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Exponential Permutable Semigroups

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ABSTRACT

A semigroup S is called a permutable semigroup if $\rho \cdot \sigma = \sigma \cdot \rho$ is satisfied for all congruences ρ and σ of S. A semigroup is called an exponential semigroup if it satisfies the identity $(xy)^n = x^n y^n$ for every integer $n \geq 2$. In this paper we deal with permutable exponential semigroups. We describe the archimedian exponential semigroups and non-archimedian exponential semigroups which are semilattices of a group and a nilpotent semigroup.

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1. Introduction

A semigroup S is called a permutable semigroup if the congruences of S commute with each other, that is $\rho\cdot\sigma=\sigma\cdot\rho$ is satisfied for all congruences ρ and σ of S.

Lemma1[3]. Every group is permutable.

Lemma2[3]. If S is a permutable semigroup and S is homomorphiconto T, then T is a permutable semigroup.

Lemma3[3]. Let Γ be a semilattice. Γ is permutable if and only if $|\Gamma|\le 2$.

Theorem1[3]. If a semigroup S has a proper ideal and a proper group congruence then S is not permutable.

Theorem2[3]. If S is a permutable semigroup then the ideals of S form a chain.

Lemma 4[2]. If a permutable semigroup is a semilattice of two semilattice indecomposable subsemigroups S_1 and S_0 such that $S_0S_1\subseteq S_0$ then S_1 is simple.

Defenition. A semigroup is called an exponential semigroup if it satisfies the identity $(xy)^n=x^ny^n$ for every integer $n\ge 2$.

Theorem3[4]. If a semigroup satisfies the identity $(xy)^2=x^2y^2$ then it satisfies the identity $(xy)^n=x^ny^n$ for all positive integers $n\ge 4$.

Corollary1[4]. A semigroup S is exponential if and only if it satisfies the identities $(xy)^2=x^2y^2$ and $(xy)^3=x^3y^3$.

Theorem4[4]. Every exponential semigroup is a semilattice of exponential archimediansemigroups.

Theorem5[4]. A semigroup is a simple exponential semigroup if and only if it is a completely simple exponential semigroup with a zero adjoined.

Theorem6[4]. S is an exponential archimediansemigroup containing at least one idempotent element if and only if S is a retract extension of a rectangular abelian group by an exponential nil semigroup.

Lemma5[5]. A nil semigroup S is a permutable if and only if the ideals of S form a chain with respect to inclusion.

Theorem7[6]. An exponential archimediansemigroup S without idempotents has a non trivial group homomorphic image.

Lemma 6[5]. A Rees matrix semigroup $S=\mathrm{M}(I,G,\Lambda;P)$ is permutable if and only if $|I|\le 2$ and $|\Lambda|\le 2$.

1.Exponentialarchimediansemigroups

Let ρ and σ be congruences on a semigroup S then $\rho\vee\sigma=\rho\cdot\sigma$ if and only if $\rho\cdot\sigma=\sigma\cdot\rho$, where

$$\rho\cdot\sigma=\{(x,y)\in S\times S\,|\,(\exists z\in S)(x,z)\in\rho;(z,y)\in\sigma\}.$$

If ρ is a congruence on rectangular exponential grup $S=I\times G\times\Lambda$ (I is a left zero semigroup, G is abelian group, Λ is a right zero semigroup) then

$$\rho_I=\{(i,j)\in I\times I\,|\,(\exists g_1,g_2\in G,\lambda_1,\lambda_2\in\Lambda)(i,g_1,\lambda_1)\rho(j,g_2,\lambda_2)\}$$

$$\rho_G=\{(h,k)\in G\times G\,|\,(\exists i_1,i_2\in I,\lambda_1,\lambda_2\in\Lambda)(i_1,h,\lambda_1)\rho(i_2,k,\lambda_2)\}$$

$$\rho_\Lambda=\{(\lambda,\mu)\in\Lambda\times\Lambda\,|\,(\exists i_1,i_2\in I,g_1,g_2\in G)(i_1,g_1,\lambda)\rho(i_2,g_2,\mu)\}.$$

Lemma7[2]. If ρ and σ are congruences on S then,

(i) $\rho_I,\rho_G,\rho_\Lambda,\sigma_I,\sigma_G,\sigma_\Lambda$ are conguences.

