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Results on n-tupled fixed points in complete ordered metric spaces.

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Abstract: The aim of this paper is to study n-tupled coincidence and n-tupled fixed point results, a new notion propounded by M.Imdad et.al.[13] for compatible maps in partially ordered metric spaces. Our results generalize, extend and improve the coupled fixed point results of Bhaskar and Lakshmikantham, Nonlinear Analysis: Theory, Methods and Applications, vol.65, no.7, 2006, pp. 1379-1393, V. Lakshmikantham and L. Ciric, Nonlinear Analysis, Theory, Method and Applications, vol. 70, no12, 2009, pp. 4341-4349, tripled fixed point theorems by Berinde and Borcut, Nonlinear Analysis, Volume 74, Issue 15, October 2011, Pages 4889-4897, Quadruple fixed point theorems by E. Karapınar and V. Berinde, *Banach Journal of Mathematical Analysis*, vol. 6, no. 1, pp. 74–89, 2012 and multidimensional fixed point results by Muzeyyen Erturk and Vatan Karakaya, Journal of Inequalities and Applications 2013, 2013:196, pp. 1-19, M. Imdad, A. H. Soliman, B. S. Choudhary and P. Das, Journal of Operators, Volume 2013, Article ID 532867, pp. 1-8 and M. Paknazar, M. E. Gordji, M. D. L. Sen and S. M. Vaezpour, *Fixed Point Theory and Applications* 2013, **2013**:11 etc.

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1. Introduction

The Banach contraction principle is the most natural and significant result of fixed point theory. It has become one of the most fundamental and powerful tools of nonlinear analysis because of its wide range of applications to nonlinear equations arising in physical and biological processes ensuring the existence and uniqueness of solutions. It is widely considered as the source of metric fixed point theory. Also, its significance lies in its vast applicability in a number of branches of mathematics. Generalization of the above principle has been done by various mathematicians see [1,3,7,8,20-26]. Existence of a fixed point for contraction type mappings in partially ordered metric space and applications have been considered by many authors.There already exists an extensive literature on this topic, but keeping in view the relevance of this paper, we merely refer to [3-6,10-13,15,18,19,25-26].

Bhaskar and Lakshmikantham [25] introduced the notions of mixed monotone property and coupled fixed point for the contractive mapping $F : X \times X \to X$, where X is a partially ordered metric space and proved some coupled fixed point theorems for a mixed monotone operator. As an application of the coupled fixed point theorems, they determined the existence and uniqueness of the solution of a periodic boundary value problems. It is very natural to extend the definition of 2-dimensional fixed point (coupled fixed point), 3-dimensional fixed point (tripled fixed point), 4-dimensional fixed point (quadrupled fixed point) and so on to multidimensional fixed point (n-tuple fixed point) ,(see also [4,5,9,16, 17,19]). The last remarkable result of this trend was given by M.Imdad et al. [13] by introducing the notion of multidimensional fixed points.(see also[1,4,12,15,18,24]).

The purpose of this paper is to establish some n-tupled coincidence and fixed point results for compatible maps in complete partially ordered metric spaces. Our results generalize and improve the results of [4,7,8,12,13,15,19,25,26].

2. Prilimaries:

As usual, this section is devoted to preliminaries which include some basic definitions and results related to coupled fixed point and n-tupled fixed point in partially ordered metric spaces.

Definition 2.1 [26] Let (X, ≤) be a partially ordered set equipped with a metric d such that

(X, d) is a metric space. Further, equip the product space $X \times X$ with the following partial ordering:

For $(x, y), (u, v) \in X \times X$, define $(u, v) \leq (x, y) \Leftrightarrow x \geq u, y \leq v$.

Definition 2.2 [26] Let (X, ≤) be a partially ordered set and F: X →X then F enjoys the mixed monotone property if F(x, y) is monotonically non-decreasing in x and monotonically non-increasing in y, that is, for any x, y ∈X,

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y) \text{ and } y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Definition 2.3 [26] Let (X, ≤) be a partially ordered set and $F: X \times X \to X$, then (x, y) $\in X \times X$ is called a coupled fixed point of the mapping F if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.4 [26] Let (X, ≤) be a partially ordered set and $F: X \times X \to X$ and $g : X \to X$ then F enjoys the mixed g-monotone property if $F(x, y)$ is monotonically g-non-decreasing in x and monotonically g-non-increasing in y, that is ,for any $x, y \in X$,

$x_1, x_2 \in X, g(x_1) \leq g(x_2) \Rightarrow F(x_1, y) \leq F(x_2, y)$, for any $y \in X$,

$y_1, y_2 \in X, g(y_1) \leq g(y_2) \Rightarrow F(x, y_1) \geq F(x, y_2)$, for any $x \in X$.

Definition 2.5 [26] Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$, then $(x, y) \in X \times X$ is called a coupled coincidence point of the maps F and g if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 2.6 [26] Let $(X, \leq)$ be a partially ordered set, then $(x, y) \in X \times X$ is called a coupled fixed point of the maps $F : X \times X \to X$ and $g : X \to X$ if $gx = F(x, y) = x$ and $gy = F(y, x) = y$.

M. Imdad et.al [13] propounded the idea of n-tupled coincidence points and n-tupled fixed points by taking r as even natural number as follows:

Definition 2.7[13] Let (X, ≤) be a partially ordered set and $F: \prod_{i=1}^{r} X^i \to X$ then F is said to have the mixed monotone property if F is non-decreasing in its odd position arguments and non-increasing in its even positions arguments , that is, if,

(i) For all $x_1^1, x_2^1 \in X, x_1^1 \leq x_2^1 \Rightarrow F(x_1^1, x^2, x^3, \dots, x^r) \leq F(x_2^1, x^2, x^3, \dots, x^r)$,

(ii) For all $x_1^2, x_2^2 \in X, x_1^2 \leq x_2^2 \Rightarrow F(x^1, x_1^2, x^3, \dots x^r) \geq F(x^1, x_2^2, x^3, \dots, x^r)$,

(iii) For all $x_1^3, x_2^3 \in X, x_1^3 \leq x_2^3 \Rightarrow F(x^1, x^2, x_1^3, x^4, \dots, x^r) \leq F(x^1, x^2, x_2^3, x^4, \dots, x^r)$,

…

For all $x_1^r, x_2^r \in X, x_1^r \leq x_2^r \Rightarrow F(x^1, x^2, x^3, \dots, x_1^r) \geq F(x^1, x^2, x^3, \dots, x_2^r)$.

Definition 2.8[13] Let (X, ≤) be a partially ordered set and $F: \prod_{i=1}^{r} X^i \to X$ and g: X→X be two maps. Then F is said to have the mixed g-monotone property if F is g-non-decreasing in its odd position arguments and g-non-increasing in its even positions arguments , that is, if ,

(i) For all $x_1^1, x_2^1 \in X, gx_1^1 \leq gx_2^1 \Rightarrow F(x_1^1, x^2, x^3, \dots, x^r) \leq F(x_2^1, x^2, x^3, \dots, x^r),$

(ii) For all $x_1^2, x_2^2 \in X, gx_1^2 \leq gx_2^2 \Rightarrow F(, x^1, x_1^2, x^3, \dots, x^r) \geq F(x^1, x_2^2, x^3, \dots, x^r),$

(iii) For all $x_1^3, x_2^3 \in X, gx_1^3 \leq gx_2^3 \Rightarrow F(x^1, x^2, x_1^3, \dots, x^r) \leq F(x^1, x^2, x_2^3, \dots, x^r),$

…

For all $x_1^r, x_2^r \in X, gx_1^r \leq gx_2^r \Rightarrow F(x^1, x^2, x^3, \dots, x_1^r) \geq F(x^1, x^2, \dots, x_2^r).$

Definition 2.9[13] Let X be a nonempty set. An element $(x^1, x^2, x^3, \dots \dots .., x^r) \in \prod_{i=1}^{r} X^i$ is called an r-tupled fixed point of the mapping $F: \prod_{i=1}^{r} X^i \to X$ if

$$x^1 = F(x^1, x^2, x^3, \dots, x^r),$$

$$x^2 = F(x^2, x^3, \dots, x^r, x^1),$$

$$x^3 = F(x^3, \dots, x^r, x^1, x^2),$$

….

