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MHD DYNAMIC BOUNDARY LAYER FLOW OVER A PLANE PLAQUE

Promise Mebine[1]
[1]Department of Mathematics/Computer Science
Niger Delta University
Wilberforce Island
Bayelsa State
NIGERIA
E-mail: p.mebine@yahoo.com; pw.mebine@ndu.edu.ng

ABSTRACT

The problem of MHD dynamic boundary layer fluid sliding flow over a plane plaque is investigated. The von Karman's integral method is applied to integrating the governing system of partial differential equations over the boundary layer thickness. Quantities of physical interest such as the boundary layer thickness, local or wall shear stress, friction drag force and coefficient of friction drag is derived. Comparisons with available literature give excellent agreements. The applicability of large or small magnetic fields in many industrial and electrical devices leads to the derivation of asymptotic results for the slip velocity. Pictorial representation made to the boundary layer thickness indicates that increasing magnetic parameter increases it as a result of the retarding force. The coefficient of friction drag is analyzed for various values of MHD and velocity slip parameters. It was observed that both the MHD and velocity slip or fluid slide parameters have retarding effect on the coefficient of friction drag, with the MHD playing a much more dominant role.

Keywords:

Boundary layer; Integral method; MHD; Fluid sliding

Academic Discipline and Sub-Disciplines

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SUBJECT CLASSIFICATION

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1. Introduction

Magnetohydrodynamics (MHD) (magneto fluid dynamics or hydromagnetics), the study of the dynamics of electrically conducting fluids, such as plasmas, liquid metals, and salt water or electrolytes, has many fundamental applications. MHD applications are very broad and bound in geophysics, where the Coriolis effect resulting geomagnetic dynamo is considered; astrophysics and cosmology since over 99% of baryonic matter content of the universe is made up of plasma; engineering, where MHD is related to engineering problems such as plasma confinement, liquid-metal cooling of nuclear reactors, electromagnetic casting, MHD generators and pumps; medicine, where magnetic drug targeting is an important task in cancer research, and the boundary layer control in the field of aerodynamics [1 - 3]. Sharma and Chaudhary [4] classified magnetic fields according to their various applications, namely, terrestrial magnetic field, which is maintained by fluid motion in the earth's core, the solar magnetic field which generates sunspots and solar flares, and the galactic field which influences the formation of stars. Qian and Bau [5] gave valuable and extensive review of recent advances and applications of MHD - based microfluidic devices.

It is important to state that there are applications of a fluid in contact with a plaque that slides on its surface, otherwise known as velocity slip condition. In boundary layer analyses, it is observed that the no-slip condition is not consistent with all physical characteristics [6]. Therefore, it becomes imperative and necessary to replace the no-slip boundary condition by the slip boundary condition. We must note here that when fluid flows in micro-fluidic devices such as micro-electromechanical system (MEMS), the no-slip condition at the solid-fluid interface becomes inapplicable. It is natural that a fluid slides on the surface of a dynamic or static solid embedded in a particulate fluid such as emulsions, suspensions, foams, polymer solutions, and hence the slip condition. To this end, the slip flow model describes more accurately the non-equilibrium region near the interface. There are many related literatures that make use of the velocity slip condition under different flow scenarios. Anderson [7] worked on slip flow past a stretching surface, while Fang and Lee [8] looked at a moving –wall boundary layer flow of a slightly rarefied gas free stream over a moving flat plate. Fang et al. [9] gave a solution to slip MHD viscous flow over a stretching sheet. Hydrodynamic and thermal slip flow boundary layers over a flat plate with constant heat flux boundary condition was considered by Aziz [10].

In this work, Von Karman boundary layer integral method is used to tackle the problem of MHD dynamic boundary layer slip flow over a plane plaque. The Von Karman boundary layer integral method is frequently termed as an approximate analysis of the boundary layer. It should be emphasized, however, that the integral equations themselves are exact within the boundary layer assumptions [11]. The solutions of these equations are approximate only to the extent that the profiles for the flow distributions (that is, velocity and temperature) chosen are not exact. The integral technique in many cases provides satisfactory results, which are used to validate similarity conditions [12]. It is, therefore, the objective of this work to examine the influence a magnetic field may have on dynamic boundary slip flow via the technique of Von Karman integral method. The treatment here is cursory; the application of the integral method is only intended to give insights to the basic physical mechanism of the processes involved in the problem.

2. Mathematical Formulation

Consider a semi-finite plane plague situated on the *Ox* axis, having the edge at *O*, attached under a null angle. The flow is a plane steady two-dimensional laminar flow of a viscous incompressible and electrically conducting fluid of density ρ , near the leading edge of a flat plate, where *Oxy* is the plane of the flow such that (*x, y*) is the Cartesian coordinates of any point in the domain of the flow, where *x* – axis is along the plate and *y* – axis is normal to the plate. Figure 1 gives a description of the flow configuration of the hydromagnetic boundary layer flow with slip velocity over a plane plaque. The slip flow is possible at lower gas pressures, when the free mean path of the gas molecules, say λ approximates the characteristic dimension L_1 of the surface, where the fluid slides upon. That is, when $\lambda \approx L_1$, the flow seems to "slip" along the surface and $u \neq 0$ at $y = 0$ such that $u = L_1 \frac{\partial u}{\partial y}$. This situation is appropriately called slip flow.

