_k(z)| = \left|\frac{H(z)}{H'(t_k)(z-t_k)}\right|$$

Using (2.5) and $|z| = 1$ and after some computation, we get

$$\left|l_k(z)\right| = \left|\frac{(1-x^2)^{\frac{1}{2}} P_n^{(\alpha,\beta)'}(x)\left[1-xu_k+(1-x^2)^{\frac{1}{2}}(1-u_k^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}}{\sqrt{2}(1-u_k^2)P_n^{(\alpha,\beta)'}(u_k)(x-u_k)}\right|$$

$$\leq \frac{(1-x^2)^{\frac{1}{2}}\left|P_n^{(\alpha,\beta)'}(x)\right|(1-xu_k)^{\frac{1}{2}}}{(1-u_k^2)\left|P_n^{(\alpha,\beta)'}(u_k)\right|(x-u_k)} = G_k(x)$$

Also, $\left|l_{n+k}(z)\right| \leq G_k(x)$

Similarly, for $\pi \leq \arg z < 2\pi$ and k=1, 2, … ,n-1

$\left|l_k(z)\right| \leq G_k(x)$, $\left|l_{n+k}(z)\right| \leq G_k(x)$

Therefore, for a fixed $z = x + iy$ and $|z| = 1$ and $-1 < x < 1$

$$\lambda_n(z) \leq 2\sum_{k=1}^{n} G_k(x) + |l_0(z)| + |l_{2n+1}(z)|$$

$$= 2\sum_{|u_k-x|\geq\frac{1}{2}(1-u_k^2)^{\frac{1}{2}}} G_k(x) + 2\sum_{|u_k-x|<\frac{1}{2}(1-u_k^2)^{\frac{1}{2}}} G_k(x) + 2$$

Using (2.12) and (2.13), we get the required result.

Lemma 3: For $C_k(z)$ be given by (4.5) and $|z| \leq 1$, we have

(5.3) $\sum_{k=1}^{2n-2}|C_k(z)| \leq c\log n \quad , \quad \alpha,\beta \leq -\frac{1}{2}$

where ,c is a constant independent of n and z.

Proof: $\sum_{k=1}^{2n-2}|C_k(z)| = \left|\frac{R^2(z)l_k(z)}{R^2(t_k)}\right| = \frac{(z^2-1)^2W^2(z)l_k(z)}{(t_k^2-1)^2W^2(t_k)}$

Using (2.8), (2.10), (2.12) and Lemma 2, we get the required result.

Lemma 4: For $B_k(z)$ be given by (4.6) and $|z| \leq 1$, we have

(5.4) $\sum_{k=0}^{2n+1}|B_k(z)| \leq \frac{c\log n}{n}; \ \alpha,\beta \leq -\frac{1}{2}$

where c is a constant independent of n and z.

Proof: Proof of this Lemma is similar to Lemma 3.

Lemma 5: For $A_k(z)$ be given by (4.7) and $|z| \leq 1$, we have

(5.5**)** $\sum_{k=0}^{2n+1}|A_k(z)| \leq c\log n \quad , where \quad \alpha,\beta \leq -\frac{1}{2}$

where, c is a constant independent of n and z.

Proof: For $|z| \leq 1$, we get

(5.6) $\sum_{k=0}^{2n+1}\frac{H(z)L_k^2(z)}{H(z_k)} \leq \sum_{k=0}^{2n+1}\frac{c}{k^{-3\alpha+\frac{3}{2}}}$

and

(5.7) $\sum_{k=0}^{2n+1}\left|\frac{H'(z_k)}{H(z_k)} + 2L'_k(z_k)\right| \leq cn$

Using (5.6) and (5.7) in (4.7), we get the required result.

6. CONVERGENCE:

Let *f (z)* be continuous in $|z| \leq 1$ and analytic in $|z| < 1$ and $\omega(f,\delta)$ be the modulus of continuity of $f(e^{ix})$.

Theorem 5: Let *f (z)* be continuous in $|z| \leq 1$ and analytic in $|z| < 1$. Let the arbitrary numbers β_k's be such that:

(6.1) $\quad |\beta_k| = o(n\omega(f, n^{-1}))\ , \ k = 0(1)2n+1$

Then $\{Q_n\}$ be defined by:

(6.2) $$Q_n(z) = \sum_{k=0}^{2n+1} f(z_k) A_k(z) + \sum_{k=0}^{2n+1} \beta_k B_k(z) + \sum_{k=1}^{2n-2} f(t_k) C_k(z)$$

satisfies the relation

(6.3) $$|Q_n(z) - f(z)| = o(\omega(f, n^{-1}) \log n)$$

where, $\omega(f, n^{-1})$ is the modulus of continuity of *f(z).*

Remark:

To prove theorem 5, we shall need the following:

Let *f (z)* be continuous in $|z| \leq 1$ and analytic in $|z| < 1$. Then there exists a polynomial $F_n(z)$ of degree 6n+1 satisfying Jackson's inequality

(6.4) $$|f(z) - F_n(z)| \leq c\omega(f, n^{-1}) \quad , \quad z = e^{i\theta}(0 \leq \theta < 2\pi)$$

And also an inequality due to O. Kiš [3]

(6.5) $$|F_n^{(m)}(z)| \leq cn^m \omega(f, n^{-1}) \quad , \quad \text{for } m \in I^+$$

Proof: Since $Q_n(z)$be given by (6.2) is a uniquely determined polynomial of degree $\leq 6n+1$, the polynomial $F_n(z)$ satisfying (6.4) and (6.5) can be expressed as:

$$F_n(z) = \sum_{k=0}^{2n+1} F_n(z_k) A_k(z) + \sum_{k=0}^{2n+1} F_n{}'(z_k) B_k(z) + \sum_{k=1}^{2n-2} F_n(t_k) C_k(z)$$

Then,

$$|Q_n(z) - f(z)| \leq |Q_n(z) - F_n(z)| + |F_n(z) - f(z)|$$

$$\leq \sum_{k=0}^{2n+1} |f(z_k) - F_n(z_k)|\,|A_k(z)| + \sum_{k=0}^{2n+1} \{|\beta_k| + |F_n{}'(z_k)|\}\,|B_k(z)| + \sum_{k=1}^{2n-2} |f(t_k) - F_n(t_k)|\,|C_k(z)|$$
$$+ |F_n(z) - f(z)|$$

Using $z = e^{i\theta}(0 \leq \theta < 2\pi)$, (6.1), (6.4), (6.5) and Lemma 2, 3 and 5, we get (6.3).

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