$$x^r = F(x^r, x^1, x^2, \dots, x^{r-1}).$$

Definition 2.10[13] Let X be a nonempty set. An element $(x^1, x^2, x^3, \dots, x^r) \in \prod_{i=1}^{r} X^i$ is called an r-tupled coincidence point of the maps $F: \prod_{i=1}^{r} X^i \to X$ and $g : X \to X$ if

$$gx^1 = F(x^1, x^2, x^3, \dots, x^r),$$

$$gx^2 = F(x^2, x^3, \dots, x^r, x^1),$$

$$gx^3 = F(x^3, \dots, x^r, x^1, x^2),$$

…

$$gx^r = F(x^r, x^2, x^3, \dots, x^{r-1}).$$

Definition 2.11[13] Let X be a nonempty set. An element $(x^1, x^2, x^3, \dots \dots .., x^r) \in \prod_{i=1}^{r} X^i$ is called an r-tupled fixed point of the maps $F: \prod_{i=1}^{r} X^i \to X$ and $g : X \to X$ if

$$x^1 = gx^1 = F(x^1, x^2, x^3, \dots, x^r),$$

$$x^2 = gx^2 = F(x^2, x^3, \dots, x^r, x^1)$$

…

$$x^r = gx^r = F(x^r, x^1, x^2, \dots, x^{r-1}).$$

Imdad et al. [13], assuming r as even natural number proved the following theorem:

Theorem 3.1 Let (X, ≤) be a partially ordered set equipped with a metric d such that (X, d) is a complete metric space. Assume that there is a function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) < t$ and $\lim_{r \to t^+} \varphi(r) < t$ for each t > 0. Further let $F: \prod_{i=1}^{r} X^i \to X$ and g: X→X be two maps such that F has the mixed g-monotone property satisfying the following conditions:

(i) $F(\prod_{i=1}^{r} X^i) \subseteq g(X),$

(ii) g is continuous and monotonically increasing ,

(iii) the pair (g, F) is commuting,

(iv) $d\big(F(x^1, x^2, x^3, \dots, x^r), F(y^1, y^2, y^3, \dots, y^r)\big) \leq \varphi\left(\frac{1}{r} \sum_{n=1}^{r} d\big(g(x^n), g(y^n)\big)\right)$

for all $x^1, x^2, x^3, \dots, x^r, y^1, y^2, y^3, \dots, y^r \in X$, with $gx^1 \leq gy^1$, $gx^2 \geq gy^2$,

$gx^3 \leq gy^3, \dots, gx^r \geq gy^r$. Also, suppose that either

(a) F is continuous or

(b) X has the following properties:

(i) If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \preceq x$ for all $n \geq 0$.

(ii) If a non-inecreasing sequence $\{y_n\} \to y$ then $y \preceq y_n$ for all $n \geq 0$.

If there exist $x_0^1, x_0^2, x_0^3, \ldots, x_0^r \in X$ such that

(iv)
$$gx_0^1 \preceq F(x_0^1, x_0^2, x_0^3, \ldots, x_0^r),$$
$$gx_0^2 \succeq F(x_0^2, x_0^3, \ldots, x_0^r, x_0^1),$$
$$gx_0^3 \preceq F(x_0^3, \ldots, x_0^r, x_0^1, x_0^2),$$
$$\ldots$$
$$gx_0^r \succeq F(x_0^r, x_0^1, x_0^2, x_0^3, \ldots, x_0^{r-1}).$$

Then F and g have a r-tupled coincidence point, i. e there exist $x^1, x^2, x^3, \ldots, x^r \in X$ such that

(v)
$$gx^1 = F(x^1, x^2, x^3, \ldots, x^r),$$
$$gx^2 = F(x^2, x^3, \ldots, x^r, x^1),$$
$$\mathrm{gx}^3 = \mathrm{F}(\mathrm{x}^3, \ldots, \mathrm{x}^r, \mathrm{x}^1, \mathrm{x}^2),$$
$$\ldots$$
$$gx^r = F(x^r, x^1, x^2, x^3, \ldots, x^{r-1}).$$

Main Results:

Remark 3.1 Regarding the definitions (2.9) and (2.10), we notice that, in the case n = 3,

$$gx^1 = F(x^1, x^2, x^3),$$
$$gx^2 = F(x^2, x^3, x^1),$$
$$gx^3 = F(x^3, x^1, x^2),$$

do not extend the notion of tripled coincidence point by Brinde and Borcut [2]. Therefore their results are not extensions of well known results in tripled case and hence we can say that the odd case is not well posed.

Remark 3.2 Also, we see that the system of equations defined in (2.7) is not suitable to work with the classical mixed monotone property when r is odd . For example, if r = 5 and F is monotone non-decreasing in its odd arguments and monotone non-increasing in its even arguments, then the equations

$x^1 = F(x^1, x^2, x^3, x^4, x^5)$ (x^1 and x^5 are placed in non-decreasing arguments of F) and

$x^2 = F(x^2, x^3, x^4, x^5, x^1)$ (x^1 and $x^5$are placed in arguments of different monotone type of F)

Do not let us to show the existence of fixed points using the classical mixed monotone property.

To make the paper free from these flaws, we recall here the concept of multidimensional fixed point/ coincidence point introduced by Roldan et. al [1], which is an extension of Berzig and Samet's notion given in [18].

Henceforth, X will denote a non-empty set and X^r will denote the product space $X \times X \times \ldots . \times X$ and r as a general natural number. Also, fix a partion $\{A, B\}$ of $A_n = \{1,2, \ldots, n\}$, that is $A_n = A \cup B$ and $A \cap B = \emptyset$ where A and B are non-empty sets. We will denote:

$\Omega_{A,B} = \{\sigma: A_n \to A_n : \sigma(A) \subseteq A \ and \ \sigma(B) \subseteq B\}$ and

$\Omega'_{A,B} = \{\sigma: A_n \to A_n : \sigma(A) \subseteq B \ and \ \sigma(B) \subseteq A\}$.

If $(X, \preceq)$ is a partially ordered space, $x, y \in X$ and $i \in A_n$, we will use the following notation:

$$x \preceq_i y \Leftrightarrow \begin{Bmatrix} x \preceq y, if \ \ i \in A, \\ x \succeq y, if \ \ i \in B. \end{Bmatrix}$$

Consider on the product space X^r, the following partial order:

$$x = (x^1, x^2, \ldots, x^r), y = (y^1, y^2, \ldots, y^r) \in X^r, \quad x \sqsubseteq y \Leftrightarrow x_i \preceq_i y_i \text{ , for all } i.$$

We say that two points x and y are comparable if $x \sqsubseteq y$ or $y \sqsubseteq x$.

Definition3.1 Let $(X, \preceq)$ be a partially ordered space with the maps $F: X^r \to X$ and $g: X \to X$. We say that F has the mixed g-monotone property $(w.r.t \ \{A, B\})$ if F is monotone g- nondecreasing in arguments of A and monotone g-non increasing in arguments of B, i.e, for all $x_1, x_2, \ldots, x_n, y, z \in X$ for all i,

$$gy \preceq gz \Rightarrow F(x^1, x^2, \ldots, x^{i-1}, y, x^{i+1}, \ldots, x^r) \preceq_i F(y^1, y^2, \ldots, y^{i-1}, z, y^{i+1}, \ldots, y^r).$$

Henceforth, let $\sigma_1, \sigma_2, \dots, \sigma_n: A_n \to A_n$ be n mappings from A_n into itself and let γ be the n-tupple $(\sigma_1, \sigma_2, \dots, \sigma_n)$. The main aim of this paper is to study the following class of multidimensional fixed points.

Definition 3.2 A point $(x^1, x^2, \dots, x^r) \in X^r$ is called a γ-fixed point of the mapping F if

$F(x^{\sigma_i(1)}, x^{\sigma_i(2)}, \dots x^{\sigma_i(r)}) = x^i$ for all i.

