ISSN 2347-1921

# Spectral relationships of Carleman integral equation in some different domains

M.A.Abdou, R.O. Abd El Rahman
Department of Mathematics, Faculty of Education, Alexandria University, Egypt
Department of Mathematics, Faculty of Science, Damanhur University, Egypt
abdella_777@yahoo.com, ragab_537yahoo.com


## Abstract

In this paper, the potential theory method (**PTM**) is used, in some different domains, to obtain the solution of Fredholm integral equation (**FIE**) of the first kind with Carleman kernel. The solution is obtained in the form of spectral relationships (**SRs**). Many new and important relationships are established and discussed from the work.


**Keywords:** Potential theory method; Carleman function; boundary value problem (BVP); spectral relationships.

Council for Innovative Research
Peer Review Research Publishing System
**Journal:** Journal of Advances in Mathematics
Vol 9, No 1
editor@cirjam.org
www.cirjam.com, www.cirworld.com

## 1. Introduction:

Singular integral equation have received considerable interest the mathematical applications in different areas of sciences, for example see Constanda [1], Venturino [2], Kangro [3], Diego [4], and Anastasia [5]. The solution of these problems can be obtained analytically, using the theory developed by Muskhelishvili [6]. The books edited by Popov [7], Tricomi [8], Hocthstab [9] and Green [10] contained many different methods to solve the integral equations analytically. More recently, since analytical method on practical problem often fail, numerical solution of these equations is a much studied subjected of numerous works. The interested reader should consult the fine exposition by Atkinson [11], Delves and Mohamed [12], Golberg [13] and Linz [14] for numerical methods. The solution of integral equation, using potential theory method, attracts the interest of many authors; see Alexandrov et al. [15], Abdou [16-17] and Abdou et al. [18].

Here, the existence and uniqueness of solution of **FIE** of the first kind with Carleman kernel is considered. Then, the **PTM** is used to obtain the **SRs** of the integral equation.

Consider the following integral relation:

$$\int_{\ell} \frac{\varphi(t)\,dt}{|x-t|^{\mu}} = f(x) \qquad (0<\mu<1) \tag{1}$$

under the static condition

$$\int_{\ell} \varphi(t)\,dt = P \qquad \text{(P is constant)}, \tag{2}$$

where $\ell$ is considered in the following three cases

$$(i)\quad \ell=\{(x,y)\in\ell : y=0, |x|\le a\}, \qquad (ii)\quad \ell=\{y=0, |x|\ge 0\},$$

$$(iii)\quad \ell=\{y=0,\ \ 0\le x<\infty\} \tag{3}$$

In Eq. (1), $f(x)$ is a known function where, $\phi(x)$ is unknown function.

In order to guarantee the existence of a unique solution of (1) under (2), we assume the following conditions

(1) The singular kernel $k(|x-y|)=|x-y|^{-\mu}$, $(0<\mu<1)$, satisfies the Fredholm condition.

$$[\int_{\ell}\int_{\ell} k^2(|x-y|)dxdy]^{\frac{1}{2}} \le E \text{ , (E is a constant).}$$

(2) The given continuous function (free term), $f(x)\in C[\ell]$, $x\in[\ell]$ with its first derivatives.

(3) The unknown function $\phi(x)$ satisfies the Lipschitz condition.

**2. Potential Theory Method:** For using the **PTM**, to obtain the solution of integral equation (1), under (2), we introduce the general Carleman potential function

$$\int_{\ell} \frac{\varphi(t)\,dt}{[(x-y)^2+y^2]^{\mu/2}} = U(x,y) \tag{4}$$

The solution of Eq. (1) under (2) with the aid of (4), is equivalent to the following **BVP**:

$$\Delta U + \frac{\mu}{y}\frac{\partial U}{\partial y} = 0 \qquad \left((x,y)\notin\ell\ ,\ \ \Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

$$U(x,y)\Big|_{y=0} = f(x,0) \qquad (x,y)\in\ell$$

$$U(x,y) \cong \Pr^{-\mu} \to \mathit{finite} \quad \text{as } r\to\infty \quad (r=\sqrt{x^2+y^2}) \tag{5}$$

The complete solution of (1) will be considered from the following equivalence condition, see Abdou [16]

$$\phi(x) = \lambda \lim_{y\to 0} \operatorname{sgn} y |y|^{\eta} \frac{\partial U}{\partial y} \qquad (x \in \ell; \quad \lambda = \frac{\Gamma(\mu/2)}{2\sqrt{\pi}\,\Gamma((1+\mu)/2)}) \qquad (6)$$

where $\Gamma(t)$ is the gamma function.

