ISSN 2347-1921

APPROXIMATION OF FOURIER SERIES OF A FUNCTION OF LIPCHITZ CLASS BY PRODUCT MEANS

[1]M.Misra, [2]P.Palo, [3]B.P.Padhy, [4]P.Samanta and [5]U.K.Misra

[1]Department of Mathematics, B.A.College,Berhampur, Odisha, India
E-mail: mahendramisra2007@gmail.com

[2] Khetra Mohan College, Narendrapur,Ganjam, Odisha,India
E-mail: prafula_palo@gmail.com

[3]Roland Institute of Technology,Golanthra, Ganjam, Odisha,India
E-mail: iraady@gmail.com

[4] Department of Mathematics, Berhampur University, Odisha,India
E-mail: dr.pns.math@gmail.com

[5]National Institute of Scince and Technology,Ganjam, Odisha,India
E-mail: umakanta_misra@yahoo.com

ABSTRACT

Lipchitz class of function had been introduced by McFadden[5}.Recently dealing with degree of approximation of Fourier series of a function of Lipchitz class Nigam[12] and Misra et al.[13] have established certain theorems. Extending their results in this paper a theorem on degree of approximation of a function $f \in Lip(\xi(t), r)$ by product summability $(E, s)(N, p_n, q_n)$ has been established.

Keywords: Degree of Approximation, $Lip(\xi(t), r)$ class of function, $(E, s)(N, p_n, q_n)$ product mean, Fourier series, Lebesgue integral.

2010-Mathematics subject classification: 42B05, 42B08.

ISSN 2347-1921

1. INTRODUCTION:

2. Let $\sum a_n$ be a given infinite series with sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ and $\{q_n\}$ be sequences of positive real numbers such that

(1.1) $$P_n = \sum_{\upsilon=0}^{n} p_\upsilon \quad \text{and} \quad Q_n = \sum_{\upsilon=0}^{n} q_\upsilon .$$

Let

(1.2) $$t_n = \frac{1}{r_n} \sum_{\upsilon=0}^{n} p_{n-\upsilon} q_\upsilon s_\upsilon ,$$

where $r_n = p_0 q_n + p_1 q_{n-1} + \ldots + p_n q_0 (\neq 0)$, $p_{-1} = q_{-1} = r_{-1} = 0$.

Then $\{t_n\}$ is called the sequence of (N, p_n, q_n) mean of the sequence $\{s_n\}$. If

(1.3) $$t_n \to s \quad \text{, as} \quad n \to \infty ,$$

then the series $\sum a_n$ is said to be (N, p_n, q_n) summable to s.

The necessary and sufficient conditions for regularity of (N, p_n, q_n) method are [3]:

(1.4) (i) $\frac{p_{n-\nu} q_\nu}{r_n} \to 0$ for each integer $\nu \geq 0$ as $n \to \infty$ and

(1.5) (ii) $\sum_{\nu=0}^{n} |p_{n-\nu} q_\nu| < H |r_n|$ where H is a positive number independent of n.

The sequence –to-sequence transformation [5],

(1.6) $$T_n = \frac{1}{(1+q)^n} \sum_{\upsilon=0}^{n} \binom{n}{\upsilon} q^{n-\upsilon} s_\upsilon ,$$

defines the sequence $\{T_n\}$ of the (E, q) mean of the sequence $\{s_n\}$. If

(1.7) $$T_n \to s \text{ , as } n \to \infty ,$$

then the series $\sum a_n$ is said to be (E, q) summable to s .Clearly (E, q) method is regular[5].

Further, the (E, q) transform of the (N, p_n, q_n) transform of $\{s_n\}$ is defined by

$$\tau_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} t_k$$

(1.8) $$= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{r_k} \sum_{\upsilon=0}^{k} p_{k-\upsilon} q_\upsilon s_\upsilon \right\}$$

If

(1.9) $$\tau_n \to s \text{ , as } n \to \infty,$$

then $\sum a_n$ is said to be $(E,q)(N,p_n,q_n)$-summable to s.