Example 1 Let (R, d) be a partial ordered metric space under natural setting and let $F: \prod_{i=1}^{r} X^i \to X$ be mapping defined by

$F(x^1, x^2, x^3, \dots, x^r) = \sin(x^1 \cdot x^2 \cdot x^3 \cdot \dots \cdot x^r)$, for any $x^1, x^2, x^3, \dots, x^r \in X$, then (0,0,0,…,0) is an r-tupled fixed point of F.

Definition 3.3 A point $(x^1, x^2, \dots, x^r) \in X^r$ is called a γ-coincidence point of the mappings $F: X^r \to X$ and $g: X \to X$ if $F(x^{\sigma_i(1)}, x^{\sigma_i(2)}, \dots x^{\sigma_i(r)}) = gx^i$ for all i.

Example 2 Let (R, d) be a partial ordered metric space under natural setting and let $F: \prod_{i=1}^{r} X^i \to X$ and $g: X \to X$ be maps defined by

$F(x^1, x^2, x^3, \dots, x^r) = \sin x^1 \cdot cosx^2 \cdot \sin x^3 \cdot \cos x^4 \cdot \dots \cdot \sin x^{r-1} \cdot \cos x^r$,

$g(x) = \sin x$,

for any $x^1, x^2, x^3, \dots, x^r \in X$, then $\{(x^1, x^2, x^3, \dots, x^r), x^i = m\pi, m \in N, 1 \le i \le r\}$ is an r-tupled coincidence point of F and g.

Definition 3.4 A point $(x^1, x^2, \dots, x^r) \in X^r$ is called a γ-fixed point of the mappings $F: X^r \to X$ and $g: X \to X$ if $F(x^{\sigma_i(1)}, x^{\sigma_i(2)}, \dots x^{\sigma_i(r)}) = gx^i = x^i$ for all i.

Definition 3.5 An ordered metric space $\{X, d\}$ is said to have the sequential g-monotone property if it saitisfies:

(i) If $\{x_m\}$ is a non-decreasing sequence and $\{x_m\} \overset{d}{\to} x$, then $gx_m \le gx$ for all m.

(ii) If $\{x_m\}$ is a non-increasing sequence and $\{x_m\} \overset{d}{\to} x$, then $gx_m \ge gx$ for all m.

If g is the identity mapping, then X is said to have sequential monotone property.

Now, we define the concept of compatible maps for r-tupled maps.

Definition 3.6 [24] Let (X, $\preceq$) be a partially ordered set, then the maps $F: X^r \to X$ and $g: X \to X$ are called compatible if

$$\lim_{n\to\infty} g\left(F\left(x_n^{\sigma_i(1)}, x_n^{\sigma_i(2)}, \dots, x_n^{\sigma_i(r)}\right), F\left(gx_n^{\sigma_i(1)}, gx_n^{\sigma_i(2)}, \dots, gx_n^{\sigma_i(r)}\right)\right) = 0, \text{ for all } i,$$

whenever, $\{x_n^1\}, \{x_n^2\}, \{x_n^3\}, \dots, \{x_n^r\}$ are sequences in X such that

$\lim_{n\to\infty} F\left(x_n^{\sigma_i(1)}, x_n^{\sigma_i(2)}, x_n^{\sigma_i(3)}, \dots, x_n^{\sigma_i(r)}\right) = \lim_{n\to\infty} g(x_n^i) = x^i$, for all i, for some $x^1, x^2, x^3, \dots, x^r \in X$.

Remark 3 If one represent a mapping $\sigma: A_n \to A_n$ throughout its order image, that is, $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(r))$, then

(i) G-Bhaskar and Lakshmikantham's election in $n = 2$ is $\sigma_1 = \tau = (1,2)$ and $\sigma_2 = (2,1)$

(ii) Berinde and Borcut's election in $n = 3$ is $\sigma_1 = \tau = (1,2,3)$, $\sigma_2 = (2,1,2)$ and $\sigma_3 = (3,2,1)$

(iii) Karapinar's election in $n = 4$ is $\sigma_1 = \tau = (1,2,3,4)$, $\sigma_2 = (2,3,4,1)$, $\sigma_3 = (34,1,2)$ and $\sigma_4 = (4,1,2,3)$

These cases consider A as the odd numbers in $\{1,2, \dots, n\}$ and B as its even numbers. However, for Berzig and Samet [18], use $A = \{1,2, \dots, m\}, B = \{m+1, \dots, n\}$ and arbitrary mappings.

For our main result, we state the following lemma:

Lemma 1 [4] If $\{x_m\}_{m\in N}$ is a sequence in a metric space (X, d) that is not Cauchy, then there exists $\varepsilon_0 > 0$ and two subsequences $\{x_{m(k)}\}_{k\in N}$ and $\{x_{n(k)}\}_{k\in N}$ such that, for all $k \in N$,

$k < m(k) < n(k) < m(k+1), d(x_{m(k)}, x_{n(k)}) \ge \varepsilon_0$ and $d(x_{m(k)}, x_{n(k)-1}) < \varepsilon_0$.

Now, we prove our main result as follows:

Theorem 3.1 Let (X, $\preceq$) be a complete ordered metric space. Let $\gamma = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$ be n-tuple of mappings from $\{1,2, \dots, n\}$ into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Assume that there is a function $\varphi: [0, \infty) \to [0, \infty)$ with $\varphi(t) < t$ and $\lim_{r\to t^+} \varphi(r) < t$ for each t > 0. Further let $F: X^r \to X$ and g: X → X be two maps such that F has the mixed g-monotone property satisfying the following conditions:

(3.1) $F(X^r) \subseteq g(X)$,

(3.2) g is continuous,

(3.3) the pair (g, F) is compatible,

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(3.4) $d(F(x^1,x^2,x^3,\dots,x^r),F(y^1,y^2,y^3,\dots,y^r)) \le \varphi(max\{\, d(g(x^n),g(y^n))\})$,

for all $x^1,x^2,x^3,\dots,x^r,y^1,y^2,y^3,\dots,y^r \in X, n=1,2,\dots,r$ and $gx^i \le_i gy^i$, for all i.

Also, suppose that either

(c) F is continuous or

(d) X has the sequential g-monotone property.

If there exist $x_0^1,x_0^2,x_0^3,\dots,x_0^r \in X$ such that

$gx_0^i \le_i F\left(x_0^{\sigma_i(1)},x_0^{\sigma_i(2)},\dots,x_0^{\sigma_i(r)}\right)$ for all i, then F and g have atleast one γ - coincidence point.

Proof. Starting with $x_0^1,x_0^2,x_0^3,\dots,x_0^r \in X$, we define the sequences $\{x_n^1\},\{x_n^2\},\{x_n^3\},\dots,\{x_n^r\}$ in X as follows:

(3.5) $gx_{n+1}^i = F\left(x_n^{\sigma_i(1)},x_n^{\sigma_i(2)},x_n^1,\dots,x_n^{\sigma_i(r)}\right)$, for all n and all i.

Now, by induction we can prove that

(3.6) $gx_n^i \le_i gx_{n+1}^i$ for all n,

As $gx_0^i \le_i F\left(x_0^{\sigma_i(1)},x_0^{\sigma_i(2)},\dots,x_0^{\sigma_i(r)}\right) = gx_1^i$, for all i, (where i take all the values in A or B at the same time). Suppose that (3.6)holds for n and we are going to prove it for n+1. The induction hypothesis is

(3.7) $gx_n^i \le_i gx_{n+1}^i \Leftrightarrow \begin{Bmatrix} gx_n^i \le gx_{n+1}^i, i \in A \\ gx_n^i \ge gx_{n+1}^i, i \in B \end{Bmatrix}$

Now, we want to prove that $gx_{n+1}^i \le_i gx_{n+2}^i$ for all i, i.e $gx_{n+1}^i \le gx_{n+2}^i, i \in A$ and $gx_{n+1}^i \ge gx_{n+2}^i, i \in B$. Therefore, we have to distinguish between whether $i \in A \; or \; i \in B$. Suppose that $i \in A$ and F is g-monotone non-decreasing in A-arguments with the first inequality of (3.7) and deduce that, for all $a_1,a_2,\dots,a_n \in X$:

$gx_n^j \le gx_{n+1}^j \Rightarrow F(a_1,a_2,\dots,a_{s-1},x_n^j,a_{s+1},\dots,a_m) \le F(a_1,a_2,\dots,a_{s-1},x_{n+1}^j,a_{s+1},\dots,a_m)$, if $j,s \in A$, and F is g-monotone non-increasing in B-arguments with the second inequality of (3.7)

$gx_n^j \ge gx_{n+1}^j \Rightarrow F(a_1,a_2,\dots,a_{s-1},x_n^j,a_{s+1},\dots,a_m) \le F(a_1,a_2,\dots,a_{s-1},x_{n+1}^j,a_{s+1},\dots,a_m)$, if $j,s \in B$.