We go now to discuss the solution of Eq. (5) in the three cases of the domain $\ell$ of Eq. (3)

### 2.1. Case (i): $\ell = \{(x,y) \in \ell : y = 0, |x| \le 1\}$.

In this case, we eliminate the term $\frac{\partial U}{\partial y}$ of Eq. (5), by assuming

$$U(x,y) = |y|^{-\mu/2} V_o(x,y) \qquad (7)$$

Then, we use the transformation mapping

$$z = x + iy = \frac{w(\xi)}{2} = \frac{1}{2}(\xi + \xi^{-1}) \ , \ \xi = \zeta + i\eta = \rho e^{i\theta}, \qquad i = \sqrt{-1}\ . \qquad (8)$$

The mapping function (8) maps the region in $(x, y)$ plane into the region outside the unit circle $\gamma$, such that $\dot{\omega}(\xi)$ does not vanish or becomes infinite outside the unit circle $\gamma$. Also the function (8) maps the upper and lower half- plane $((x, y) \in [-1,1])$ into the lower and the upper of the semi circle $\rho = 1$, respectively. Moreover the point $z = \infty$ will be mapped onto the point $\xi = 0$.

Hence, the **BVP** of (5), after using (8) and (7), becomes

$$\Delta V(\rho,\theta) + \mu(2-\mu)\left[\frac{1}{(\rho^2-1)^2} + \frac{1}{4\rho^2 \sin\theta}\right] V(\rho,\theta) = 0 \quad (\rho \le 1),$$

$$[\tfrac{1}{2}(\rho - \tfrac{1}{\rho})\sin\theta\,]^{-\mu/2} = f(\cos\theta), \qquad (\rho < 1)$$

$$V(\rho,\theta)\Big|_{\rho=0} = 0, \quad (-\pi < \theta \le \pi; \quad \Delta = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}). \qquad (9)$$

Where, $V(\rho,\theta) = V_0(x,y) = V_o\left(\frac{1}{2}(\rho + \frac{1}{\rho})\cos\theta,\ \frac{1}{2}(\rho - \frac{1}{\rho})\sin\theta\right)$.

Using the separation of variables

$$V(\rho,\theta) = R(\rho)\,\Phi(\theta) \qquad (10)$$

The first formula of (9) can be written in the following:

$$\rho^2 \frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + \left[\mu(2-\mu)\frac{\rho^2}{(\rho^2-1)^2} - \lambda^2\right] R = 0 \quad (0 \le \rho \le 1) \qquad (11)$$

$$\frac{d^2\Phi}{d\theta^2} + \left[\lambda^2 + \frac{\mu(2-\mu)}{4\sin^2\theta}\right]\Phi = 0 \qquad (-\pi < \theta \le \pi) \qquad (12)$$

where $\lambda$ is the constant of separation of variables.

From the third formula of (9) and (10), we deduce that $R(0)=0$. Moreover, the differential equation of the formula (12) yields us to consider the periodic condition $\Phi(\theta)=\Phi(\theta+2\pi)$. This periodic condition leads us to assume the solution of Eq. (12) in the form:

$$\Phi(\theta)=\sqrt{|\sin\theta|}\ G(\theta) \qquad (0<\pi<\theta) \tag{13}$$

With the aid of periodic condition and the properties of Legendre polynomial see Bateman et.al [19], we have $\lambda=n+\mu/2 \quad (n=0,1,2,...)$ . Also, it is easily to prove that the even function $(\sin\theta)^{\upsilon}P_{n-\upsilon}^{\upsilon}(\cos\theta)$, $(\upsilon=\frac{1-\mu}{2},\ 0<\theta<\pi)$ and the odd function $(\sin\theta)^{\upsilon}Q_{n-\upsilon}^{\upsilon}(\cos\theta)$ $(-\pi<\theta<0)$ satisfy the differential equation (12). Hence the unique solution of (12), which satisfies the periodic solution, takes the form:

$$\Phi(\theta)=\sqrt{|\sin\theta|}\ P_{n-\upsilon}^{\upsilon}(\cos\theta) \qquad (-\pi<\theta\le\pi;\ n=1,2,...\ ) \tag{14}$$

