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Asymptotic Behavior of Third Order Nonlinear Difference Equations with Mixed Arguments

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Abstract

In this paper, we established criteria for asymptotic properties of nonlinear difference equation with mixed arguments of the form

$$\Delta^2\left(a_n\left(\Delta x_n\right)^{\alpha}\right)+q_n f\left(x_{n-\ell}\right)+p_n h\left(x_{n+m}\right)=0,\quad n\in N_0$$

where $\{a_n\}$, $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences, α is a ratio of odd positive integer, and ℓ and m are positive integers. We deduce the properties of studied equation by establishing new comparison theorem, so that some asymptotic properties of nonoscillatory solutions are resulted from the oscillation of a set of first order difference equations. Some examples are provided to illustrate the main results.

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1. Introduction

In this paper we are concerned with the following third order nonlinear difference equation with mixed arguments of the form

$$\Delta^2\left(a_n\left(\Delta x_n\right)^{\alpha}\right)+q_n f\left(x_{n-\ell}\right)+p_n h\left(x_{n+m}\right)=0,\quad n\in N_0 \tag{1.1}$$

Where $N_0=\{n_0,n_0+1,n_0+2,...\}$, n_0 is a nonnegative integer, ℓ and m are positive integers. $f,h:R\to R$ is continuous and nondecreasing and Δ is the forward difference operator defined by $\Delta x_n = x_{n+1}-x_n$, subject to the following conditions.

(H_1) α is a ratio of odd positive integers;

(H_2) $\{a_n\}$ is a positive real nondecreasing sequences;

(H_3) $\{P_n\}$ and $\{q_n\}$ are nonnegative real sequences;

(H_4) $uf(u)>0$ and $uh(u)>0$ for $u\neq 0$;

(H_5) $-f(-uv)\geq f(uv)\geq f(u)f(v)$ for $uv>0$;

(H_6) $-h(-uv)\geq h(u)h(v)$ *for* $uv>0$.

Let $\theta=\max\{\ell,m\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ which is defined for $n\geq n_0-\theta$ and satisfies equation (1.1) for all $n\in N_0$. A nontrivial solution $\{x_n\}$ of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

The oscillatory behavior of solutions of third order difference equations with or without delay have been investigated by several authors, see for example [3, 4, 5, 7, 13], and the references quoted therein.

In [8, 12], the authors studied the oscillatory behavior of third order difference equation with mixed arguments. Following this trend in this paper we discuss the nonoscillatory behavior of equation (1.1) by comparing it with first order delay / advance difference equation. In Section 2, we establish some results on the nonoscillatory properties of equation (1.1) and in Section 3, we provide some examples to illustrate the main results.

2. Main Results

Through the paper it is assumed that

$$R_n=\sum_{s=n_0}^{\infty}a_s^{-1/\alpha}=\infty. \tag{2.1}$$

It is convenient to prove our main results by means of a series of lemmas as follows.

Lemma 2.1. *Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Then $\{x_n\}$ satisfies one of the following*

(I) $x_n\Delta x_n>0,\ x_n\Delta\left(a_n\left(\Delta x_n\right)^{\alpha}\right)>0$ *and* $x_n\Delta^2\left(a_n\left(\Delta x_n\right)^{\alpha}\right)<0;$

(II) $x_n\Delta x_n<0,\ x_n\Delta\left(a_n\left(\Delta x_n\right)^{\alpha}\right)>0$ *and* $x_n\Delta^2\left(a_n\left(\Delta x_n\right)^{\alpha}\right)<0,$

eventually.

Proof: Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1) and assume without loss of generality $x_n>0$ for all $n\geq n_0$. It follows from (1.1),

$$\Delta^2\left(a_n\left(\Delta x_n\right)^{\alpha}\right)<0.$$

Thus $\Delta\left(a_n\left(\Delta x_n\right)^\alpha\right)$ is decreasing and of fixed sign. We claim that $\Delta\left(a_n\left(\Delta x_n\right)^\alpha\right)>0$ for all $n\geq n_0$. If $\Delta\left(a_n\left(\Delta x_n\right)^\alpha\right)<0$ for $n\geq n_1\geq n_0$, then from (1.1) and condition (2.1), we have $a_n\left(\Delta x_n\right)^\alpha<0$ which implies $x_n<0$, a contradiction. Therefore we conclude that $\Delta\left(a_n\left(\Delta x_n\right)^\alpha\right)>0$ eventually. Hence $a_n\left(\Delta x_n\right)^\alpha$ is of fixed sign for all n. Thus either case (I) or case (II) holds.