This means that, if $j,s \in \{1,2,\dots,n\}$ verify $j,s \in A$ or $j,s \in B$. Then

$F(a_1,a_2,\dots,a_{s-1},x_n^j,a_{s+1},\dots,a_m) \le F(a_1,a_2,\dots,a_{s-1},x_{n+1}^j,a_{s+1},\dots,a_m)$, for all $a_1,a_2,\dots,a_n \in X$. As $\sigma_i \in \Omega_{A,B}$:

$$\begin{aligned} gx_{n+1}^{\sigma_i} &= F\left(x_n^{\sigma_i(1)},x_n^{\sigma_i(2)},\dots,x_n^{\sigma_i(r)}\right) \le (1,\sigma_i(1) \in A \; or \; 1,\sigma_i(1) \in B\,) \\ &\le F\left(x_{n+1}^{\sigma_i(1)},x_n^{\sigma_i(2)},\dots,x_n^{\sigma_i(r)}\right) \le (2,\sigma_i(2) \in A \; or \; 2,\sigma_i(2) \in B\,) \\ &\le F\left(x_{n+1}^{\sigma_i(1)},x_{n+1}^{\sigma_i(2)},\dots,x_n^{\sigma_i(r)}\right) \le (2,\sigma_i(2) \in A \; or \; 2,\sigma_i(2) \in B\,) \\ &\le \dots \le F\left(x_{n+1}^{\sigma_i(1)},x_{n+1}^{\sigma_i(2)},\dots,x_{n+1}^{\sigma_i(r)}\right) = gx_{n+2}^{\sigma_i}. \end{aligned}$$

Hence $gx_{n+1}^{\sigma_i} \le gx_{n+2}^{\sigma_i}$ when $j \; or \; \sigma_i(j) \in A$ and (3.7) is true if $i \in A$. Now suppose that $j \in B$ (so $\sigma_i(j) \in B$. In this case, we apply that F is g-monotone non-increasing in B-argument with the second inequality of (3.7) and deduce that, for all $a_1,a_2,\dots,a_n \in X$, that, if $j,s \in \{1,2,\dots,n\}$ verify $j \in A$ or $s \in B$ or $j \in B$ or $s \in A$. Then

$F(a_1,a_2,\dots,a_{s-1},x_n^j,a_{s+1},\dots,a_m) \ge F(a_1,a_2,\dots,a_{s-1},x_{n+1}^j,a_{s+1},\dots,a_m)$.

Since $\sigma_i \in \Omega_{A,B}^{/}$. Therefore,

$$\begin{aligned} gx_{n+1}^{\sigma_i} &= F\left(x_n^{\sigma_i(1)},x_n^{\sigma_i(2)},\dots,x_n^{\sigma_i(r)}\right) \ge (1 \in A,\sigma_i(1) \in B \; or \; 1 \in B,\sigma_i(1) \in A\,) \\ &\ge F\left(x_{n+1}^{\sigma_i(1)},x_n^{\sigma_i(2)},\dots,x_n^{\sigma_i(r)}\right) \ge (2 \in A,\sigma_i(2) \in B \; or \; 2 \in B,\sigma_i(2) \in A\,) \\ &\ge F\left(x_{n+1}^{\sigma_i(1)},x_{n+1}^{\sigma_i(2)},\dots,x_n^{\sigma_i(r)}\right) \ge (3 \in A,\sigma_i(3) \in B \; or \; 3 \in B,\sigma_i(3) \in A\,) \\ &\ge \dots \ge F\left(x_{n+1}^{\sigma_i(1)},x_{n+1}^{\sigma_i(2)},x_{n+1}^1,\dots,x_{n+1}^{\sigma_i(r)}\right) = gx_{n+2}^{\sigma_i}. \end{aligned}$$

Hence $gx_{n+1}^{\sigma_i} \ge gx_{n+2}^{\sigma_i}$ when $j \in B$ and hence (3.7) is true.

Define

(3.8) $\gamma_n = max_{1 \le j \le r}\left\{ d\left(g\left(x_n^j\right),g\left(x_{n+1}^j\right)\right)\right\} = max_{1 \le i,j \le r}\left\{ d\left(g\left(x_n^{\sigma_i(j)}\right),g\left(x_{n+1}^{\sigma_i(j)}\right)\right)\right\}$.

Firstly, suppose that there exist $n_0 \in N$ such that $\gamma_{n_0} = 0$. Then

$gx^i_{n_0} = gx^i_{n_0+1} = F\left(x^{\sigma_i(1)}_{n_0}, x^{\sigma_i(2)}_{n_0}, \dots, x^{\sigma_i(r)}_{n_0}\right)$ for all i, so $(x^1_{n_0}, x^2_{n_0}, \dots, x^r_{n_0})$ is a γ-coincidence point of F and g and we are nothing to prove. Therefore, we may reduce to the case in which $\gamma_n > 0$ for all n that is

$\forall\, n, \exists\, j$ such that $gx^j_n \neq gx^j_{n+1}$. Using (3.4) and (3.5)

$$(3.9)\quad d\left(g(x^j_n), g(x^j_{n+1})\right)$$
$$= d\left(F\left(x^{\sigma_i(1)}_{n-1}, x^{\sigma_i(2)}_{n-1}, \dots, x^{\sigma_i(r)}_{n-1}\right), F\left(x^{\sigma_i(1)}_{n}, x^{\sigma_i(2)}_{n}, \dots, x^{\sigma_i(r)}_{n}\right)\right)$$
$$\leq \varphi\left(\max\left\{d\left(g\left(x^{\sigma_i(j)}_{n-1}\right), g\left(x^{\sigma_i(j)}_{n}\right)\right)\right\}\right)$$
$$= \varphi\left(\max\left\{d\left(g(x^j_{n-1}), g(x^j_n)\right)\right\}\right), \text{ for all } n \text{ and for all } j.$$

Taking maximum on j, we deduce that

$$(3.10)\quad \gamma_n = max\left\{d\left(g(x^j_n), g(x^j_{n+1})\right)\right\}$$
$$\leq \varphi\left(\max\left\{d\left(g(x^j_{n-1}), g(x^j_n)\right)\right\}\right) = \varphi(\gamma_{n-1}).$$

Since $\varphi(t) < t$ for all t > 0, therefore, $\gamma_n \leq \gamma_{n-1}$ for all n so that $\{\gamma_n\}$ is a non-increasing sequence. Since it is bounded below, there is some $\gamma \geq 0$ such that

$$(3.11)\quad \lim_{n\to\infty} \gamma_n = +\gamma.$$

We shall show that $\gamma = 0$. Suppose, if possible $\gamma > 0$. Taking limit as $n \to \infty$ of both sides of (3.10) and keeping in mind our supposition that $\lim_{r\to t^+} \varphi(r) < t$ for all t > 0, we have

$$(3.12)\quad \gamma = \lim_{n\to\infty} \gamma_m \leq \varphi(\gamma_{m-1}) = \varphi(\gamma) < \gamma,$$

this contradiction gives $\gamma = 0$ and hence

$$(3.13)\quad \lim_{n\to\infty}\left[max\left\{d\left(g(x^j_n), g(x^j_{n+1})\right)\right\}\right] = 0, \text{ for all } j.$$