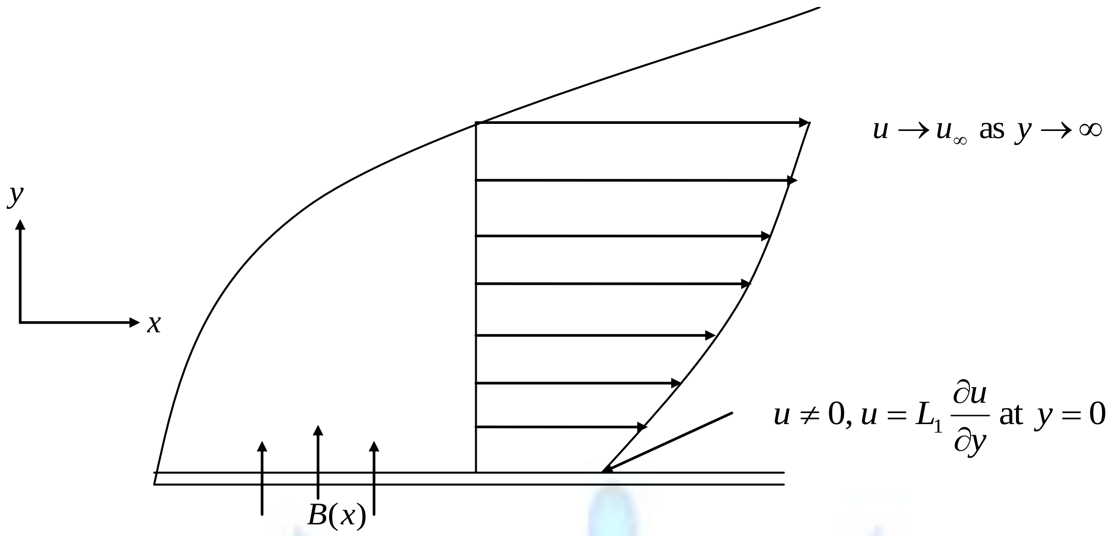

Figure 1: Flow Configuration of Hydromagnetic Boundary Layer

It is assumed that *u* and *v* are the respective velocity component in the *x* and *y* directions, and a constant magnetic field B_0 is applied normally to the plate. Within the boundary layer approximations, the continuity and momentum equations are written in the usual notations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B(x)^2}{\rho}u\,, \tag{2.2}$$

with the associated boundary conditions:

$$y = 0: \quad u(x,0) = L_1\frac{\partial u}{\partial y}(x,0), v(x,0) = 0, \tag{2.3a}$$

$$y \to \infty: \quad u(x,\infty) = u_\infty\,, \tag{2.3b}$$

where ν is the kinematic viscosity, σ is the electrical conductivity of the fluid, $B(x)$ is the magnetic field intensity and ρ is the fluid density.

The first boundary condition at $y = 0$ signifies the fact that the fluid in contact with the plaque, slides on its surface. As usual with boundary layer analysis with the application of integral boundary layer method, additional conditions at $y = 0$ and as $y \to \infty$ are defined in accordance to equations (2.1 – 2.3), and these are the compatibility conditions:

$$y = 0: \ \frac{\partial^2 u}{\partial y^2}(x,0) = \frac{\sigma B(x)^2}{\rho\nu}u(x,0), \tag{2.4a}$$

$$y \to \infty: \ \frac{\partial u}{\partial y}(x,\infty) = 0, \quad \frac{\partial^2 u}{\partial y^2}(x,\infty) = 0. \tag{2.4b}$$

It is pertinent to note here that in the absence of the magnetic field intensity (i.e. the case $B(x) = 0$), an approximation solution to the equations (2.1 – 2.3) is obtained in literature [13, 14], while Blasius [15] deduced exact numerical solution.

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3. Integral Boundary Layer Method

From equation (2.1) we obtain

$$v == -\int_0^y \frac{\partial u}{\partial x}\, dy\,. \tag{3.1}$$

Putting equation (3.1) into equation (2.2) and by integration with respect to y from $y = 0 \cdots y = \delta(x)$, we obtain the equation:

$$\rho\, u_\infty^2 \frac{d}{dx}\int_0^{\delta(x)} \frac{u}{u_\infty}\left(\frac{u}{u_\infty} - 1\right) dy + \int_0^{\delta(x)} \sigma B_0^2 u dy = -\tau_w \ , \tag{3.2}$$

where $\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$ and $\mu\left(\frac{\partial u}{\partial y}\right)_{y=\delta(x)} = 0$. The equation (3.2) is known as the MHD integral boundary layer equation within the boundary, and where $\delta(x)$ is the boundary layer thickness. Also, equation (3.2) provides a quantitative connection between the shear stress on the plaque and the velocity profile in the boundary layer. In other words, the shear stress applied by the plaque on the boundary layer can be calculated directly as long as the velocity profile is known.