Definition 2.1. The equation (1.1) has property (A) if every nonoscillatory solution $\{x_n\}$ of equation (1.1) satisfies case (II) of Lemma 2.1.

Lemma 2.2. *Suppose* $\{p_n\}$ *and* $h(u)$ *satisfies* (H_3) *and* (H_4) *respectively. If the first order advanced difference inequality*

$$\Delta z_n - p_n h\left(z_{n+m}\right)\geq 0 \tag{2.2}$$

has an eventually positive solution, then so does the advanced difference equation

$$\Delta z_n - p_n h\left(z_{n+m}\right)=0. \tag{2.3}$$

Proof: Let $\{z_n\}$ be a positive solution of (2.2) for all $n\geq N\in N_0$. Then z_n satisfies the inequality

$$z_n\geq z_N+\sum_{s=N}^{n-1}p_s h\left(z_{s+m}\right).$$

Let $y_1(n)=z_n$ and

$$y_k(n)\geq z_N+\sum_{s=N}^{n-1}p_s h\left(y_{k-1}(S+m)\right),\ k=2,3,..$$

From the definition of $y_k(n)$ and (H_4) that the sequence $\{y_k(n)\}$ has the property

$$z_n=y_1(n)\geq y_2(n)\geq y_3(n)\geq\ldots\geq z_N,\quad n\geq N.$$

Hence $\lim_{k\to\infty}y_k(n)=y_n$, where $z_n\geq y_n\geq z_N$, $n\geq N$. Let $h_k(n)=p_n h\left(y_k(n+m)\right), k=1,2,\ldots$. Then $h_1(n)\geq h_2(n)\geq\ldots\geq 0$. Since $h_1(n)$ is summable on $[N,n]$ and $\lim_{k\to\infty}h_k(n)=p_n h\left(y_{n+m}\right)$. By Lebegue dominated convergence theorem

$$y_n=z_N+\sum_{s=N}^{n-1}p_s h\left(y_{s+m}\right).$$

Thus y_n satisfies (2.3).

Lemma 2.3. *Assume* $A\geq 0,\ B\geq 0,\ \gamma\geq 1$ *then*

$$(A+B)^\gamma\geq A^\gamma+B^\gamma.$$

Lemma 2.4. *Assume* $A\geq 0,\ B\geq 0,\ \gamma\geq 1$ *then*

$$(A+B)^\gamma\geq\frac{A^\gamma+B^\gamma}{2^{1-\gamma}}.$$

The proof of Lemmas 2.3 and 2.4 can be found in [12].

Lemma 2.5. *Assume* $z_n>0,\ \Delta z_n>0,$ *and* $\Delta\left(a_n\left(\Delta z_n\right)^\alpha\right)>0$ *eventually. Then for arbitrary* $k\in(0,1)$

$$\frac{z_{n+m}}{z_n} \geq k\frac{R_{n+m}}{R_n} \tag{2.4}$$

eventually.

Proof: Let $w_n = a_n(\Delta z_n)^\alpha$. Then, we have

$$z_{n+m} - z_n = \sum_{s=n}^{n+m-1} w_s^{1/\alpha} a_s^{-1/\alpha} \geq w_n^{1/\alpha} \sum_{s=n}^{n+m-1} a_s^{-1/\alpha} \geq w_n^{1/\alpha}(R_{n+m} - R_n)$$

or

$$\frac{z_{n+m}}{z_n} \geq 1 + \frac{w_n^{1/\alpha}}{z_n}(R_{n+m} - R_n). \tag{2.5}$$

On the otherhand, since $z_n \to \infty$ as $n \to \infty$, then for any $k \in (0,1)$ and N as large as possible, such that

$$kz_n - \leq z_n - z_N = \sum_{s=N}^{n-1} \Delta z_s \leq \sum_{s=N}^{n-1} w_s^{1/\alpha} a_s^{-1/\alpha} \leq w_n^{1/\alpha} \sum_{s=N}^{n-1} a_s^{-1/\alpha}.$$

Therefore

$$\frac{w_n^{1/\alpha}}{z_n} \geq \frac{k}{R_n}. \tag{2.6}$$

Using (2.6) in (2.5), we get

$$\frac{z_{n+m}}{z_n} \geq 1 + \frac{k}{R_n}(R_{n+m} - R_n) \geq k\frac{R_{n+m}}{R_n}.$$

The proof is now complete.