Using the relation between the Legendre polynomial $P_{n-\upsilon}^{\upsilon}(\cos\theta)$ and the Gegenbauer polynomial $C_n^{\alpha}(x)$, see Bateman et.al [19], Eq. (14) yields

$$\Phi(\theta)=|\sin\theta|^{\varpi/2}C_n^{\mu/2}(\cos\theta) \qquad (-\pi<\theta\le\pi;\ n=1,2,...\ ) \tag{15}$$

Also, after using the Riemann Scheme, see Gradstein et.al [20]

$$M(u)=u^{\lambda/2}(1-u)^{\mu/2}P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & \mu/2 & 0 & u \\ -\lambda & \lambda+\mu/2 & 1-\mu & \end{Bmatrix},$$

$$(u=\rho^2,\ R(\sqrt{u})=M(u),\quad 0\le u<1) \tag{16}$$

in the Gauss hypergeametric function form F(a, b; c; z), the solution of Eq. (11) takes the form

$$R(\rho)=\rho^{n+\mu/2}(1-\rho^2)F(\mu/2,n+\mu;n+1+\mu/2\ ;\rho^2\ ),\ (0\le\rho<1) \tag{17}$$

Using (16) and (17) in (10), the solution of the **BVP** (9) takes the form:

$$V(\rho,\theta)=\rho^{n+\mu}F(\mu/2,n+\mu;n+1+\mu/2\ ;\rho^2\ )C_n^{\mu/2}(\cos\theta)$$

$$(0\le\rho<1\ ,\quad -\pi<\theta\le\pi\quad ,\quad n\ge 0\quad ) \tag{18}$$

The complete solution of integral equation (1) can be obtained after adapting the equivalence condition (4) in the form

$$\Phi(\cos\theta)=\frac{\Gamma(\mu/2)(\sin\theta)^{\mu-1}}{\sqrt{\pi}\,2^{\mu+1}\Gamma((\mu+1)/2)}\lim_{\rho\to 1}(1-\rho^2)^{\mu}\frac{\partial V(\rho,\theta)}{\partial\rho} \qquad (0<\theta<\pi) \tag{19}$$

Then, using the formula (18) to obtain

$$\Phi(\cos\theta)=\frac{\Gamma(\mu)\Gamma(n+1+\mu/2)}{\sqrt{\pi}\,2^{\mu}\Gamma(\mu+1)\Gamma((\mu+1)/2)}(\sin\theta)^{\mu-1}C_n^{\mu/2}(\cos\theta) \qquad (0<\theta<\pi\ ) \tag{20}$$

The given function $f(x)$ of Eq. (1) can be written in the Gegenbauer polynomial, after using the second condition of Eqs. (5), (7) with the aid of (8) we have

ISSN 2347-1921

$$f(\cos\theta) = \frac{\Gamma(1-\mu)\,\Gamma(n+1+\mu/2)}{n!\,\Gamma(1-\mu/2)}\,C_n^{\mu/2}(\cos\theta) \qquad (0<\theta<\pi) \tag{21}$$

Inserting (20) and (21) in the integral equation (1), we obtain the following **SRs** of integral equation of the first kind with Carleman kernel

$$\int_{-1}^{1} \frac{C_n^{\mu/2}(t)\ dt}{|x-t|^{\mu}(1-t^2)^{(1-\mu)/2}} = \alpha_n C_n^{\mu/2}(x) \qquad (\,|t|<1) \tag{22}$$

where the eigenvalues $\alpha_n$ are given by

$$\alpha_n = \pi\Gamma(n+\mu)\left[n!\,\Gamma(\mu)\cos(\pi\mu/2)\right]^{-1} \qquad (0<\mu<1,\ \ n\ge 0\ \ ) \tag{23}$$

From the parametric equation of the transformation mapping (2.5), the inequality $|x|>1$ leads us to take $\theta=0$ when $x>1$ and $\theta=\pi$ when $x<-1$

Also, we have

$$\rho = |x| - \sqrt{x^2-1}\ , \qquad (|x|>1) \tag{24}$$

Substituting $\theta=0$ into (18), then using the result to evaluate Eq. (20) at $\theta=0$, also using $\theta=\pi$ in (6), with the aid of (24), we get

$$\int_{-1}^{1} \frac{C_n^{\mu/2}(t)\ dt}{|x-t|^{\mu}(1-t^2)^{(1-\mu)/2}} = \upsilon_n\left[H(x)+(-1)^n H(-x)\right]\left(|x|-\sqrt{x^2-1}\right)^{n+\mu} \times$$

$$F(\mu/2,\, n+\mu;\, n+1+\mu/2;\, 2x^2-2|x|\sqrt{x^2-1}-1)\ , \qquad (\,|x|<1\,) \tag{25}$$

where $H(x)$ is the Heaviside function and

$$\upsilon_n = \frac{\sqrt{\pi}\,2^{\mu}\,\Gamma^2(n+\mu)\,\Gamma((1+\mu)/2)}{n!\ \ \Gamma^2(\mu)\Gamma(n+1+\mu/2)} \tag{26}$$