(ii) $(\rho\cdot\sigma)_I=\rho_I\cdot\sigma_I$, $(\rho\cdot\sigma)_G=\rho_G\cdot\sigma_G$ and $(\rho\cdot\sigma)_\Lambda=\rho_\Lambda\cdot\sigma_\Lambda$

(iii) If $\rho_I=\sigma_I,\rho_G=\sigma_G,\rho_\Lambda=\sigma_\Lambda$ then $\rho=\sigma$.

*Lemma8[2].*Let I be left zero semigroup, G abelian group, Λ right zero semigroup. Then rectangular group $S=I\times G\times\Lambda$ is permutable if and only if $|I|\le 2,|\Lambda|\le 2.$

Lemma9 [5]. Every finite nil semigroup is nilpotent.

Lemma10[5]. A finite semigroup is an archimedian permutable semigroup if and only if it is either a cyclic nilpotent semigroup or a permutable completely simple semigroup.

Lemma11[6]. Let S be an exponential semigroup and $a\in S$. Then S_a is the least unitary reflexive subsemigroup of S that contains a. If S is also archimedian then the principal right congruence determined by S_a is a group congruence.

Theorem 8. An exponential archimedian permutable semigroup S has an idempotent element.

Proof. Suppose S has no idempotents. By theorem 7.and because S is a permutable semigroup then S is a simple semigroup with non-trivial group homomorphic images, these images are abelian. The group congruences on S contain the least commutative congruence ρ on S. Since S/ρ is commutative and simple it is a group. Hence ρ is the least group congruence on S. So ρ is the principal right congruence determined by the least unitary subsemigroup A of S. A is reflexive by $(xy)^3=x(yx)^2y=xy^2x^2y=xy(yx)xy$ and has empty right residue since S is archimedian. By lemma 11. $A=S_a=S_b$ for $a,b\in A$ so by

$$S_a=\{x\in S\,|\,a^ixa^j=a^k \text{ for } i,j,k\in Z^+\} \qquad ....(1)$$

A isarchimedian. Clearly A is exponential. For $d\in A$ let A_d be the unitary subsemigroup of A defined by (1).

A_d is unitary in S, if $u,v\in S$ and $u,uv\in A_d$ then $u,uv\in A$ so $v\in A$ and hence $v\in A_d$. But A is the least unitary subsemigroup of S so $A=A_d$ for all $d\in A$. Hence A is an exponential archimediansemigroup with only trivial group homomorphic image. By theorem 7, A and S have an idempotent. This is a contradiction. ∎

Theorem 9. Let S be an exponential archimediansemigroup. S is permutable if and only if S is an exponential nil semigroup whose principal ideals form a chain with respect to inclusion and a rectangular group.

Proof. Let S be an exponential archimedian nilsemigroup then by lemma8 and lemma5, S is permutable.

Conversely, assume that S is exponential archimedian permutable semigroup.Then by theorem 8 contain an idempotent element. Hense by theorem 6 is a retract extension of a rectangular abelian group J by an exponential nil semigroup N.

Assume $J=I\times G\times\Lambda$ (I is a left zero semigroup, G is abelian group, Λ is a right zero semigroup). Suppose $N\ne\{0\}$, then S is non simple and by Lemma 8 [7], S and $J=I\times\Lambda$ have only trivial group homomorphic image. Hence $|G|=1$ and may consider S an ideal extension of an rectangular band by N. Let

$$\rho=\{(x,y)\in S\times S\,|\,x^n=y^n \text{ for some intiger } n>0\}$$

ρ is an equivalence relation.

For $x,y\in S^1$ and $(x,y)\in\rho$ there is an integer $n>0$ such that $x^n=y^n$ and

$$(ayb)^n=a^ny^nb^n=a^nx^nb^n=(axb)^n$$

so ρ is a congruence on S. For $x \in S$ there is an integer $m > 0$ so that $x^m \in J$. Hence there are classes of ρ that contain elements of N and exatly one element of J. So ρ is comparable with the Rees congruence modulo J only if $|J|=1.$ Hence $S = N$ is a nil exponential semigroup.