Next we establish some properties of nonoscillatory solutions of equation (1.1). Define

$$Q_1(n) = a_n^{-1/\alpha}(n - N_1)^{1/\alpha}\left[\sum_{s=n}^{\infty} q_{s+\ell}\right]^{1/\alpha}, \tag{2.7}$$

$$P_1(n) = a_n^{-1/\alpha}(n - N_1)^{1/\alpha}\left[\sum_{s=n}^{\infty} p_s\right]^{1/\alpha}, \tag{2.8}$$

and

$$E_1(n) = \prod_{s=N_1}^{n}(1 + Q_1(s)). \tag{2.9}$$

Theorem 2.1. *Let* $0 < \alpha \leq 1$, *assume that* (H_6) *holds and*

$$\frac{f(u)}{u^\alpha} \geq 1 \text{ for } u \neq 0. \tag{2.10}$$

If the first order advanced difference equation

$$\Delta z_n - P_1(n)(E_1(n))^{-1} h^{1/\alpha}(E_1(n+m-1)) h^{1/\alpha}(z_{n+m}) = 0 \tag{2.11}$$

Is oscillatory then equation (1.1) has property (A).

Proof: Assume the contrary, let $\{x_n\}$ be a nonoscillatory solution of equation (1.1) satisfying case (I) of Lemma 2.1.

We assume that $x_n > 0$ for $n \geq N$. Summing equation (1.1) from n to ∞ and using (2.10), we obtain

$$\Delta\left(a_n(\Delta x_n)^\alpha\right) \geq \sum_{s=n}^{\infty} q_s x_{s-\ell}^\alpha + \sum_{s=n}^{\infty} p_s h(x_{s+m})$$

$$\geq \sum_{s=n}^{\infty} q_s x_{s-\ell}^\alpha + h(x_{n+m}) \sum_{s=n}^{\infty} p_s . \tag{2.12}$$

On the otherhand we substitute $s-\ell = u$, we have

$$\sum_{s=n}^{\infty} q_s x_{s-\ell}^\alpha = \sum_{u=n-\ell}^{\infty} q_{u+\ell} x_u^\alpha \geq \sum_{s=n}^{\infty} q_{s+\ell} x_s^\alpha \geq x_n^\alpha \sum_{s=n}^{\infty} q_{s+\ell} .$$

From (2.12) and the last inequality, we have

$$\Delta\left(a_n(\Delta x_n)^\alpha\right) \geq x_n^\alpha \sum_{s=n}^{\infty} q_{s+\ell} + h(x_{n+m}) \sum_{s=n}^{\infty} p_s \tag{2.13}$$

Since $\Delta\left(a_n(\Delta x_n)^\alpha\right)$ is decreasing, we have

$$a_n(\Delta x_n)^\alpha \geq \sum_{s=N_1}^{n-1} \Delta\left(a_s(\Delta x_s)^\alpha\right) \geq \Delta\left(a_n(\Delta x_n)^\alpha\right)(n-N_1). \tag{2.14}$$

Combining (2.13) and (2.14) we obtain

$$a_n(\Delta x_n)^\alpha \geq x_n^\alpha (n-N_1) \sum_{s=n}^{\infty} q_{s+\ell} + h(x_{n+m})(n-N_1) \sum_{s=n}^{\infty} p_s. \tag{2.15}$$

By Lemma 2.3, we have

$$\Delta x_n \geq x_n a_n^{-1/\alpha} (n-N_1)^{1/\alpha} \left(\sum_{s=n}^{\infty} q_{s+\ell}\right)^{1/\alpha} + h^{1/\alpha}(x_{n+m}) a_n^{-1/\alpha} (n-N_1)^{1/\alpha} \left(\sum_{s=n}^{\infty} p_s\right)^{1/\alpha}$$

or

$$\Delta x_n \geq Q_1(n) x_n + P_1(n) h^{1/\alpha}(x_{n+m}).$$

By setting $x_n = z_n \prod_{s=N_1}^{n-1}(1+Q_1(s))$, we can easily verify that z_n is the positive solution of the advanced difference inequality

$$\Delta z_n - P_1(n)(E_1(n))^{-1} h^{1/\alpha}(E_1(n+m-1)) h^{1/\alpha}(z_{n+m}) \geq 0.$$

By Lemma 2.2, the corresponding difference equation inequality (2.11) has a positive solution, which is a contradiction. This completes the proof.