Let $f(t)$ be a periodic function with period 2π, L-integrable over $(-\pi,\pi)$, The Fourier series associated with f at any point x is defined by

(1.10) $$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x).$$

Let $s_n(f;x)$ be the n-th partial sum of (1.10). The L_∞-norm of a function $f : R \to R$ is defined by

(1.11) $$\|f\|_\infty = \sup\{|f(x)| : x \in R\}$$

and the L_υ-norm is defined by

(1.12) $$\|f\|_\upsilon = \left(\int_0^{2\pi} |f(x)|^\upsilon \right)^{\frac{1}{\upsilon}} , \ \upsilon \geq 1.$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial $P_n(x)$ of degree n under norm $\| \cdot \|_\infty$ is defined by [14]

(1.13) $$\|P_n - f\|_\infty = \sup\{|P_n(x) - f(x)| : x \in R\}$$

and the degree of approximation $E_n(f)$ of a function $f \in L_\upsilon$ is given by [12]

(1.14) $$E_n(f) = \min_{P_n} \|P_n - f\|_\nu .$$

This method of approximation is called Trigonometric Fourier approximation.

A function $f \in Lip\alpha$ if [7]

(1.15) $$|f(x+t) - f(x)| = O(|t|^\alpha) , 0 < \alpha \leq 1.$$

and $f \in Lip(\alpha, r)$, for $0 \leq x \leq 2\pi$, if [7]

(1.16) $$\left(\int_0^{2\pi} |f(x+t)-f(x)|^r\,dx\right)^{\frac{1}{r}} = O\left(|t|^{\alpha}\right),\ 0<\alpha\le 1, r\ge 1, t>0.$$

For a positive increasing function $\xi(t)$ and an integer $p>1$, we define[13], $f\in Lip(\xi(t),r)$ if

(1.17) $$\left(\int_0^{2\pi} |f(x+t)-f(x)|^r\,dx\right)^{\frac{1}{r}} = O(\xi(t)).$$

We use the following notation throughout this paper:

(1.18) $$\phi(t)=f(x+t)+f(x-t)-2f(x),$$

and

(1.19) $$K_n(t)=\frac{1}{2\pi(1+s)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}q_{\nu}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}.$$

Further, the method $(E,q)(N,p_n,q_n)$ is assumed to be regular and this case is supposed throughout the paper.

3. KNOWN THEOREMS:

Bernestein[2], Alexits[1], Sahney and Goel[10], Chandra[4] and several others have determined the degree of approximation of the Fourier series of the function $f\in Lip\alpha$ by $(C,1)$, (C,δ), (N,p_n) and $(\bar{N},p_n)$ means. Subsequently, working on the same direction Sahney and Rao[12], and Khan[6] have established results on the degree of approximation of the function belonging to the class $Lip\alpha$ and $Lip(\alpha,r)$ by (N,p_n) and (N,p_n,q_n) means respectively. However, dealing with product summability Nigam et al [10] proved the following theorem on the degree of approximation by the product $(E,q)(C,1)$-mean of Fourier series.

Theorem 2.1:

If a function $f\ is 2\pi$- periodic and of class $Lip\alpha$, then its degree of approximation by $(E,q)(C,1)$ summability mean on its Fourier series $\sum_{n=0}^{\infty}A_n(t)$ is given by $\left\|E_n^qC_n^1-f\right\|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right)$, $0<\alpha<1$, where $E_n^qC_n^1$ represents the (E,q) transform of $(C,1)$ transform of $s_n(f;x)$.

Subsequently Misra et al [8] have established the following theorem on degree of approximation by the product mean $(E,q)(N,p_n)$ of the Fourier series:

Theorem 2.2:

If f is a $2\pi-$ Periodic function of class $Lip(\alpha,r)$, then degree of approximation by the product $(E,q)(N,p_n)$ summability means on its Fourier series (defined above) is given by $\|\tau_n - f\|_\infty = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right)$ $,0<\alpha<1,\ r\geq 1$, where τ_n as defined in (1.8) .

Recently Misra et al [9] have established the following theorem on degree of approximation by the product mean $(E,s)(N,p_n,q_n)$ of the Fourier series:

Theorem 2.3:

If f is a $2\pi-$ Periodic function of the class $Lip(\alpha,l)$, then degree of approximation by the product $(E,s)(N,p_n,q_n)$ summability means on its Fourier series (1.10) is given by $\|\tau_n - f\|_\infty = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{l}}}\right)$ $,0<\alpha<1, l\geq 1.$, where τ_n is as defined in (1.8).