Next we show that all the sequences $\{g(x^1_m)\}, \{g(x^2_m)\}, \{g(x^3_m)\}, \dots\dots\dots, and\ \{g(x^r_m)\}$ are Cauchy sequences. If possible, suppose that $\{gx^{i_1}_m\}, \{gx^{i_2}_m\}, \dots, \{gx^{i_s}_m\}$, $(s \geq 1)$ are not Cauchy sequenes and $\{gx^{i_{s+1}}_m\}, \{gx^{i_{s+2}}_m\}, \dots, \{gx^{i_r}_m\}$ are Cauchy being $\{i_1, i_2, \dots, i_r\} = \{1,2,\dots,r\}$. By lemma [1], for all $t \in \{1,2,\dots,s\}$, there exists $\varepsilon_t > 0$ and subsequences $\{gx^{i_t}_{m_t(k)}\}$ and $\{gx^{i_t}_{n_t(k)}\}, k \in N$ such that

$k < m_t(k) < n_t(k),$

$$(3.14)\quad d\left(gx^{i_t}_{m_t(k)}, gx^{i_t}_{n_t(k)}\right) \geq \varepsilon_t \text{ and } d\left(gx^{i_t}_{m_t(k)}, gx^{i_t}_{n_t(k)-1}\right) < \varepsilon_t\ \forall\, k \in N$$

Now, let $\varepsilon_0 = max(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_s) > 0$ and $\varepsilon_0{}' = min(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_s) > 0$. Since $\{gx^{i_{s+1}}_m\}, \{gx^{i_{s+2}}_m\}, \dots, \{gx^{i_r}_m\}$ are Cauchy, there exist $n_0 \in N$ such that if $n, m \geq n_0$. Then

$d(gx^j_m, gx^j_n) < \frac{\varepsilon_0{}'}{2}$ for all $j \in \{i_{s+1}, \dots, i_n\}$.

Let $k_0 \in N$ such that $n_0 < min\{m_1(k_0), m_2(k_0), \dots, m_s(k_0)\}$ and $m(1) = min\{m_1(k_0), m_2(k_0), \dots, m_s(k_0)\}$. As $m(1) = m_t(k_0)$ for some $t \in \{1,2,\dots,s\}$, there exist $n_t(k_0)$ such that $d\left(gx^{i_t}_{m_t(k_0)}, gx^{i_t}_{n_t(k_0)}\right) \geq \varepsilon_t \geq \varepsilon_0$. Thus, we can consider the numbers $m(1)+1, m(1)+2, \dots$, until finding the positive integer $n(1) > m(1)$ verifying

$$(3.15)\quad \max d\left(gx^{i_t}_{m(1)}, gx^{i_t}_{n(1)}\right) \geq \varepsilon_0 \geq \frac{\varepsilon_0}{2},$$
$$d\left(gx^{i_j}_{m(1)}, gx^{i_j}_{n(1)-1}\right) < \frac{\varepsilon_0}{2}\ \forall\, j \in \{1,2,\dots,s\}.$$

Now, let $k_1 \in N$ such that $n(1) < min\{m_1(k_1), m_2(k_1), \dots, m_s(k_1)\}$ and $m(2) = min\{m_1(k_1), m_2(k_1), \dots, m_s(k_1)\}$. Since $m(2) \in \{m_1(k_1), m_2(k_1), \dots, m_s(k_1)\}$, we can consider the numbers $m(2)+1, m(2)+2, \dots$, until finding the positive integer $n(2) > m(2)$ verifying

$$(3.16)\quad \max d\left(gx^{i_t}_{m(2)}, gx^{i_t}_{n(2)}\right) \geq \varepsilon_0 \geq \frac{\varepsilon_0}{2},$$
$$d\left(gx^{i_j}_{m(2)}, gx^{i_j}_{n(2)-1}\right) < \frac{\varepsilon_0}{2}\ \forall\, j \in \{1,2,\dots,s\}.$$

Repeating this process, we can find sequences such that, for all $k \geq 1$,

$n_0 < m(k) < n(k) < m(k+1)$,

(3.17) $\max d\left(gx^{i_t}_{m(k)}, gx^{i_t}_{n(k)}\right) \geq \varepsilon_0 \geq \frac{\varepsilon_0}{2}$,

$d\left(gx^{i_j}_{m(k)}, gx^{i_j}_{n(k)-1}\right) < \frac{\varepsilon_0}{2}\ \forall j \in \{1,2,\ldots,s\}$.

Since $n_0 < m(k) < n(k)$, we know that $d\left(gx^{j}_{m(k)}, gx^{j}_{n(k)}\right), d\left(gx^{j}_{m(k)}, gx^{j}_{n(k)-1}\right), d\left(gx^{j}_{m(k)-1}, gx^{j}_{n(k)-1}\right) < \frac{\varepsilon_0/}{2}$ for all $j \in \{i_{s+1},\ldots,i_n\}$. Therefore, for all k,

(3.18) $max_{1\leq j\leq r} d\left(gx^{j}_{m(k)}, gx^{j}_{n(k)}\right) = max_{1\leq j\leq r} d\left(gx^{i_t}_{m(k)}, gx^{i_t}_{n(k)}\right) \geq \varepsilon_0$,

$max_{1\leq j\leq r} d\left(gx^{j}_{m(k)}, gx^{j}_{n(k)-1}\right) < \varepsilon_0/$.

Note that for $k > k_1$,

(3.19) $d\left(gx^{j}_{m(k)-1}, gx^{j}_{n(k)-1}\right) \leq d\left(gx^{j}_{m(k)-1}, gx^{j}_{m(k)}\right) + d\left(gx^{j}_{m(k)}, gx^{j}_{n(k)-1}\right) < \frac{\varepsilon_0}{2}$

Then for all j and all $k > k_1$,

(3.20)
$$d\left(gx^{j}_{m(k)}, gx^{j}_{n(k)}\right) = d\left(F\left(x^{\sigma_i(1)}_{m(k)-1},\ldots,x^{\sigma_i(r)}_{m(k)-1}\right), F\left(x^{\sigma_i(1)}_{m(k)-1},\ldots,x^{\sigma_i(r)}_{m(k)-1}\right)\right)$$
$$\leq \varphi\left(max\left\{d\left(gx^{\sigma_i(1)}_{m(k)-1}, gx^{\sigma_i(1)}_{n(k)-1}\right)\right\}\right)$$
$$= \varphi(max\{d(gx^{i}_{m(k)-1}, gx^{i}_{n(k)-1})\}) < \varphi\left(\frac{\varepsilon_0}{2}\right) < \frac{\varepsilon_0}{2},$$

this contradict (3.17) since $\max d\left(gx^{i_t}_{m(k)}, gx^{i_t}_{n(k)}\right) \geq \frac{\varepsilon_0}{2}$. This contradiction shows that $\{g(x^i_m)\}$, for all i, is Cauchy. Since the metric space (X, d) is complete, so there exist $x^1, x^2, \ldots\ldots, x^r \in X$ such that

(3.21) $\lim_{m\to\infty} g(x^i_m) = x^i$, for all i.

As g is continuous, so from (3.21), we have

(3.22) $\lim_{m\to\infty} g(g(x^i_m)) = g(x^i)$, for all i. By the compatibility of g and F, we have

(3.23) $\lim_{m\to\infty} d\left(g\left(F\left(x^{\sigma_i(1)}_m, x^{\sigma_i(2)}_m,\ldots,x^{\sigma_i(r)}_m\right)\right), F\left(g\left(x^{\sigma_i(1)}_m\right), g\left(x^{\sigma_i(2)}_m\right),\ldots,g\left(x^{\sigma_i(r)}_m\right)\right)\right) = 0$, for all i.