Corollary 2.1. Let $0 < \alpha \leq 1$. *Assume that* (H_6) *and (2.10) hold. If*

$$\frac{h^{1/\alpha}(u)}{u} \geq 1, \ |u| \geq 1, \tag{2.16}$$

and

$$\liminf_{n\to\infty} \sum_{s=n}^{n+m-1} P_1(s)(E_1(s))^{-1} E_1(s+m-1) > \left(\frac{m}{m+1}\right)^{m+1} \tag{2.17}$$

then equation (1.1) has property (A).

Proof: We see that (2.17) implies

$$\sum_{s=n}^{\infty} P_1(s)\left(E_1(s)\right)^{-1} E_1\left(s+m-1\right)=\infty. \tag{2.18}$$

By Theorem 2.1, it is sufficient to prove that equation (2.11) is oscillatory. Suppose, let (2.11) has a positive solution z_n. Then $\Delta z_n > 0$ and $z_{n+m} > c > 0.$

Summing (2.11) from N_1 to $n-1$ we obtain inview of (2.16)

$$z_n \geq \sum_{s=N_1}^{n-1} P_1(s)\left(E_1(s)\right)^{-1} h^{1/\alpha}\left(E_1\left(s+m-1\right)\right)h^{1/\alpha}\left(z_{s+m}\right)$$
$$\geq c\sum_{s=N_1}^{n-1} P_1(s)\left(E_1(s)\right)^{-1} E_1\left(s+m-1\right).$$

By (2.18) implies

$$\lim_{n\to\infty} z_n = \infty.$$

Using (2.16) in (2.11) we have $\{z_n\}$ is a positive solution of difference inequality

$$\Delta z_n - P_1(n)\left(E_1(n)\right)^{-1} E_1\left(n+m-1\right) z_{n+m} \geq 0. \tag{2.19}$$

On the otherhand Corollary 2.2 in [6] and (2.17) guarantees that (2.19) has no positive solution. This contradiction shows that equation (1.1) has property (A).

Theorem 2.2. *Let* $\alpha \geq 1$*. Assume condition* (H_6) and (2.10) hold. If the first order advanced difference equation

$$\Delta z_n - P_1\left(n\right)\left(2^{(1-\alpha)/\alpha} E_1\left(n\right)\right)^{-1} h^{1/\alpha}\left(2^{(1-\alpha)/\alpha} E_1\left(n+m-1\right)\right)h^{1/\alpha}\left(z_{n+m}\right)=0 \tag{2.20}$$

is oscillatory then equation (1.1) has property (A).

Proof: Suppose $\{x_n\}$ is an eventually positive solution of equation (1.1) satisfies case (I) of Lemma 2.1. Then from inequality (2.15) and Lemma 2.3, we have

$$\Delta x_n \geq 2^{(1-\alpha)/\alpha}\left[Q_1(n)x_n + P_1(n)h^{1/\alpha}(x_{n+m})\right]. \tag{2.21}$$

Denote $x_n = z_n 2^{(1-\alpha)/\alpha}\prod_{s=N_1}^{n-1}(1+Q_1(s)).$ It is easy to see that z_n is the positive solution of the advanced difference inequality

$$\Delta z_n - P_1\left(n\right)\left(2^{(1-\alpha)/\alpha} E_1\left(n\right)\right)^{-1} h^{1/\alpha}\left(2^{(1-\alpha)/\alpha} E_1\left(n+m-1\right)\right)h^{1/\alpha}\left(z_{n+m}\right)\geq 0.$$

By Lemma 2.2 we deduce that the corresponding difference equation inequality (2.19) has a positive solution, which is a contradiction. Hence the proof is complete.

Corollary 2.2 *Let* $\alpha \geq 1$*. Assume condition* (H_6), (2.10) and (2.16) are hold*. If*

$$\liminf_{n\to\infty} \sum_{s=n}^{n+m-1} P_1(s)\left(2^{(1-\alpha)/\alpha} E_1(s)\right)^{-1} E_1\left(s+m-1\right) > \left(\frac{m}{m+1}\right)^{m+1}\frac{1}{2^{(1-\alpha)/\alpha}}, \tag{2.22}$$

then equation (1.1) has property (A).