3. MAIN THEOREM:

In this paper, we have studied a theorem on degree of approximation by the product mean $(E,s)(N,p_n,q_n)$ of the Fourier series of a function of class $Lip(\xi(t),r)$. We prove:

Theorem -3.1:

For a positive increasing function $\xi(t)$ and an integer $l>1$, if f is a $2\pi-$ Periodic function of the class $Lip(\xi(t),l)$, then degree of approximation by the product $(E,s)(N,p_n,q_n)$ summability means on its Fourier series (1.10) is given by $\|\tau_n - f\|_\infty = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{l}}}\right)$ $,0<\alpha<1, l\geq 1.$, where τ_n is as defined in (1.8).

4.REQUIRED LEMMAS:

We require the following Lemma for the proof the theorem.

Lemma -4.1:

$$|K_n(t)| = O(n) \quad ,0\leq t\leq \frac{1}{n+1}.$$

Proof of Lemma-4.1:

For $0 \le t \le \frac{1}{n+1}$, we have $\sin nt \le n \sin t$ then

$$|K_n(t)| = \frac{1}{2\pi(1+s)^n} \left| \sum_{k=0}^{n} \binom{n}{k} s^{n-k} \left\{ \frac{1}{r_k} \sum_{\upsilon=0}^{k} p_{k-\upsilon} q_\upsilon \frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\} \right|$$

$$\le \frac{1}{2\pi(1+s)^n} \left| \sum_{k=0}^{n} \binom{n}{k} s^{n-k} \left\{ \frac{1}{r_k} \sum_{\upsilon=0}^{k} p_{k-\upsilon} q_\upsilon \frac{(2\upsilon+1)\sin\frac{t}{2}}{\sin\frac{t}{2}} \right\} \right|$$

$$\le \frac{1}{2\pi(1+s)^n} \left| \sum_{k=0}^{n} \binom{n}{k} s^{n-k} (2k+1) \left\{ \frac{1}{R_k} \sum_{\upsilon=0}^{k} p_{k-\upsilon} q_\upsilon \right\} \right|$$

$$\le \frac{(2n+1)}{2\pi(1+s)^n} \left| \sum_{k=0}^{n} \binom{n}{k} s^{n-k} \right|$$

$$= O(n).$$

This proves the lemma.

Lemma-4.2:

$$|K_n(t)| = O\left(\frac{1}{t}\right), \; for \; \frac{1}{n+1} \le t \le \pi .$$

Proof of Lemma-4.2:

For $\frac{1}{n+1} \le t \le \pi$, we have by Jordan's lemma, $\sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}$, $\sin nt \le 1$.

Then

$$|K_n(t)| = \frac{1}{2\pi(1+s)^n} \left| \sum_{k=0}^{n} \binom{n}{k} s^{n-k} \left\{ \frac{1}{r_k} \sum_{\upsilon=0}^{k} p_{k-\upsilon} q_\upsilon \frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\} \right|$$

$$\leq \frac{1}{2\pi(1+s)^n}\left|\sum_{k=0}^{n}\binom{n}{k} s^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k}\frac{\pi\ p_{k-\upsilon}q_{\upsilon}}{t}\right\}\right|$$

$$= \frac{1}{2(1+s)^n t}\left|\sum_{k=0}^{n}\binom{n}{k} s^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}q_{\upsilon}\right\}\right|.$$

$$= \frac{1}{2(1+s)^n t}\left|\sum_{k=0}^{n}\binom{n}{k} s^{n-k}\right|$$

$$= O\left(\frac{1}{t}\right).$$

This proves the lemma.

5. Proof of Theorem 3.1:

Using Riemann –Lebesgue theorem, for the n-th partial sum $s_n(f;x)$ of the Fourier series (1.10) of $f(x)$ and following Titchmarch [15], we have

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\,\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\,dt \quad .$$

Using (1.2), the (N, p_n, q_n) transform of $s_n(f;x)$ is given by

$$t_n - f(x) = \frac{1}{2\pi r_n}\int_0^{\pi}\varphi(t)\sum_{k=0}^{n} p_{n-k}q_{\upsilon}\,\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\,dt\,.$$