Now, we show that F and g have an r-tupled coincidence point. To accomplish this, suppose (a) holds, i. e F is continuous. In this case $\left\{g\left(x^{\sigma_i(j)}_m\right)\right\} \to x^{\sigma_i(j)} = x^i$ then using (3.23) and (3.5), we see that

$$d\left(g(x^i), F(x^{\sigma_i(1)}, x^{\sigma_i(2)},\ldots,x^{\sigma_i(r)})\right)$$
$$= \lim_{n\to\infty} d\left(g\left(g\left(x^{\sigma_i(j)}_{m+1}\right)\right), F\left(g\left(x^{\sigma_i(1)}_m\right), g\left(x^{\sigma_i(2)}_m\right),\ldots,g\left(x^{\sigma_i(r)}_m\right)\right)\right)$$
$$= \lim_{n\to\infty} d\left(g\left(F\left(x^{\sigma_i(1)}_m, x^{\sigma_i 2}_m,\ldots,x^{\sigma_i(r)}_m\right)\right), F\left(g\left(x^{\sigma_i(1)}_m\right), g\left(x^{\sigma_i(2)}_m\right),\ldots,g\left(x^{\sigma_i(r)}_m\right)\right)\right) = 0,$$

which gives $g(x^i) = F(x^{\sigma_i(1)}, x^{\sigma_i(2)},\ldots,x^{\sigma_i(r)})$, for all i, that is $(x^1, x^2,\ldots,x^r)$ is a γ - coincidence point of F and g.

If (b) holds, that is (X,d) has sequential g-monotone property and by (3.6), we have $gx^i_n \leq_i gx^i_{n+1}$ for all n and i. This means that the sequence $\{gx^i_n\}$ is monotone. As $x^i = \lim_{n\to\infty} gx^i_n$, we deduce that $gx^i_n \leq_i gx^i$, for all n and i. This condition implies that , for all n and j,

(3.24) either $g\left(g\left(x^{\sigma_i(j)}_n\right)\right) \leq_i g(x^{\sigma_i(j)})$, for all i

or $g(x^{\sigma_i(j)}) \leq_i g\left(g\left(x^{\sigma_i(j)}_n\right)\right)$, for all i

(the first case occurs when $j \in A$ and second one when $j \in B$. Fix $j \in \{1,2,\ldots,r\}$ and since $\{ggx^i_n\} \to gx^i$.

Also F is g-compatible and g is continuous, by (3.23) and (3.5), we have

$$\lim_{n\to\infty} ggx^i_n = gx^i = lim_{n\to\infty} g\left(F(x^{\sigma_j(1)}, x^{\sigma_j(2)},\ldots,x^{\sigma_j(r)})\right) = lim_{n\to\infty}\left(F(gx^{\sigma_j(1)}, gx^{\sigma_j(2)},\ldots,gx^{\sigma_j(r)})\right)$$

(3.25) $d\left(g(x^j), F(x^{\sigma_j(1)}, x^{\sigma_j(2)}, \dots, x^{\sigma_j(r)})\right)$

$$\leq d\left(gx^j, ggx_{n+1}^{\sigma_j(i)}\right) + d\left(ggx_{n+1}^{\sigma_j(i)}, F(x^{\sigma_j(1)}, x^{\sigma_j(2)}, \dots, x^{\sigma_j(r)})\right)$$

$$= d\left(gx^j, ggx_{n+1}^{\sigma_j(i)}\right) + d\begin{pmatrix} g\left(F\left(x_n^{\sigma_j(1)}, x_n^{\sigma_j(2)}, \dots, x_n^{\sigma_j(r)}\right)\right), \\ F(x^{\sigma_j(1)}, x^{\sigma_j(2)}, \dots, x^{\sigma_j(r)}) \end{pmatrix}$$

$$= d\left(gx^j, ggx_{n+1}^{\sigma_j(i)}\right) + d\begin{pmatrix} \left(F\left(gx_n^{\sigma_j(1)}, gx_n^{\sigma_j(2)}, \dots, gx_n^{\sigma_j(r)}\right)\right), \\ F(x^{\sigma_j(1)}, x^{\sigma_j(2)}, \dots, x^{\sigma_j(r)}) \end{pmatrix}$$

$$\leq d\left(gx^j, ggx_{n+1}^{\sigma_j(i)}\right) + \varphi\left(max\left\{d\left(ggx_n^{\sigma_j(i)}, gx_n^{\sigma_j(i)}\right)\right\}\right), \text{ for all } j$$

$$\to 0 \; as \; n \to \infty.$$

Therefore, $g(x^j) = F(x^{\sigma_j(i)}, x^{\sigma_j(i)}, \dots, x^{\sigma_j(i)})$.

Now, we furnish our theorem by an example.

Example 3. Let $X = [0, \infty)$ be complete metric space under metric $d(x, y) = min\{x, y\}$ and natural ordering $\leq$ of real numbers. Define the maps $F: \prod_{i=1}^{r} X^i \to X$ and g: X→X as follows

$g(x) = \begin{Bmatrix} 0, & 0 \leq x < 1 \\ x-1, & x \geq 1 \end{Bmatrix}$ and $F(x^1, x^2, x^3, \dots, x^r) = \begin{Bmatrix} 1, & x^{i+1} \leq x^i, \; i \text{ is odd} \\ 0 & otherwise, \end{Bmatrix}$

Also set $\varphi(t) = t$.

Then g is continuous and F enjoys the mixed g-monotone property. Also F is g-compatible in X and the contractive condition is also satisfied and $(0,0,\dots,0)$ is the unique fixed point of the maps F and g.

The following corollary is a generalization of corollary 1[10] and theorem 2.1[9]

Corollary 3.1 Thesis of Theorem 3.1 also holds if one replaces the contractivity condition (3.4) by the following (for which $gx^i \leq_i gy^i$)

(i) $\left(d\left(F(x^1, x^2, \dots, x^r), F(y^1, y^2, \dots, y^r)\right) \leq k\left(max\left\{d\left(g(x^n), g(y^n)\right)\right\}\right), n = 1,2,\dots,r\right.$

(ii) $\left(d\left(F(x^1, x^2, \dots, x^r), F(y^1, y^2, \dots, y^r)\right) \leq \sum_{i=1}^{r} \alpha_i d\left(g(x^n), g(y^n)\right), n = 1,2,\dots,r\right.$

where $\alpha_1, \alpha_2, \dots, \alpha_r \in [0,1)$ such that $\alpha_1 + \alpha_2 + \dots + \alpha_r < 1$.

$and \; k \in [0,1)$.Then F and g have a r-tupled coincidence point.

Proof: If we put $\varphi(t) = k.t$ with $k \in [0,1)$ in theorem 3.2, we get the corollary.

Corollary 3.2 Thesis of Theorem 3.1 also holds if one replaces the contractivity condition (3.4) by the following (for which $gx^i \leq_i gy^i$)

(i) $\left(d\left(F(x^1, x^2, \dots, x^r), F(y^1, y^2, \dots, y^r)\right) \leq \sum_{i=1}^{r} \alpha_i d\left(g(x^n), g(y^n)\right), n = 1,2,\dots,r\right.$

where $\alpha_1, \alpha_2, \dots, \alpha_r \in [0,1)$ such that $\alpha_1 + \alpha_2 + \dots + \alpha_r < 1$.

Proof: If $k = \alpha_1 + \alpha_2 + \dots + \alpha_r < 1$, then

$$\left(d\left(F(x^1, x^2, \dots, x^r), F(y^1, y^2, \dots, y^r)\right) \leq \sum_{i=1}^{r} \alpha_i d\left(g(x^n), g(y^n)\right)\right)$$
$$\leq \sum_{i=1}^{r} \alpha_i max\{d\left(g(x^n), g(y^n)\right)\}$$
$$= k\left(max\{d\left(g(x^n), g(y^n)\right)\}\right)$$

Uniqueness of r-tupled fixed point

For the uniqueness of a fixed point, we need the following notion. Consider on the product space X^r the following partial order: for

For all $(x^1, x^2, \dots, x^r), (y^1, y^2, \dots, y^r) \in X^r$,

$$(x^1, x^2, \dots, x^r) \leq (y^1, y^2, \dots, y^r) \Leftrightarrow x^i \leq_i y^i, \forall \, i.$$

We say that $(x^1, x^2, \dots, x^r) \; and \; (y^1, y^2, \dots, y^r)$ are comparable if

$(x^1, x^2, \dots, x^r) \leq (y^1, y^2, \dots, y^r)$ or $(x^1, x^2, \dots, x^r) \geq (y^1, y^2, \dots, y^r)$

Theorem 3.2 In addition to the hypothesis of theorem 3.1, suppose that for every

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$$(x^1, x^2, \dots, x^r), (y^1, y^2, \dots, y^r) \in X^r$$

There exist $(z^1, z^2, \dots, z^r) \in X^r$ such that $F(z^{\sigma_i(1)}, z^{\sigma_i(2)}, \dots, z^{\sigma_i(r)})$ is comparable to $F(x^{\sigma_i(1)}, x^{\sigma_i(1)}, \dots, x^{\sigma_i(1)})$ and $(y^{\sigma_i(1)}, y^{\sigma_i(2)}, \dots, y^{\sigma_i(r)})$, for all i. Then F and g have a unique γ -coincidence point, which is a fixed point of $g: X \to X$ and $F: X^r \to X$. That is there exists a unique $(u^1, u^2, \dots, u^r) \in X^r$ such that

(3.31) $u^i = g(u^i) = F(u^{\sigma_i(1)}, u^{\sigma_i(2)}, \dots, u^{\sigma_i(r)})$ for all $i \in \{1,2, \dots, r\}$.