The proof of Corollary 2.2 is similar to that of Corollary 2.1, so it can be omitted.

For our next result, we define

$$A_n = q_n\left(\sum_{s=n-\ell-k}^{n-\ell-1} a_s^{-1/\alpha}\right)^{\alpha}, \tag{2.23}$$

$$P_2(n) = h\left(\sum_{j=s+m-k}^{s+m-1} a_j^{-1/\alpha}\right)\sum_{s=n}^{\infty} p_s, \tag{2.24}$$

$$Q_2(n) = \sum_{s=n}^{\infty} A(s+\ell+k), \tag{2.25}$$

and

$$E_2(n) = \prod_{s=N_1}^{n}(1+Q_2(s)) \tag{2.26}$$

where k is a positive integer.

Theorem 2.3. *Let (H_6) holds. If the first order advanced difference equation*

$$\Delta z_n - P_2(n)\left(E_2(n)\right)^{-1} h\left(E_2(n+m-k-1)\right)^{1/\alpha} h\left(z_{n+m-k}^{1/\alpha}\right) = 0 \tag{2.27}$$

is oscillatory then equation (1.1) has property (A).

Proof: Let $\{x_n\}$ be a positive solution of equation (1.1) satisfying case (I) of Lemma 2.1. By monotonicity of $y_n = a_n(\Delta x_n)^\alpha > 0$ implies

$$x_n \geq \sum_{s=n-k}^{n-1} y_s^{1/\alpha} a_s^{-1/\alpha} \geq y_{n-k}^{1/\alpha} \sum_{s=n-k}^{n-1} a_s^{-1/\alpha} \tag{2.28}$$

eventually. Combining (2.28) together with (2.12) we have

$$\Delta y_n \geq \sum_{s=n}^{\infty} q_s x_{s-\ell}^{\alpha} + h(x_{n+m})\sum_{s=n}^{\infty} P_s$$

$$\geq \sum_{s=n}^{\infty} q_s y_{s-\ell-k}\left(\sum_{j=s-\ell-k}^{s-\ell-1} a_j^{-1/\alpha}\right)^{\alpha} + h\left(y_{n+m-k}^{1/\alpha}\right)P_2(n)$$

or

$$\Delta y_n \geq \sum_{s=n}^{\infty} A_s y_{s-\ell-k} + P_2(n)h\left(y_{n+m-k}^{1/\alpha}\right). \tag{2.29}$$

By arguing as in the proof of Theorem 2.1, we see that $s-\ell-k=u$ leads to

$$\sum_{s=n}^{\infty} A_s y_{s-\ell-k} = \sum_{u=n}^{\infty} A(u+\ell+k)y_u = y_n\sum_{u=n}^{\infty} A(u+\ell+k) = Q_2(n)y_n.$$

Using the last inequality in (2.29) we have

$$\Delta y_n - Q_2(n)y_n - P_2(n)h\left(y_{n+m-k}^{1/\alpha}\right) \geq 0. \tag{2.30}$$

By putting $y_n = z_n \prod_{s=N_1}^{n-1}(1+Q_2(s))$. It is easy to check that z_n be a positive solution of the advanced difference inequality

$$\Delta z_n - P_2(n)\left(E_2(n)\right)^{-1} h\left(E_2(n+m-k-1)\right)^{1/\alpha} h\left(z_{n+m-k}^{1/\alpha}\right) \geq 0.$$

It follows from Lemma 2.2, the corresponding difference equation (2.25) has a positive solution, a contradiction to the assumption. So we conclude that equation (1.1) has property (A).

By using similar argument as in the proof of Corollary 2.1, we easily verified that the following result holds.

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Corollary 2.3. *Assume that* (H_6) *and (2.10) hold. If*

$$\frac{h(u^{1/\alpha})}{u} \geq 1, \ |u \geq 1, \tag{2.31}$$

and

$$\liminf_{n\to\infty} \sum_{s=n}^{n+m-k-1} P_2(s)\left(E_2(s)\right)^{-1} E_2\left(s+m-k-1\right) > \left(\frac{m}{m+1}\right)^{m+1} \tag{2.32}$$

then equation (1.1) has property (A).