Denoting the $(E,q)(N,p,q)$ transform of $s_n(f;x)$ by τ_n, we have

$$\|\tau_n - f\| = \frac{1}{2\pi(1+s)^n}\int_0^{\pi}\varphi(t)\sum_{k=0}^{n}\binom{n}{k} s^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}q_{\upsilon}\,\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt$$

$$= \int_0^{\pi}\phi(t)\ K_n(t)\,dt$$

$$= \left\{ \int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right\} \phi(t)\ K_n(t)\ dt$$

(5.1) $$= I_1 + I_2 \text{, } say$$

Now

$$|I_1| = \frac{1}{2\pi(1+s)^n} \left| \int_0^{1/n+1} \varphi(t) \sum_{k=0}^{n} \binom{n}{k} s^{n-k} \left\{ \frac{1}{r_k} \sum_{\upsilon=0}^{k} p_{k-\upsilon} q_\upsilon \frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\} dt \right|$$

$$\leq \left| \int_0^{\frac{1}{n+1}} \varphi(t) K_n(t) dt \right|$$

$$= \left(\int_0^{\frac{1}{n+1}} \left| \frac{\varphi(t)}{\xi(t)} \right|^l \right)^{\frac{1}{l}} \left(\int_0^{\frac{1}{n+1}} \left| \xi(t) K_n(t) \right|^m \right)^{\frac{1}{m}} \text{, where } \frac{1}{l} + \frac{1}{m} = 1 \text{, using Holder's inequality}$$

$$= O(1) \left(\int_0^{\frac{1}{n+1}} \xi(t) n^m dt \right)^{\frac{1}{m}}$$

$$= O\left(\xi\left(\frac{1}{(n+1)} \right) \right) \left(\frac{n^m}{n+1} \right)^{\frac{1}{m}}$$

$$= O\left(\xi\left(\frac{1}{(n+1)} \right) \frac{1}{(n+1)^{\frac{1}{m}-1}} \right).$$

$$= O\left(\xi\left(\frac{1}{(n+1)} \right) \frac{1}{(n+1)^{-\frac{1}{l}}} \right)$$

(5.2) $$= O\left(\xi\left(\frac{1}{(n+1)}\right)(n+1)^{\frac{1}{l}}\right)$$

Next

$$|I_2| \le \left(\int_{\frac{1}{n+1}}^{\pi} \left|\frac{\phi(t)}{\xi(t)}\right|^l dt\right)^{\frac{1}{l}} \left(\int_{\frac{1}{n+1}}^{\pi} |\xi(t) K_n(t)|^m dt\right)^{\frac{1}{m}}$$,using Holder's inequality, as above.

$$= O(1)\left(\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\xi(t)}{t}\right)^m dt\right)^{\frac{1}{m}}$$,using Lemma 4.2

(5.3) $$= O(1)\left(\int_{\frac{1}{\pi}}^{n+1} \left|\left(\frac{\xi\left(\frac{1}{y}\right)}{\frac{1}{y}}\right)\right|^m dy\right)^{\frac{1}{m}}$$

Since $\xi(t)$ is a positive increasing function, so is $\left(\frac{\xi\left(\frac{1}{y}\right)}{\frac{1}{y}}\right)$. Using second mean value theorem we get

$$= O\left((n+1)\xi\left(\frac{1}{n+1}\right)\right)\left(\int_{\delta}^{n+1} \frac{1}{y^2} dy\right)^{\frac{1}{m}}, \text{ for some } \frac{1}{\pi} \le \delta \le n+1$$

$$= O\left((n+1)^{\frac{1}{l}} \xi\left(\frac{1}{n+1}\right)\right)$$

Then from (5.2) and (5.3) , we have

$$|\tau_n - f(x)| = O\left((n+1)^{\frac{1}{l}} \xi\left(\frac{1}{n+1}\right)\right), \text{ for } l \ge 1 \quad .$$

Hence

$$\|\tau_n - f(x)\|_\infty = \sup_{-\pi<x<\pi} |\tau_n - f(x)| = O\left((n+1)^{\frac{1}{l}} \xi\left(\frac{1}{n+1}\right)\right) \quad , l \ge 1.$$

This completes the proof of the theorem.

REFERENCES

[1] G.Alexits : $\ddot{U}$ ber die Annanherung einer stetigen function durch die Cesarochen Mittel in hrer Fourier reihe, Math. Annal 100(1928), 264-277

[2] S.Bernstein : Sur l' order de la Melleure approximation des function continue par des polynomes de degree' donne'e, Memories Acad. Roy-Belyique 4(1912), 1-104.