Suppose $N = \{0\}$ then, by lemma8 S is an rectangular semigroup with $|I| \leq 2$ and $|\Lambda| \leq 2.$ ■

3. Exponential non-archimedian permutable semigroup

Remark 1. As every semigroup is a semilattice indecomposable semigroups, lemma 3 and lemma 4 together imply that every permutable semigroup is either semilattice indecomposable or a semilattice of two semilattice indecomposable semigroups S_0 and S_1 such that $S_0S_1 \subseteq S_0$.

Lemma 12. If S is a finite non-archimedian exponential permutable semigroup then it is a semilattice of a completely simple semigroup $S_1 = \mathrm{M}(I, G, \Lambda; P)$ such that $|I| \leq 2, |\Lambda| \leq 2$ and a semigroup S_0 such that $S_0S_1 \subseteq S_0$ and S_0 is an ideal extension of a completely simple semigroup K by a nilpotent semigroup.

Proof. Let S be a finite permutable non-archimedian exponential semigroup. Then by lemma 3 and Lemma 4, S is a semilattice of two archimedian semigroups S_0 and S_1 such that $S_0S_1 \subseteq S_0$.

As the Rees factor $S_1^0 = S / S_0$ is permutable then by Lemma 3, S_1 is a permutable archimedian semigroup. By lemma 4 and Lemma 10, S_1 is completely simple. Then S_1 is a Rees matrix semigroup $S_1 = \mathrm{M}(I, G, \Lambda; P)$ and by Lemma 6 $|I| \leq 2, |\Lambda| \leq 2$. By Theorem 6, S_0 is an ideal extension of a completely simple semigroup K by a nilpotent Rees factor semigroup $N = S_0 / K.$ ■

Lemma 13.

S_1 is ortogonal group so, $S_1 = I \times G \times \Lambda$ where I is a semigroup of the left zeros with $|I| \leq 2, G$ is an abelian group and Λ is a semigroup of the right zeros with $|\Lambda| \leq 2.$

Proof.

The proof. flow from the Theorem 8 and 9.

Lemma14.

S_0 is an orthogonal ore a nil- exponential semigroup tape(strap)

The proof. flow directly from the theomre 8 and 9.

Ongoing we will see which are the characteristics of a non-archimedian exponential permutable semigroup who S_0 is a orthogonal band.

Lemma 15[7].

If S_0 is a semigroup of the right zeros than,

(i) For each $a \in S_0$ have that $S_1 a = a$.

(ii) For each $a \in S_0, i \in I, g \in G$ exists $\lambda \in \Lambda$ so that $a(i, g, \lambda) = a$.

Remark 2.

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For the semigroup $S = S_0 \cup S_1$ where $S_0 \cap S_1 = \varnothing$ and ρ congurenc on S it's clear that for $x \in S_0, y \in S_1$ from $(x, y) \in \rho$ flow $\rho = S \times S$ that meaning that two congurences of the semigroup S are permutable if and only if , if theyr closely on S_0 and S_1 are permutable.

Now before we give the Theorem 10 we take the groupoid $(S,*)$ of the kind (i) and (ii):

$(i) \, |S_0| \leq 2, |I| \leq 2, |\Lambda| \leq 2; \; x * y = xy$ for $x, y \in S$

or

$x, y \in S_1$ have : $a * S_1 = S_1 * a = a$ for each $a \in S_0$.

$(ii) \, |S_0| = 3, |I| = 2, |\Lambda| \leq 2$, $x * y = xy$ if $x, y \in S_0$

or

$x, y \in S_1$ have ($S_0 = \{a, b, c\}, I = \{i, j\}$)

$$a * S_1 = (\{i\} \times G \times \Lambda) * a = (\{i\} \times G \times \Lambda) * b = a$$

$$b * S_1 = a * (\{j\} \times G \times \Lambda) = (\{j\} \times G \times \Lambda) * b = b$$

$$c * S_1 = S_1 * c = c \, .$$

In the paper [7] " A family of permutable completely regular semigroups" are prove that these groupoids are permutable.

Theorem 10.

The given conditions are equivalence.