**2.2. Case (ii)** $\ell=\{\,y=0, |x|\ge 0\}$: The integral equation (1) with its static condition become

$$\left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right) \frac{\phi(t)\ dt}{|x-t|^{\mu}} = f(x) \tag{27}$$

and

$$\left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right) \phi(t)\ dt = P < \infty \tag{28}$$

The transformation mapping (8) maps the all points $|x|>1$ into the half space $\eta>0$, in the $\xi$- plane. Hence, for $y>0$, we have $\{\rho>1,\quad 0<\theta<\pi\}$, while for $y<1$, we have $\{\rho>1,\quad 0<\theta<\pi\}$.Therefore we have the following **BVP**:

$$\Delta V + \mu(2-\mu)\left[\frac{1}{(\rho^2-1)^2} + \frac{1}{4\rho^2 \sin\theta}\right] V = 0 \qquad (\rho < \infty, \quad 0 < \theta < \pi)$$

$$(\frac{1}{2}\left|\rho - \frac{1}{\rho}\right| \sin\theta)^{-\mu/2} V(\rho,\theta)\Big|_{\theta=0,\theta=\pi} = f[\frac{1}{2}(\rho + \frac{1}{\rho})\cos\theta]\Big|_{\theta=0,\theta=\pi}, (1 < \rho < \infty)$$

$$(\frac{1}{2}\left|\rho - \frac{1}{\rho}\right| \sin\theta)^{-\mu/2} V\big(\rho,\theta\big)\Big|_{\rho\to 0,\,\rho\to\infty} \approx \Pr^{-\mu} \quad (r \to \infty) \tag{29}$$

To discuss and obtain the solution of (29), we use the formula (10) to separate the variables. In this aim, we have the same two differential equations of (11) and (12) after removing the constant $\lambda^2$ by $-\lambda^2$ . Then solving these two formulas, with the aid of Bateman et.al [19], we obtain

$$U(x,y) = U_0(\rho,\theta) = (\frac{1}{2}\left|\rho - \frac{1}{\rho}\right| \sin\theta)^{\upsilon}.[\, AP_\alpha^\upsilon(\cos\theta) + BQ_\alpha^\upsilon(\cos\theta)\,] \times$$

$$\{ CP_\alpha^\upsilon[\frac{1}{2}(\rho + \frac{1}{\rho})] + DP_\alpha^{-\upsilon}[\frac{1}{2}(\rho + \frac{1}{\rho})]\},$$

$$(\rho < \infty, 0 < \theta < \pi,\ \alpha = -\frac{1}{2} + i\lambda,\ \upsilon = \frac{1-\upsilon}{2},\ \ \lambda > 0\ ) \tag{30}$$

where $A, B$ are complex constants, while $C$ and $D$ are real constants.

To discuss the even and odd functions of the variables $\theta\ (at\theta = \frac{\pi}{2})$ of Eq. (30), we assume

$$G(\theta) = AP_\alpha^\upsilon(\cos\theta) + BQ_\alpha^\upsilon(\cos\theta) \tag{31}$$

With the help of the two formulas (14) and (15); pp.145 0f Bateman et.al [19], where $G(\theta) = G(\pi - \theta),\ (0 < \theta \le \frac{\pi}{2})$.

And then, for the even function at $\theta = \frac{\pi}{2}$, we have

$$A\{\cos[\pi(\frac{\mu}{2} - i\lambda] - 1\} + \frac{\pi B}{2}\sin[\pi(\frac{\mu}{2} - i\lambda)] = 0$$

$$\frac{2A}{\pi}\sin[\pi(\frac{\mu}{2} - i\lambda)] - B\{1 + \cos[\pi(\frac{\mu}{2} - i\lambda]\} = 0$$

Solving the two previous formulas, we get

$$B = \frac{2A}{\pi}\tan\left[\frac{\pi}{2}(\frac{\mu}{2} - i\,\lambda)\right] \tag{32}$$