We offer another criteria for oscillation of equation (1.1) based on the properties of first order delay difference equation. Let us denote,

$$P_3(n) = p_n \beta \frac{R_{n+m}^{\alpha}}{R_n^{\alpha}} \left(\sum_{s=N}^{n-\ell-1} a_s^{-1/\alpha} (s-N_1)^{1/\alpha} \right)^{\alpha}, \tag{2.33}$$

$$Q_3(n) = q_n f\left(\sum_{s=N}^{n-\ell-1} a_s^{-1/\alpha} (s-N_1)^{1/\alpha} \right), \tag{2.34}$$

and

$$E_3(n) = \prod_{s=N_1}^{n} (1 - P_3(s)) \tag{2.35}$$

where $\beta \in (0,1)$ *is arbitrary.*

Theorem 2.4. *Assume that* (H_5) *and* (H_6) *hold. Let*

$$\frac{h(u^{1/\alpha})}{u} \geq 1 \text{ for } u \neq 0. \tag{2.36}$$

If for some $\beta \in (0,1)$, *the first order delay difference equation*

$$\Delta w_n + Q_3(n)\left(E_3(n)\right)^{-1} f\left(\left(E_3\left(n-\ell-1\right)\right)^{1/\alpha}\right) f\left(w_{n-\ell}^{1/\alpha}\right) = 0 \tag{2.37}$$

is oscillatory then equation (1.1) has property (A).

Proof: Assume that contradiction. Then there exists a nonoscillatory solution $\{x_n\}$ of equation (1.1) satisfies case (I) of Lemma 2.1. We assume that $x_n > 0$ and let $\beta_1 = \beta^{1/\alpha}$. Appling Lemma 2.5 we see that $\{x_n\}$ satisfies

$$x_{n+m} \geq \beta_1 \frac{R_{n+m}}{R_n} x_n \tag{2.38}$$

eventually. Let $n \geq N_1$, we denote $z_n = \Delta\left(a_n\left(\Delta x_n\right)^{\alpha}\right)$. It is easy to verify that

$$\Delta z_n + q_n f\left(x_{n-\ell}\right) + p_n h\left(\beta_1 \frac{R_{n+m}}{R_n} x_n\right) \leq 0 \tag{2.39}$$

and (2.14) can be written in the form,

$$a_n\left(\Delta x_n\right)^{\alpha} \geq \Delta\left(a_n\left(\Delta x_n\right)^{\alpha}\right)\left(n-N_1\right) \geq z_n\left(n-N_1\right).$$

Summing from N_1 to $n-1$, we lead to

$$x_n \geq \sum_{s=N_1}^{n-1} a_s^{-1/\alpha} z_s^{1/\alpha} (s-N_1)^{1/\alpha} \geq z_n^{1/\alpha} \sum_{s=N_1}^{n-1} a_s^{-1/\alpha} (s-N_1)^{1/\alpha}.$$

Combining the last inequality with (2.39), we obtain

$$\Delta z_n + q_n f\left(z_{n-\ell}^{1/\alpha} \sum_{s=N_1}^{n-\ell-1} a_s^{-1/\alpha} (s-N_1)^{1/\alpha} \right) + p_n h\left(\beta_1 \frac{R_{n+m}}{R_n} z_n^{1/\alpha} \sum_{s=N_1}^{n-1} a_s^{-1/\alpha} (s-N_1)^{1/\alpha} \right) \leq 0$$

or

$$\Delta z_n + P_3(n) z_n + Q_3(n) f\left(z_{n-\ell}^{1/\alpha} \right) \leq 0.$$

By setting $z_n = w_n \prod_{s=n}^{n-1} (1-P_3(s))$. It follows that z_n is a positive solution of delay difference inequality

$$\Delta w_n + Q_3(n)\left(E_3(n)\right)^{-1} f\left(\left(E_3\left(n-\ell-1\right)\right)^{1/\alpha} \right) f\left(w_{n-\ell}^{1/\alpha} \right) \leq 0.$$

By Lemma 3 in [10], the corresponding delay difference equation (2.37) has a positive solution, which is a contradiction. This complete the proof.

Corollary 2.4. *Assume that (H_5), (H_6) and (2.36) hold. If*

$$\frac{f(u^{1/\alpha})}{u} \geq 1, \quad 0 < |u| \leq 1 \tag{2.40}$$

and for some $\beta \in (0,1)$,

$$\lim_{n\to\infty} \inf \sum_{s=n-\ell}^{n-1} Q_3(s)\left(E_3(s)\right)^{-1} E_3\left(s-\ell-1\right) > \left(\frac{\ell}{\ell+1}\right)^{\ell+1} \tag{2.41}$$

then equation (1.1) has property (A).