Let be S a groupoid of the kind (i) and (ii) its evident that the semigroup S a non-archimedin exponential semigroup created from two archimedian components S_0 and S_1 . Also ρ is congurenc on S so that $(x, y) \in \rho$ have that $\rho = S \times S$ .So,if $|S_0| \leq 2$ (from Lemma 8 flow that S_0 and S_1 are permutable.) than from remark 2 flow that S is permutable.

If $|S_0| = 3$ It can be proof. easy that the closely on S_0 of a congruences of S is equal with $S_0 \times S_0$ ore with the congruences how are bound with the divisions S_0 in classes of $\{a, b\}$ and $\{c\}$.

Since those closely are two by two permutable and as S_1 is permutable fwol that and S is permutable.

Conversely, let be S a non-archimedian exponential permutable semigroup where S_0 is a semigroup of the left zeros.

First we suppose that for each $a \in S_0$ applis $aS_1 = a$ (S_0, S_1 are archimedian componets). In this case we see that the semigroup S is permutable and S_0 is permutable that flow that the semigroup S is groupoid of the first (i) kind.

Now we suppose that exists $x \in S_0$ so that $xS_1 \neq x.$ In this case we have $|S_0| \geq 2$ in S_1 have $|I| = 2$.

We take $I = \{i, j\}$ and we write:

$$\begin{aligned} x(\{i\} \times G \times \Lambda) &= x \, (sipas \quad lemës 15.(ii)) \\ x(i, g, \lambda) &= y \qquad i \in I, g \in G, y \in S_0, y \neq x) \end{aligned} \qquad ....(1)$$

From (1) flow:

$$x(\{j\} \times G \times \Lambda) = y, y(\{i\} \times G \times \Lambda) = x, y((\{j\} \times G \times \Lambda) = y \qquad .. ..(2)$$

Examine the equivalence relation ρ that separating S in classes:

$\{i\}\times G\times\Lambda, \quad \{j\}\times G\times\Lambda, \quad \{s\}$,

where $s\in S_0$ so that $sS_1 = s,\{s,t\}$, $s,t\in S_0$ that exists $v,w\in S_1$ for those applies the identity $sv=t, tw=s.$

When we see (1) and (2) and from lemma 15 (i), we proof that ρ is congurenc.

The semigroup $\Sigma = S/\rho$ is a non-archimedian exponential permutable semigroup where (Σ_0 and Σ_1 are archimedina componets of Σ) Σ_0 -is a semigroup of the left zeros.One the other side for each $\alpha\in\Sigma_0$ have that $\alpha\Sigma_1=\alpha$ so Σ_0 is permutable.So from lemma 7 and 8 have $|\Sigma_0|\leq 2.$

Other cases :

If $|\Sigma_0|=1$ than $|S_0|=2.$

So $S_0=\{a,b\}$ from we take that

$$aS_1 = (\{i\}\times G\times\Lambda)a = (\{i\}\times G\times\Lambda)b = a$$

$$bS_1 = (\{j\}\times G\times\Lambda)a = (\{j\}\times G\times\Lambda)b = b$$

theequivalence relation that separating S in classes:

$$(\{i\}\times G\times\Lambda)\cup\{a\},\ (\{i\}\times G\times\Lambda)\cup\{b\}$$

is a non-permutable congurenc ,with a Rees congruenc modulo S_0 which is contrary with our suppose that the semigroup S is permutable.

If $|\Sigma_0|=2$ than $|S_0|=4$ ore $|S_0|=3.$

If $|S_0|=4$ than $S_0=\{a,b,c,d\}$ and we suppose that:

$$\Sigma_0=\{a,b\},\{c,d\}$$

$$aS_1 = (\{i\}\times G\times\Lambda)a = (\{i\}\times G\times\Lambda)b = a$$

$$bS_1 = (\{j\}\times G\times\Lambda)a = (\{j\}\times G\times\Lambda)b = b$$

$$cS_1 = (\{i\}\times G\times\Lambda)c = (\{i\}\times G\times\Lambda)d = c$$

$$dS_1 = (\{j\}\times G\times\Lambda)c = (\{j\}\times G\times\Lambda)d = d$$

In this case the congurenc we take as an equivalences calasses.

$(\{i\}\times G\times\Lambda)\cup\{a,c\},(\{j\}\times G\times\Lambda)\cup\{b,d\}$

With the Rees modulo S_0 congurenc aren't permutabel which are contraly with us suppose that the semigroup S is permutable.