With the help of formula (32) we can obtain the real even function of Eq. (31) in the form

$$G(\theta) = (\sin\theta)^\upsilon G_\lambda^+(\theta), \qquad (0 < \theta < \pi),$$

$$G_\lambda^+(\theta) = P_\alpha^\upsilon(\cos\theta) + \frac{1}{\pi}\Big\{\tan\,[\pi(\frac{\mu}{4} - \frac{i\lambda}{2})]Q_\alpha^\upsilon(\cos\theta) + \tan[\pi(\frac{\mu}{4} + \frac{i\lambda}{2})]\ Q_\alpha^\upsilon(\cos\theta)\Big\} \tag{33}$$

Also, for the odd real function, we have

$$G(\theta) = (\sin\theta)^\upsilon G_\lambda^-(\theta) \qquad (0 < \theta < \pi)$$

$$G_\lambda^-(\theta) = P_\alpha^\upsilon(\cos\theta) - \frac{1}{\pi}\Big\{\cot[\pi(\frac{\mu}{4} - \frac{i\lambda}{2})]Q_\alpha^\upsilon(\cos\theta) + \cot[\pi(\frac{\mu}{4} + \frac{i\lambda}{2})]Q_\alpha^\upsilon(\cos\theta)\Big\} \tag{34}$$

Using (33) and (34) in (20), after putting $C=1, D=0$, we have the solution of the **BVP** in the form:

$$U(\rho,\theta)=(\tfrac{1}{2}\left|\rho-\tfrac{1}{\rho}\right|)^{\upsilon}G(\theta)\,P_{\alpha}^{\upsilon}[\tfrac{1}{2}(\rho+\tfrac{1}{\rho})] \qquad ,(\rho<\infty\ ,\ 0<\theta<\pi) \quad (35)$$

Also, for $C=0, D=1$ we have

$$U_0(\rho,\theta)=(\tfrac{1}{2}\left|\rho-1/\rho\right|)^{\upsilon}G_0(\theta)\,P_{\alpha}^{-\upsilon}[\tfrac{1}{2}(\rho+\tfrac{1}{\rho})] \quad (\rho<\infty\,,\ 0<\theta<\pi) \quad (36)$$

The reader, after some calculus, can establish from the formula (35) at $\theta=0$, the relation

$$U_0(\rho,\theta)=\frac{2^{\upsilon}}{\Gamma((1+\mu)/2)}\left[\frac{1+\cos(\pi\mu/2)\cosh(\lambda\pi)}{\cos(\pi\mu/2)[\cosh(\lambda\pi)+\cos(\pi\mu/2)]}\right]\times$$

$$\left[1/2\left|\rho-1/\rho\right|\right]^{\upsilon}P_{\alpha}^{\upsilon}[\tfrac{1}{2}(\rho+\tfrac{1}{\rho})]=f(\tfrac{1}{2}(\rho+\tfrac{1}{\rho})) \quad (37)$$

To obtain the equivalence relation of the potential function, we can adapt the right hand side of (6) in polar coordinates, then using (35) to obtain

$$\phi(x)=E_{\lambda}^{\mu}\,[\tfrac{1}{2}\left|\rho-1/\rho\right|]^{\upsilon}P_{\alpha}^{\upsilon}[\tfrac{1}{2}(\rho+1/\rho)] \quad ,\quad (x=\tfrac{1}{2}(\rho+1/\rho),\ \rho>1),$$

$$E_{\lambda}^{\mu}=\frac{\Gamma(\mu/2)}{2^{\upsilon}\sqrt{\pi}}\left[\frac{1+\cos(\pi\mu/2)\cosh(\lambda\pi)}{\{\left|\Gamma(\mu/2+i\lambda)\right|[\cosh(\lambda\pi)+\cos(\pi\mu/2)]\}^{2}}\right] \quad (38)$$

Substituting from (37) and (38) into (27) we have the following **SRs**:

$$\int_{1}^{\infty}\left(\frac{1}{(x-t)^{\mu}}\pm\frac{1}{(x+t)^{\mu}}\right)(t^{2}-1)^{-\upsilon}\Psi_{+}(t,\lambda)dt=\sigma_{\pm}(\lambda)\Psi_{+}(t,\lambda),\ (x>1,\ \lambda>0) \quad (39)$$