[3] D.Borwein : On product of sequences, Journal of London Mathematical Society, 33(1958), 352-357.

[4] P.Chandra : On degree of approximation of functions belonging to Lipchitz class, Nanta Math. 80(1970), 88-89.

[5] G.H. Hardy: Divergent series, First edition, Oxford University press (1949).

[6] Huzoor H. Khan :On degree of approximation of function belonging to the class , Indian Journal of pure and applied Mathematics, 13(1982), 132-136.

[7] L. McFadden : Absolute $\ddot{N}$ orlund summabilty, Duke Maths. Journal, 9(1942),168-207.Brown, L. D., Hua, H., and Gao, C. 2003. A widget framework for augmented interaction in SCAPE.

[8] U.K.Misra, M. Misra, B.P. Padhy and M.K. Muduli: On degree of approximation by product mean $(E,q)(N,p_n)$ of Fourier series, Gen. Math. Notes ISSN 2219 – 7184, Vol.6, No.2 (2011),

[9] U.K.Misra, M. Misra, B.P. Padhy and P.C.Das: "Degree of approximation of the Fourier Series of a function of Lipchitz class by Product Means", Communicated to Journal of Mathematical Modeling, SciKnow Publication

[10] H.K. Nigam and Ajay Sharma: On degree of Approximation by product means, Ultra Scientist of Physical Sciences, Vol.22 (3) M, 889-894, (2010).

[11] U.K.Misra, M. Misra, B.P. Padhy and P.C.Das: "Degree of approximation of the Fourier Series of a function of Lipchitz class by Product Means", Communicated to Journal of Mathematical Modeling, SciKnow Publication

[12] B .N. Sahney and D.S.Goel : On degree of approximation of continuous functions, Ranchi University Mathematical Journal, 4(1973), 50-53.

[13] B .N. Sahney and G.Rao : Errors bound in the approximation function, Bulletin of Australian Mathematical Society, 6(1972)

[14] A.H.Sidiqui : Ph.D. Thesis, Aligarh Muslim University, Aligarh, a967.

[15] E.C. Titchmarch: The theory of functions, oxford university press, p.p402-403(1939).

[16] A . Zygmund : Trigonometric Series , second Edition ,Vol.I , Cambridge University press ,Cambridge , (1959).

Authors Biography

Mahendra Misra has got his Ph.D in the year 1997 from Berhampur University. He has published around 30 research articles in various International and National journals of repute.To his credit he has produced three Ph.Ds under his guidance. Presently he is working as a Principal at Binayak Achrya Science College, Berhampur,Odisha.

P.Palo is currently working as a lecturer in Mathematics at Khetramohan Science College,Narendrapur, Ganjam, Odisha which is under Berhampur University.He has teaching experience of more than 15yrs and pursuing research (Ph. D.) work in the field of "Summability Theory" under Berhampur University.

B. P. Padhy has got his Ph.D from Berhampur University in 2011.He has 16 years of teaching experience. He has published around 30 research articles in various International and National journals of repute. He has also published a book Entitled "Summability Methods and Applications" under Lap Lambert Academic Publishing GmbH & Co.KG, Germany. Presently he is working as a Professor in mathematics in Roland Institute of Technology,Berhampur, Odisha.

P.N.Samanta has got his Ph.D in the year 2011 from Berhampur University. He has published around 15 research articles in various International and National journals of repute.To his credit he is guiding three Ph.Ds under hm. After working around 15 years in Gopalpur Science college, Gopalpur,presently he is working as a lecturer In the P.G. department of Mathematics of Berhampur University,Bhanjabihar, Berhampur,Odisha.

U. K. Misra is working as a Professor in the department of Mathematics at National Institute of Science and Technology, Berhampur. Formerly he was the professor in the P.G.Department of Mathematics, Berhampur University. To his credit he has guided 20 Ph.Ds and 1 D.Sc .He has published more than 100 papers in various International and National journal of repute. The field of Prof. Misra's research is Summability theory, sequence space, Fourier series, Inventory control,Mathematical Modeling. He is the reviewer of Mathematical review published by American Mathematical Society.