And at last if $|S_0|=3$,than S coincides (to the izomorf) with the groupoid like (ii).So, the theorem are profed.

Remark 3.

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In the same way we can form the clain for non-archimedian exponential semigroups when S_0 is a semigroup of the right zeros, from theorem 8 and 9 we can form the new clain with the conditions (ii) like this:

(ii) $|S_0|=3, |I|=2, |\Lambda|\leq 2, x*y=xy$ for $x,y\in S_0$

or

$x,y\in S_1$ and (take $S_0=\{a,b,c\}, \Lambda=\{\lambda,\mu\}$)

$$S_1*a=a*(I\times G\times\{\lambda\})=b*(I\times G\times\{\lambda\})=a$$

$$S_1*b=a*(I\times G\times\{\mu)=b*(I\times G\times\{\mu\})=b$$

$$S_1*c=c*S_1=c. \ \blacksquare$$

Lemma 16[2]

Let be S_0 an orthogonal band and the principal ideal in S .So $S_0=V\times W$ (where V is a semigroup of the left zeros and W a semigroup of the right zeros),like :

$$\alpha=\{(a,b)\in S\times S\,|\,(\exists w\in W)a,b\in V\times\{w\}, \text{ or } a=b\}$$

$$\beta=\{(a,b)\in S\times S\,|\,(\exists v\in V)a,b\in\{v\}\times W, \text{ or } a=b\}$$

arecongruences in S. In other side ϕ and ψ are natural homomorphic in S.on S/α and S/β dhe $S_1=S\setminus S_0$ than we have:

$S_0\phi\cong W$, $S_1\phi\cong S_1$ dhe $S_0\psi\cong V$, $S_1\psi\cong S_1$. ■

Theorem 11.

An Exponential non- archimediansemigroup S is permutable if and only if S/α and S/β are exponential permutable semigroups (where α and β are defined like Lemma 16).

Proof.

By Lemma 2 as S is an exponential permutable semigroup then S/α and S/β are exponential permutable semigroups.

Prove the inverse:

Assume that S/α and S/β are exponential permutable semigroup and $S_0=V\times W$ (where V is a semigroup of the left zeros, W is a semigroup of the right zeros).

Proved that S is an exponential non-archimediansemigroup.

As S_0 and S_1 are exponential semigroups is easily provable that for $x,y\in S$ where one is from S_0 and the other from S_1, applies :

$(xy)^n=x^ny^n$ for each $n\geq 2$

True, as $(xy)^n$ and x^ny^n are part of S_0 , we write:

$(xy)^n=(v_1,w_1), x^ny^n=(v_2,w_2)$ where $v_1,v_2\in V, w_1,w_2\in W$ and $n\geq 2.$

As ϕ from Lemma 16 is a natural homomorphism of S over S/α we have

$$[(xy)^n]\phi = V \times \{w_1\}, \qquad (x^n y^n)\phi = V \times \{w_2\}$$

except this, because S/α is an exponential semigroup have :

$$[(xy)^n]\phi = [(x^n)\phi][(y^n)\phi] = (x^n y^n)\phi$$

So, $w_1 = w_2$. similar we can proof that $v_1 = v_2$.

On the other side let be : Σ_0, Σ_1 and Σ_0^*, Σ_1^* archimedian components of S/α and S/β respectively we have by:

$$\Sigma_0 = S_0\phi \square W, \qquad \Sigma_1 = S_1\phi \square S_1$$

And

$$\Sigma_0^* = S_0\psi \square V, \qquad \Sigma_1^* = S_1\psi \square S_1.$$

So, S/α is a semigroup of the kind given in **Remark 3,** also S/β is a semigroup like that given in Theorem 10.So $|V| \leq 3$ and $|W| \leq 3$.

Other cases:

If $|V| = 1$ $(|W| = 1)$ congruence α (β) is relation of the equation and so $S/\alpha \square S$ $(S/\beta \square S)$ flows that S is permutable.