Also, using the two formulas (34) and (36), we have the following **SRs**:

$$\int_{1}^{\infty}\left(\frac{1}{(x-t)^{\mu}}\pm\frac{1}{(x+t)^{\mu}}\right)(t^{2}-1)^{-\upsilon}\Psi_{-}(t,\lambda)dt=\sigma_{\pm}(\lambda)\Psi_{-}(t,\lambda),(x>1,\ \lambda>0) \quad (40)$$

Here, in (39) and (40) we assume

$$\Psi_{\pm}(x,\lambda)=(x^{2}-1)^{\upsilon/2}P_{\alpha}^{2\upsilon}(x),$$

$$\sigma_{\pm}(\lambda)=[\cosh(\pi\lambda)\pm\cos(\pi\mu/2)]\left|\Gamma(\mu/2+i\lambda)\right|^{2}[\Gamma(\mu)\cos(\pi\mu/2)]^{-1} \quad (41)$$

The two formulas of Eqs. (39), (40) can be written in following **SRs**:

$$\int_{1}^{\infty}\frac{\Psi_{+}(t,\lambda)}{\left|(x\pm t)\right|^{\mu}}(t^{2}-1)^{-\upsilon}dt=\rho_{\pm}\Psi_{+}(x,\lambda) \qquad (x>1,\ \lambda>0) \quad (42)$$

$$\int_{1}^{\infty}\frac{\Psi_{-}(t,\lambda)}{\left|(x\pm t)\right|^{\mu}}(t^{2}-1^{-\upsilon}dt=\rho_{\pm}\Psi_{-}(x,\lambda) \qquad (x>1,\ \lambda>0) \quad (43)$$

where

$$\rho_{+}(x)=\left|\Gamma(\mu/2+i\lambda)\right|^{2}[\Gamma(\mu)]^{-1}$$

$$\rho_{-}(x)=\cosh(\pi\lambda)\left|\Gamma(\mu/2+i\lambda)\right|^{2}[\Gamma(\mu)\cos(\pi\mu/2)]^{-1}$$

As important **SRs** in contact problems, we take in (42) and (43) the positive sign. Then let $\mu=1$, to get

$$\int_1^\infty \frac{P_{-\frac{1}{2}+i\lambda}(t)\,dt}{x+t} = \frac{\pi}{\cosh(\pi\lambda)} P_{-\frac{1}{2}+i\lambda}(x), \qquad (x>1). \tag{44}$$

Another **SRs** corresponding to the four **SRs** of Eq. (39) and (40), in the interval $0<x<1$, which transformed to $\{\rho<1, 0<\theta<\frac{\pi}{2}\}$ in $\xi-$plane, by using the formulas (33),(35) and (34), (36), with the equivalence condition (38), are

$$\int_1^\infty [\frac{1}{(x-t)^\mu} \pm \frac{1}{(x+t)^\mu}](t^2-1)^{-\upsilon}\Psi_+(t,\lambda)dt = \beta_\pm(\lambda)(1-x^2)^{\upsilon/2} G_\lambda^\pm(arc\cos x) \tag{45}$$

$$\int_1^\infty [\frac{1}{(x-t)^\mu} \pm \frac{1}{(x+t)^\mu}](t^2-1)^{-\upsilon}\Psi_-(t,\lambda)dt = 0 \qquad (0<x<1) \tag{46}$$

where

$$\beta_\pm(\lambda) = |\Gamma(\mu/2+i\lambda)|^2 \frac{[\cosh(\lambda\pi) \pm \cos(\pi\mu/2)]^2}{\Gamma(\mu)[1\pm\cos(\pi\mu/2)\cosh(\lambda\pi)]} \tag{47}$$

### 2.3. Case (iii) $\ell = \{y=0, \ 0\le x<\infty\}$

In this case, we consider the transformation mapping

$$z = \tfrac{1}{2}\xi^2 \tag{48}$$

The parametric equations of (48) will take the form

$$x = \tfrac{1}{2}(\xi^2-\eta^2) \quad , \qquad y = \xi\eta \tag{49}$$

In this case, the corresponding **BVP** to the formulas (5), (7) and (9) will take the form

$$\frac{\partial^2 V}{\partial\xi^2} + \frac{\partial^2 V}{\partial\eta^2} + \frac{\mu}{2}(1-\frac{\mu}{2})(\frac{1}{\xi^2}+\frac{1}{\eta^2})V(\xi,\eta) = 0 \qquad (\eta>0)$$

$$(|\xi|\,\eta)^{-\mu/2} V(\xi,\eta)\Big|_{\eta=0} = f(\xi^2/2) \qquad (0<\xi<\infty)$$