Proof: It is easy to see that (2.41) implies

$$\sum_{s=n-\ell}^{n-1} Q_3(s)\left(E_3(s)\right)^{-1} E_3\left(s-\ell-1\right) = \infty. \tag{2.42}$$

By Theorem 2.4, it is sufficient to prove that (2.37) is oscillatory. Assume that (2.37) have an eventually positive solution w_n, then $\Delta w_n < 0$. We claim that $\lim_{n\to\infty} w_n = 0$. If not, there exists some $r > 0$ such that $w_{n-\ell} > r$. Summing (2.37) from N_1 to $n-1$, we have inview of (2.40)

$$w_{N_1} = w_n + \sum_{s=N_1}^{n-1} Q_3(s)\left(E_3(s)\right)^{-1} f\left(\left(E_3\left(s-\ell-1\right)\right)^{1/\alpha} \right) f\left(w_{s-\ell}^{1/\alpha} \right)$$

$$\geq f\left(r^{1/\alpha} \right) \sum_{s=N_1}^{n-1} Q_3(s)\left(E_3(s)\right)^{-1} E_3\left(s-\ell-1\right).$$

Letting $n \to \infty$, we get a contradiction and we conclude that $\lim_{n\to\infty} z_n = 0$. Thus $0 \leq z_n \leq 1$. By using (2.40) in (2.37), we see that z_n is a positive solution of difference inequality

$$\Delta w_n + Q_3(n)\left(E_3(n)\right)^{-1} E_3\left(n-\ell-1\right) w_{n-\ell} \leq 0. \tag{2.43}$$

By Corollary 2.2 in [6] and condition (2.41) ensure that (2.43) has no positive solution. From this contradiction, we conclude that equation (1.1) has property (A).

3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. Consider the third order nonlinear difference equation with mixed arguments

$$\Delta^2\left(n\Delta x_n\right)+\frac{n-2}{(n+1)(n+2)(n+3)}x_{n-2}+\frac{1}{(n+1)(n+2)}x_{n+3}=0,\ \ n\geq 3. \tag{3.1}$$

Here $a_n=n,\ \alpha=1, \ell=2, m=3, q_n=\frac{n-2}{(n+1)(n+2)(n+3)}$ and $p_n=\frac{1}{(n+1)(n+2)}$. Then, it is easy to see that all conditions of Corollary 2.1 are satisfied and hence equation (3.1) has property (A). In fact $\{x_n\}=\left\{\frac{1}{n}\right\}$ is one such solution of equation (3.1) having property (A).

Example 3.2. Consider the third order nonlinear difference equation with mixed arguments

$$\Delta^2\left(2n\Delta x_n\right)+\frac{2(n-1)}{(n+1)(n+2)(n+3)}x_{n-1}+\frac{2}{(n+1)(n+3)}x_{n+2}=0,\ \ n\geq 2. \tag{3.2}$$

Here $a_n=2n,\ \alpha=1, \ell=1, m=2, q_n=\frac{2(n-1)}{(n+1)(n+2)(n+3)}$ and $p_n=\frac{2}{(n+1)(n+3)}$. Then, it is easy to see that all conditions of Corollary 2.2 are satisfied and hence equation (3.2) has property (A). In fact $\{x_n\}=\left\{\frac{1}{2n}\right\}$ is one such solution of equation (3.2) having property (A).

Example 3.3. Consider the third order nonlinear difference equation with mixed arguments

$$\Delta^2\left(2^n\left(\Delta x_n\right)^3\right)+\frac{5}{2^{n+9}}x_{n-2}+\frac{1}{2^{n+1}}x_{n+4}=0,\ \ n\geq 2. \tag{3.3}$$

Here $a_n=2^n,\ \alpha=3, \ell=2, m=4, q_n=\frac{5}{2^{n+9}}$ and $p_n=\frac{1}{2^{n+1}}$. Then, it is easy to see that all conditions of Corollary 2.3 are satisfied and hence equation (3.3) has property (A). In fact $\{x_n\}=\left\{\frac{1}{2^n}\right\}$ is one such solution of equation (3.3) having property (A).

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