Now we assum that $|V| = 2$ and $|W| = 2$ from Theorem 10 and the Remark 3 we have that $|I| \leq 2$, $|\Lambda| \leq 2$ and for each $a \in S_0$ flow that :

$$aS_1 = S_1 a = a.$$

If $a \in S_0, x \in S_1$, we take :

$$a = (a_1, a_2), ax = (v, w) \text{ where } a_1, v \in V \text{ and } a_2, w \in W.$$

So,

$$a\phi = V \times \{a_2\}, \quad (ax)\phi = V \times \{w\}.$$

On the other side, from theorem 10 $\Rightarrow$

$$(ax)\phi = (a\phi)(x\phi) = a\phi = V \times \{a_2\} \text{ because that } w = a_2.$$

In the same way we can proof. that $v = a_1$ that meaning $ax = a$. And similar we have that $xa = a$.

If ρ is congruence on S than, from $(x, y) \in \rho$ where $x \in S_0$ and $y \in S_1$ have that $\rho = S \times S$.

So, from Lemma 8 S_0 and S_1 are permutable, than and S is permutable.

Let be $|V| = 2, |W| = 3.$ from Theorem 10 and Remark 3 flow $|I| = 2, |\Lambda| \leq 2.$

Take,

$$V = \{v_1, v_2\}, W = \{w_1, w_2, w_3\}, \Lambda = \{\lambda, \mu\},$$

$$a = (v_1, w_1), b = (v_2, w_2), c = (v_2, w_3)$$

Then:

$\Sigma_0 = \{a\phi, b\phi, c\phi\}$, $\Sigma_0^* = \{a\psi, b\psi\}$

And from Theorem 10 and remark 3 we have:

$$S_1(a\psi) = (a\psi)S_1 = a\psi,$$

$$S_1(b\psi) = (b\psi)S_1 = b\psi,$$

$$S_1(a\phi) = (a\phi)(I \times G \times \{\lambda\}) = (b\phi)(I \times G \times \{\lambda\}) = a\phi \qquad \text{...(3)}$$

$$S_1(b\phi) = (a\phi)(I \times G \times \{\mu\}) = (b\phi)(I \times G \times \{\mu\}) = b\phi$$

$$S_1(c\phi) = (c\phi)S_1 = c\phi$$

Ongoing, if ρ is congurenc on S so that $(x, y) \in \rho$ where $x \in S_0, y \in S_1$ flow that $\alpha \subseteq \rho$.

Really that for $x = (x_1, x_2)$ where $x_1 \in V, x_2 \in W$ have that $V \times \{x_2\} \subseteq x\rho$. In fact if $a \in V \times \{x_2\}$ than $ax = a$ and from $(x, y) \in \rho$ flow $(xax, yax) \in \rho$ that means that $(x, a) \in \rho$.

From

$$V \times \{x_2\} \subseteq x\rho$$

For each $a = (a_1, a_2)$ $(a_1 \in V$ and $a_2 \in W)$ have that:

$$(V \times \{x_2\})a = V \times \{a_2\}$$

Than:

$$V \times \{a_2\} \subseteq (xa)\rho = a\rho$$

From

$$\alpha \subseteq \rho$$

And so the binary equation

$$\rho' = \{(x', y') \in S/\alpha \times S/\beta \mid (\exists x, y \in S)(x, y) \in \rho, x' = x\phi, y' = y\phi\}$$

is congruenc.

Also from $(x, y) \in \rho, x \in S_0, y \in S_1$

flow:

$$(x\phi, y\phi) \in \rho', \qquad x\phi \in \Sigma_0, y\phi \in \Sigma_1.$$

So $\rho' = S/\alpha \times S/\beta$ from $\rho = S \times S$.]

Applied the therd (3) equation we can see that the narrowing in S_0 of a congurence in S is equal with $S_0 \times S_0$ ore with the congurenc how is related with this divisions of S_0:

$$\{(v_1, w_1), (v_1, w_2)\}, \qquad \{(v_2, w_1), (v_2, w_2)\}, \qquad \{(v_1, w_3), (v_2, w_3)\}$$

$$V \times \{w_1\}, \qquad V \times \{w_2\}, \qquad V \times \{w_3\}$$

$$V \times \{w_1, w_2\}, \qquad V \times \{w_3\}$$

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$$\{v_1\}\times W, \qquad \{v_2\}\times W.$$

With those of narrowing are two of tow permutable and from Lemma 8 S_1 is permutable than and S is permutable.