Proof. By theorem 3.1, the set of γ -coincidence points is non-empty. Now, suppose that $(x^1, x^2, \dots, x^r)$ and $(y^1, y^2, \dots, y^r)$ are two γ -coincidence points of F and g, that is

$g(x^i) = F(x^{\sigma_i(1)}, x^{\sigma_i(2)}, \dots, x^{\sigma_i(r)})$ for all i and

$g(y^i) = F(y^{\sigma_i(1)}, y^{\sigma_i(2)}, \dots, y^{\sigma_i(r)})$ for all i.

We will show that

(3.32) $g(x^i) = g(y^i)$, for all i.

By assumption, there exists $(z^1, z^2, \dots, z^r) \in X^r$ such that

$F(z^{\sigma_i(1)}, z^{\sigma_i(2)}, \dots, z^{\sigma_i(r)})$ is comparable to $F(x^{\sigma_i(1)}, x^{\sigma_i(1)}, \dots, x^{\sigma_i(1)})$ and $F(y^{\sigma_i(1)}, y^{\sigma_i(2)}, \dots, y^{\sigma_i(r)})$, for all i.

Let $z_0^i = z^i$ for all $i \in \{1,2, \dots, r\}$. Since $F(X^r) \subseteq g(X)$, we can choose $z_1^i \in X$ such that

$g(z_1^i) = F\left(z_0^{\sigma_i(1)}, z_0^{\sigma_i(2)}, \dots, z_0^{\sigma_i(r)}\right)$ for all i. By a similar reason, we can inductively define sequences $\{g(z_n^i)\}, n \in N$ for all $i \in \{1,2, \dots, r\}$ such that

$g(z_{n+1}^i) = F\left(z_n^{\sigma_i(1)}, z_n^{\sigma_i(2)}, \dots, z_n^{\sigma_i(r)}\right)$ for all i.

In addition, let $x_0^i = x^i$ and $y_0^i = y^i$ for all i and in the same way, define the sequences $\{g(x_n^i)\}$ and $\{g(y_n^i)\}, n \in N$ for all i. Since

$g(x^i) = F(x^{\sigma_i(1)}, x^{\sigma_i(2)}, \dots, x^{\sigma_i(r)})$ and $g(z^i) = F(z^{\sigma_i(1)}, z^{\sigma_i(2)}, \dots, z^{\sigma_i(r)})$ for all i, are comparable, then

$g(x_1^i) \leq_i g(z_1^i)$ or $g(x_1^i) \geq_i g(z_1^i)$ for all i .

Define $\beta_n = max_{1 \leq i \leq r}\, d(gx^i, gz_n^i)$ for all n. Following the reasoning in theorem 3.1, we can deduce that $\{ d(gx^i, gz_n^i)\} \to 0$ for all i, that is,

(3.34) $lim_{n \to \infty} gz_n^i = gx^i$.

Similarly, one can prove that

(3.35) $lim_{n \to \infty} d\left(g(y^i), g(z_{n+1}^i)\right) = 0$ for all i.

Using (3.34), (3.35) and triangle inequality we get

$d\left(g(x^i), g(y^i)\right) \leq d\left(g(x^i), g(z_{n+1}^i)\right) + d\left(g(z_{n+1}^i), g(y^i)\right) \to 0$, for all i.

As $n \to \infty$. Hence, $g(x^i) = g(y^i)$.

Since $g(x^i) = F(x^{\sigma_i(1)}, x^{\sigma_i(2)}, \dots, x^{\sigma_i(r)})$ for all i, hence, we have

(3.36) $\lim g\left(g(x^i)\right) = \lim g\left(F(x^{\sigma_i(1)}, x^{\sigma_i(2)}, \dots, x^{\sigma_i(r)})\right) = \lim F(gx^{\sigma_i(1)}, gx^{\sigma_i(2)}, \dots, gx^{\sigma_i(r)})$

Denote $gx^i = u^i$ for all i. From (3.36), we have

(3.37) $g(u^i) = g(gx^i) = F(u^{\sigma_i(1)}, u^{\sigma_i(2)}, \dots, u^{\sigma_i(r)})$ for all i.

Hence $(u^1, \dots, u^{r-1}, u^r)$ is a γ-coincidence point of F and g.

It follows $y^i = u^i$ and so $g(y^i) = g(u^i)$ for all i.

This means that $g(u^i) = u^i$ for all i.

Now, from (3.37), we have

$u^i = g(u^i) = F(u^{\sigma_i(1)}, u^{\sigma_i(2)}, \dots, u^{\sigma_i(r)})$ for all i.

Hence, $(u^1, u^2, \dots, u^r)$ is a γ -fixed point of F and a fixed point of g.

To prove the uniqueness of the fixed point, assume that $(v^1, v^2, \dots, v^r)$ is another γ -fixed point. Then, we have

$u^i = g(u^i) = v^i = g(v^i)$ for all i.

Thus, $(u^1, u^2, \dots, u^r) = (v^1, v^2, \dots, v^r)$. This completes the proof.

In the following theorem, we replace the continuity of g, the compatibility of F and g and the completeness of X by assuming that g(X) is a complete subspace of X.

Theorem 3.3 Let (X, $\preceq$) be a partially ordered set equipped with a metric d such that (X, d) is a complete metric space. Assume that there is a function $\varphi:[0,\infty)\to[0,\infty)$ with $\varphi(t)<t$ and $\lim_{r\to t^+}\varphi(r)<t$ for each t > 0. Further let $F:\prod_{i=1}^{r} X^i \to X$ and g: X → X be two maps such that F has the mixed g-monotone property and satisfying (3.1), (3.4) and the following conditions:

(3.38) g(X) is a complete subspace of X,

Also, suppose that either X has the following properties:

(i) If a non-decreasing sequence $\{x_n\}\to x$ then $x_n \preceq x$ for all $n\geq 0$.

(ii) If a non-inecreasing sequence $\{y_n\}\to y$ then $y\preceq y_n$ for all $n\geq 0$.

If there exist $x_0^1, x_0^2, x_0^3, \ldots\ldots, x_0^r \in X$ such that (3.5) holds. Then F and g have a r-tupled coincidence point.

Proof: We construct the sequences $\{x_n^1\},\{x_n^2\},\{x_n^3\},\ldots,\{x_n^r\}$ as in Theorem 3.2. As in the proof of Theorem 3.2, the sequences $\{g(x_m^1)\},\{g(x_m^2)\},\{g(x_m^3)\},\ldots\ldots\ldots, and\ \{g(x_m^r)\}$ are Cauchy sequences. Since g(X) is complete, there exist $x^1, x^2,\ldots\ldots, x^r \in X$ such that

(3.39) $\lim_{m\to\infty} g(x_m^1) = gx^1, \lim_{m\to\infty} g(x_m^2) = gx^2, \ldots\ldots\ldots\ldots\ldots, \lim_{m\to\infty} g(x_m^r) = gx^r.$

Since $g(x_m^i)$ is non-decreasing or non-increasing as i is odd or even and $g(x_m^i)\to x^i$ as $m\to\infty$, we have $g(x_m^i)\leq x^i$, when i is odd while $g(x_m^i)\geq x^i$, when i is even. Since g is monotonically increasing, therefore

(3.40) $g\left(g(x_m^i)\right)\leq g(x^i)$ when i is odd,

$g\left(g(x_m^i)\right)\geq g(x^i)$ when i is even.