$$(|\xi|\,\eta)^{-\mu/2} V(\xi,\eta) \to 0 \qquad (\xi^2+\eta^2\to\infty) \tag{50}$$

where

$$V(\xi,\eta) = V_0(\tfrac{1}{2}(\xi^2-\eta^2,\xi\eta) \tag{51}$$

Also the equivalence relation (6), in this case, takes the form

$$\frac{2\sqrt{\pi}\,\Gamma((1+\mu)/2)}{\Gamma(\mu/2)}\phi(x) = \xi^{\mu-1}\lim_{\eta\to 0}\eta^\mu \frac{\partial U_0(\xi,\eta)}{\partial\eta} \quad (x=\xi^2/2 \quad , \ \xi>0\ ) \tag{52}$$

$$U_0(\xi,\eta) = U(\frac{1}{2}(\xi^2-\eta^2,\xi\eta) = (|\xi|\,\eta)^{-\mu/2} V(\xi,\eta) \tag{53}$$

To discuss and obtain the solution of (50), we use the separation of variables

$$V(\xi,\eta) = X(\xi)Y(\eta) \tag{54}$$

To obtain

$$\frac{d^2X(\xi)}{d\xi^2} + \left[\frac{\mu(1-\mu)}{4\xi^2} + \lambda^2\right] X = 0 \qquad (-\infty<\xi<\infty) \tag{55}$$

ISSN 2347-1921

$$\frac{d^2Y(\eta)}{d\eta^2}+\left[\frac{\mu(1-\mu)}{4\eta^2}+\lambda^2\right]Y=0 \qquad (0<\eta<\infty) \tag{56}$$

where $\lambda^2$ is the constant of separation.

The independent solution of Eq. (55) can be represented in the form of two functions $\sqrt{\xi}\,J_{\pm}(\lambda\xi),\ \sqrt{\xi}\,Y_{\pm}(\lambda\xi),\ (\lambda,,\xi>0,\ \ \upsilon=\frac{\mu-1}{2})$, where $J_\alpha(x)$ is the Bessel functioning of the first kind, while $Y_\alpha(x)$ of the second kind. Since the function $|\xi|^{-\mu/2}X(\xi)$ (see the second formula of (50), in the axis $-\infty<\xi<\infty$, must be even, bounded and exist at infinity, so, the solution of Eq. (55) will takes the form

$$X(\xi)=\sqrt{\xi}\,J_\upsilon(\lambda\xi) \qquad (\lambda,\xi>0,\quad \upsilon=\frac{\mu-1}{2}) \tag{57}$$

Also, the solution of Eq. (56) takes the form

$$Y(\xi)=\sqrt{\eta}\,K_\upsilon(\lambda\eta) \qquad (\eta>0) \tag{58}$$

where $K_\alpha(x)$ is the Macdonald function

Hence, the solution of the **BVP** (50) becomes

$$V(\xi,\eta)=\sqrt{\xi\eta}\,J_\upsilon(\lambda\xi)K_\upsilon(\lambda\eta) \tag{59}$$

To obtain the equivalence relation, we use the formula (53) to obtain

$$U_0(\xi,\eta)=(\xi\eta)^{-\upsilon}\,J_\upsilon(\lambda\xi)K_\upsilon(\lambda\eta) \qquad (\lambda,\xi,\eta>0\ ) \tag{60}$$

then, finally, we have

$$\phi(x)=\frac{\lambda^{-\upsilon}\Gamma(\mu/2)}{2^{\frac{1}{4}(7-3\mu)}\sqrt{\pi}}\,x^{\upsilon/2}J_\upsilon(\lambda\sqrt{2x}) \tag{61}$$

To satisfy the second formula of (50), with the aid of (60), we use the following properties and relations of Macdonald and Bessel functions, respectively, (see Bateman et.al [19])

$$K_\upsilon=\frac{\pi}{2\sin(\pi\upsilon)}\left[I_{-\upsilon}-I_\upsilon\right];\qquad I_\upsilon(x)\ \sum_{n=0}^{\infty}\frac{(x/2)^{2n+\upsilon}}{n!\,\Gamma(n+\upsilon+1)}. \tag{62}$$

$I_\upsilon(x)$ is called the modified Bessel functions of the first kind.