In the same way we have the case when $|V|=3,|W|=2$

And in the and we assum that $|V|=3,|W|=3$

In front of Theomre 10 and remark 3 have: $|I|=2,|\Lambda|=2$

We take:

$$V=\{v_1,v_2,v_3\},W=\{w_1,w_2,w_3\},I=\{i,j\},\Lambda=\{\lambda,\mu\},$$

$$a=(v_1,w_1),b=(v_2,w_2),c=(v_2,w_3)$$

Than:

$$\Sigma_0=\{a\phi,b\phi,c\phi\}, \qquad \Sigma_0^*=\{a\psi,b\psi,c\psi\}$$

From theorem 10 and remark 3 we have:

$$S_1(a\phi)=(a\phi)(I\times G\times\{\lambda\})=(b\phi)(I\times G\times\{\lambda\})=a\phi$$

$$S_1(b\phi)=(a\phi)(I\times G\times\{\mu\})=(b\phi)(I\times G\times\{\mu\})=b\phi \qquad .....(4)$$

$$(a\psi)S_1=(\{i\}\times G\times\Lambda)(a\psi)=(\{i\}\times G\times\Lambda)(b\psi)=a\psi$$

$$(b\psi)S_1=(\{j\}\times G\times\Lambda)(a\psi)=(\{j\}\times G\times\Lambda)(b\psi)=(b\psi)$$

$$S_1(c\phi)=(c\phi)S_1=c\phi, \qquad S_1(c\phi)=(c\phi)S_1=c\phi$$

From what was said above,if ρ is congurenc in S that $(x,y)\in\rho$ where $x\in S_0$ and $y\in S_1$ have that $\alpha\subseteq\rho$ and flow that $\rho=S\times S$.

Proof. let be $(x_1,x_2)\in S_0(x_1\in V,x_2\in W)$ and $(i,g,\lambda),(j,g,\lambda)$ two elements from S_1. Now the forth (4) equation takes the form

$$\{(i,g,\lambda)(z_1,z_2)(x_1,x_2)\,|\,z_1\in V,z_2\in W\}=\{(v_1,x_2),(v_3,x_2)\}$$

$$\{(j,g,\lambda)(z_1,z_2)(x_1,x_2)\,|\,z_1\in V,z_2\in W\}=\{(v_2,x_2),(v_3,x_2)\} \qquad ...(5)$$

$$(j,u,\lambda)(v_1,x_2)=(v_2,x_2),(j,u,\lambda)(v_3,x_2)=(v_3,x_2)$$

Where u is the neutral element G.

From what was said above, let be $(x,y)\in\rho$ where $x\in S_0$ and $y\in S_1$.

We take $x=(x_1,x_2),y=(i,g,\lambda)$ and $(x,y)\in\rho$ flow $(x,yzx)\in\rho$ for each $z\in S_0$ and from the equation (5) flow $(v_1,x_2),(v_3,x_2)\in x\rho.$

We proof. that $(v_2,x_2)\in x\rho.$

Distinguish these cases:

If $x=(v_2,x_2)$ it's clear that $(v_2,x_2)\in x\rho$.

Assum that $x=(v_1,x_2)$ .From $(x,y)\in\rho$ flow $(j,u,\lambda)(v_1,x_2)\rho(j,u,\lambda)(i,g,\lambda),$

For which:

$$(v_2,x_2)\rho(j,g,\lambda)$$

for each $z\in S_0$ have:

$$(v_2,x_2)\rho(j,g,\lambda)z(v_2,x_2).$$

So, whene we considering the equation (5) have that:

$$(v_2,x_2)\rho(v_3,x_2).$$

And at last whene $x=(v_3,x_2)$ it's like before

$$V\times\{x_2\}\subseteq x\rho.$$

Whene we considering tha for each:

$$z=(z_1,z_2)\ (z_1\in V,z_2\in W)$$

have:

$$(V\times\{x_2\})z=V\times\{z_2\}$$

and flow:

$$V\times\{z_2\}\subseteq(xz)\rho.$$

So, $\alpha\subseteq\rho$.

So since the applies in S_0 in each two congurences of S are permutable and as S_1 is permutable semigroup than form Lemma 8. flow that S is permutable.

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