(3.41)
$$d\left(g(x^i), F(x^{\sigma_i(1)}, x^{\sigma_i(2)},\ldots,x^{\sigma_i(r)})\right)$$
$$\leq d\left(g(x^i), g(x_{m+1}^i)\right) + d\left(g(x_{m+1}^i), F(x^{\sigma_i(1)}, x^{\sigma_i(2)},\ldots,x^{\sigma_i(r)})\right)$$
$$\leq d\left(g(x^i), g(x_{m+1}^i)\right) + d\begin{pmatrix}\left(F\left(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)},\ldots,x_m^{\sigma_i(r)}\right)\right),\\ F(x^{\sigma_i(1)}, x^{\sigma_i(2)},\ldots,x^{\sigma_i(r)})\end{pmatrix}$$
$$d\left(g(x^i), g(x_{m+1}^i)\right) + \varphi\left(max\left\{d\left(g\left(x_m^{\sigma_i(r)}\right), g(x^{\sigma_i(r)})\right)\right\}\right)$$
$$\to 0\ as\ n\to\infty.$$

Therefore, $g(x^i) = F(x^{\sigma_i(1)}, x^{\sigma_i(2)},\ldots,x^{\sigma_i(r)})$. Similarly we can prove. Thus the theorem follows.

Conclusion : Our work sets analogues, unifies, generalizes, extends and improves several well known results existing in literature, in particular the recent results of [1-4,7-10,12,13,15,19,21,25,26] etc. in the frame work of ordered metric spaces as the notion of compatible maps is more general than commuting and weakly commuting maps. Our theorems 3.1 and 3.2 have been proved by assuming much weaker condition than in analogous results and our corollary 3.1 is a generalization of corollary 1[10] and theorem 2.1 [9]. Also, our theorem 3.3 does not need completeness of space and continuity of maps involved therein. The results concerning commuting and weakly commuting maps being extendable in the spirit of our theorems, can be extended verbatim by simply using wider class of compatibly in place of commuting and weakly commuting maps.

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References:

[1] A. Roldan, J. Martınez-Moreno, and C. Roldan, “Multidimensional fixed point theorems in partially ordered complete metric spaces,” *Journal of Mathematical Analysis and Applications*, vol. 396, no. 2, pp. 536–545, 2012.

[2] V. Berinde. and M.. Borcut, Tripled fixed point theorems for contractive type mappings in Partially ordered metric spaces, Nonlinear Analysis, Volume 74, Issue 15, October 2011, Pages 4889-4897.

[3] A. C. M. Ran and M. C. B. Reuring, A fixed point theorem in partially metric spaces and some applications in partially ordered metric spaces, Proc. Amer. Math. Soc. 132(2004), pp. 1435-1443.

[4] E. Karapınar, A. Roldan, J.M.Moreno and C.Roldan, Meir –Keller type multidimensional fixed point theorems in partially ordered metric spaces, Abstract and Applied Analysis, volume 2013,Article ID 406026,9pages.

[5] E. Karapınar and V. Berinde, “Quadruple fixed point theorems for nonlinear contractions in partially ordered metric spaces,” *Banach Journal of Mathematical Analysis*, vol. 6, no. 1, pp. 74–89, 2012.

[6] E. Karapinar, N. V. Luong, N. X. Thuan, Coupled coincidence points for mixed monotone operators in partially ordered metric spaces, Arab J. Math(2012)1, pp. 329-339.
[7] G. Jungck, Commuting mappings and fixed point, Amer. Math. Monthly, 83 (1976), pp. 261-263.
[8] G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci., 9(1986), pp. 771-779.
[9] H.S.Ding and S. Radenovic, Coupled coincidence point theorems for generalized nonlinear contractionin partially ordered metric spaces, Fixed point theory and applications 2012, 2012:96, pp. 1-10.
[10] J. Harjni, B. Lopez and K. Sadarangani, Fixed point theorems for mixed monotone operators and applications to integral equations, Nonlinear Analysis 74(2011), 1749-1760.
[11] J .J. Nieto, R. Rodriguez-Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equation, Order 22(2005), pp. 223-239.
[12] M. Paknazar, M. E. Gordji, M. D. L. Sen and S. M. Vaezpour, *N*-fixed point theorems for nonlinear contractions in partially ordered metric spaces, Paknazar et al. *Fixed Point Theory and Applications* 2013, **2013**:11
[13] M. Imdad, A. H. Soliman, B. S. Choudhary and P. Das, On n-tupled Coincidence Point Results in Metric Spaces, Journal of Operators, Volume 2013, Article ID 532867, pp. 1-8.
[14] M. Gugani, M. Agarwal and R. Chugh, Common Fixed Point Results in G-Metric Spaces and Applications, International Journal of Computer Applications (0975 – 8887) Volume 43– No.11, April 2012, pp. 38-42.
[15] Muzeyyen Erturk and Vatan Karakaya, n-tuplet fixed point theorems for contractive type mappings in partially ordered metric spaces, Journal of Inequalities and Applications 2013, 2013:196, pp. 1-19.
[16] M. Abbas, B. Ali, W. Sintunavarat and P. Kumam, Tripled fixed point and tripled coincidence point theorems in intuitionistic fuzzy normed spaces , Abbas et al. *Fixed Point Theory and Application, 2012,2012:87.*
[17] M. Abbas, B. Ali, W. Sintunavarat and P. Kumam, Tripled fixed point and tripled coincidence point theorems in intuitionistic fuzzy normed spaces , Abbas et al. *Fixed Point Theory and Application 2012,2012:87*
[18] M. Berzig and B. Samet, “An extension of coupled fixed point’s concept in higher dimension and applications,” *Computers & Mathematics with Applications*, vol. 63, no. 8, pp. 1319–1334, 2012
[19] M. Borcut, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, Appl. Math. Comput, 218(2012), no.14, 7339-734.
[20] Sumitra, Sunny Chaun, B.D Pant, Coincidence and common fixed point theorems in intutionistic Fuzzy –metric Spaces, Far East Journal of Mathematical Science, Volume79, Number 1, 2013, pp. 25-48.
[21] Sumitra Dalal**,** Sunny Chauhan and Zoran Kadelburg**,** A common fixed point theorem in metric space under general contractive condition , Journal of Applied Mathematics, Volume 2013 (2013), Article ID 510691, 7 pages http://dx.doi.org/10.1155/2013/510691
[22] Sumitra Dalal and Saurabh Manro, Coincidence and common fixed point of weakly compatible maps in fuzzy metric spaces, Applied Mathematics. To appear
[23] Sumitra Dalal, Mohammad Imdad and Sunny Chauhan, Unified fixed point theorems via common limit range property in modified intuitionistic fuzzy metric spaces, Hindawi Publishing Corporation, Abstract and Applied Analysis, Volume 2013, Article ID 413473, 11 pages
[24] Sumitra Dalal, Sunny Chaun and M.Alamgir Khan, n-tupled coincidence point theorems in patially ordered metric spaces for compatible maps, Hindawi Publishing Corporation, Abstract and Applied Analysis, To appear.
[25] T. G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis: Theory, Methods and Applications, vol.65, no.7, 2006, pp. 1379-1393.
[26] V. Lakshmikantham and L. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis, Theory, Method and Applications, vol. 70, no12, 2009, pp. 4341-4349
[27] V. Nguyen, X. Nguyen, Coupled fixed point theorems in partially ordered metric spaces, Bull. Math. Anal. Appl. 2(2010), pp. 16-24.
[28] W. Shatanawi, Coupled Fixed Point Theorems in Generalized Metric Spaces, Hacettepe, Journal of Mathematics and Statistics, Volume 40(3)(2011), pp. 441-447.