Hence, we have

$$\lim_{\eta\to 0}\eta^{-\upsilon}K_\upsilon(\lambda,\eta)=\frac{\pi\lambda^{-\upsilon}}{2^{(1+\mu)/2}\cos(\pi\mu/2)\Gamma((1+\mu)/2)}\ . \tag{63}$$

With the aid of the previous formulas (63) and (60), we get

$$u_0(\xi,\eta)\Big|_{\eta=0}=\frac{\pi\lambda^{\upsilon}\,\xi^{-\upsilon}}{2^{(1+\mu)/2}\cos(\pi\mu/2)\Gamma((1+\mu)/2)}J_\upsilon(\lambda\xi)=f(\xi^2/2) \tag{64}$$

Using the two formulas (60) and (58) in the integral equation of Carleman kernel (1), we have the following **SRs**:

ISSN 2347-1921

$$\frac{1}{\pi}\int_0^\infty \frac{t^{\upsilon/2} J_\upsilon(\lambda\sqrt{t})}{|x-t|^\mu}dt = \sigma(\lambda) x^{-\upsilon/2} J_\upsilon(\lambda\sqrt{x}) \qquad (x>0)$$

$$\sigma(\lambda) = \frac{2^{-2\upsilon}\lambda^{2\upsilon}}{\Gamma(\mu)\cos(\pi\mu/2)} \qquad (\lambda > 0, \quad \upsilon = \frac{\mu-1}{2}) \qquad (65)$$

Also, following the same previous way, we obtain the following **SRs**:

$$\int_0^\infty \frac{t^{\upsilon/2} J_\upsilon(\lambda\sqrt{t})}{|x-t|^\mu}dt = \frac{\lambda^{2\upsilon}(-x)^{-\upsilon/2}}{2^{\mu-2}\Gamma(\mu)} K_\upsilon(\lambda\sqrt{-x}) \quad (x>0) \qquad (64)$$

# References


[1] C. Constanda, Integral equation of the first kind in plane elasticity, J. Quart. Appl. Math. Vol. LIII No. 4 (1995) 783-793.

[2] E. Venturino, The Galerkin method for singular integral equations revisited. J Comp. Appl. Math. Vol. 40 (1992) 91- 103

[3] R. Kangro, P. Oja, Convergence of spline collection for Volterra integral equation, Appl. Num. Math.58 (2008) 1434 - 1447.

[4] T. Diego, P. Lima, Super convergence of collocation methods for class of weakly singular integral equations, J. Cam. Appl. Math. 218(2008) 307- 346.

[5] G. A. Anastasia, A. Aral, Generalized Picard singular integrals, Compute. Math. Appl. 57 (2009) 821-830.

[6] N. I. Muskhelishvili, Singular Integral Equations, Noordhoff, Amsterdam,1953.

[7] G. Ya. Popov, Contact Problems for a Linearly Deformable Base, Kiev, Odessa 1982

[8] F.G. Tricomi, Integral Equations, N. Y. Nelson, 1985

[9] H. Hotchstadt, Integral Equations, N. Y. London, Nelson, 1971

[10] C. D. Green, integral equation methods, N. Y. Nelson, 1969

[11] K.E. Atkinson, A Survey of Numerical Method for the Solution of Fredholm Integral Equation of Second Kind, SIAM, Philadelphia, 1976

[12] L. M. Delves, J. L. Mohamed, Computational Methods for Integral Equation, University of Cambridge, Cambridge,1985.

[13] M. A. Golberg, Numerical Solution of Integral Equation, Plenum, N .Y. SIAM, Philadelphia, 1990

[14] P. Linz, Analytical and Numerical Methods for Integral Volterra Equation, SIAM, Philadelphia, 1990

[15] V.M. Alexandrov, E.V. Kovalenko, Problems with Mixed Conditions in Continuous Mechanics, Nauka, Moscow 1986.

[16] M.A. Abdou, Fredholm integral kernel and its structure resolvent, J.Appl. Math. Comput.107 (2000) 169-180.

[17] M.A. Abdou, Spectral relationships for integral operators in contact problem of impressing stamp, J. Appl. Math. Comput. 118 (2001)

95-111.

[18]M.A.Abdou, A.A.Badr, On a method for solving an integral equation in the displacement contact problem, J. Appl. Math. Comput. 127 (2002) 65-78

[19]H. Bateman, A.Erdely. Higher Transcendental Functions, vol.II, Mc.Graw Hill, NewYork, 1985.

[20] I.S. Gradstein, I.M. Ryzhik, Tables of Integrals, Series and Products, Fifth ed. Academic Press, New York, 1994.