In the development of boundary layer analysis over a flat plate, the velocity profile has been shown to be well approximated by a non-dimensional similarity profile. Therefore, to use the integral boundary layer relationship, the applied velocity profile shape within the boundary layer is of the type:

$$\frac{u}{u_\infty}(\eta) \equiv \bar{u} = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4\,, \tag{3.3}$$

where $0 \le y \le \delta(x), \frac{u}{u_\infty}(\eta) \equiv \bar{u} = 1 \text{ for } y \ge \delta(x), \eta = \frac{y}{\delta(x)}.$

The a_i coefficients could be determined by using the appropriate conditions:

$$\eta = 0 : \bar{u} = L\frac{\partial \bar{u}}{\partial \eta}, \frac{\partial^2 \bar{u}}{\partial \eta^2} = \frac{\sigma B(x)^2\, \delta(x)^2}{\rho\nu}\bar{u}, \tag{3.4a}$$

$$\eta = 1 : \bar{u} = 1, \frac{\partial \bar{u}}{\partial \eta} = 0, \frac{\partial^2 \bar{u}}{\partial \eta^2} = 0. \tag{3.4b}$$

The application of the conditions (3.4) in the proposed non-dimensional velocity profile shape (3.3) gives the result:

$$\bar{u} = \frac{1}{2L + 1 + \frac{M}{6}}\left[2L + 2\eta + M\eta^2 - 2\left(1 + \frac{2}{3}M\right)\eta^3 + \left(1 + \frac{1}{2}M\right)\eta^4\right], \tag{3.5}$$

where $M = \frac{\sigma B_0^2}{\rho\nu}$ is the magnetic parameter, and the magnetic field intensity is now defined as

$$B(x)^2 = B_0^2\delta(x)^{-2}L^{-1} \equiv B_0^2\delta(x)^{-1}L_1^{-1}, \tag{3.6}$$

and where

$$L_1 = L\delta(x) \tag{3.7}$$

is the velocity slip factor.

The local tension between two neighbour layers $\tau = \mu\left(\frac{\partial u}{\partial y}\right)$ has the expression:

$$\tau = \frac{2\mu u_\infty}{\delta(x)\left(1+2L+\frac{M}{6}\right)}\left[1+M\eta-3\left(1+\frac{2}{3}M\right)\eta^2+2\left(1+\frac{1}{2}M\right)\eta^3\right]. \tag{3.8}$$

The local stress on the plaque has the expression:

$$\tau_w = \frac{2\mu u_\infty}{\delta(x)\left(1+2L+\frac{M}{6}\right)}. \tag{3.9}$$

Putting equations (3.5) and (3.9) into the boundary layer equation (3.2), and solving for the boundary layer thickness $\delta(x)$ due to $\delta(0)=0$ gives

$$\delta(x) = \frac{\sqrt{42}}{2}\sqrt{\left(\frac{\mu x}{\rho u_\infty}\right)\cdot\frac{240L^2+120L+240L^2M+M^3+32M^2L+13M^2+224LM+42M}{189L^2+13LM+37L+LM^2+42L^2M}}. \tag{3.10}$$

Therefore, the local stress τ_w is uniquely determined by substituting the solution (3.10) in equation (3.9), and it becomes:

$$\tau_w = \frac{4}{\sqrt{42\,\mathrm{Re}_x}}\frac{\rho u_\infty^2}{\left(1+2L+\frac{M}{6}\right)} \times\sqrt{\frac{189L^2+13LM+37L+LM^2+42L^2M}{240L^2+120L+240L^2M+M^3+32LM^2+13M^2+224LM+42M}}, \tag{3.11}$$

where $\mathrm{Re}_x = \frac{xu_\infty}{\nu}$, is the local Reynolds number.

Another important parameter is the friction drag force applied on the plane plaque, obtained by integrating equation (3.11) over the entire length L_p of the plaque:

$$F_{Df} = \int_0^{L_p}\tau_w dx = \frac{2.469}{2\sqrt{\mathrm{Re}_{L_p}}}\frac{\rho u_\infty^2 L_p}{\left(1+2L+\frac{M}{6}\right)} \times\sqrt{\frac{189L^2+13LM+37L+LM^2+42L^2M}{240L^2+120L+240L^2M+M^3+32LM^2+13M^2+224LM+42M}}, \tag{3.12}$$

where Re_{L_p} is defined similar to Re_x, but with L_p in place of x. In addition, the coefficient of friction drag for the plaque is defined by

$$C_{Df} = \frac{F_{Df}}{\rho \frac{u_\infty^2}{2} L_p} = \frac{2.469}{\sqrt{\mathrm{Re}_{L_p}}} \frac{1}{\left(1+2L+\frac{M}{6}\right)} \times \sqrt{\frac{189L^2+13LM+37L+LM^2+42L^2M}{240L^2+120L+240L^2M+M^3+32LM^2+13M^2+224LM+42M}}. \tag{3.13}$$

It is important to note that where the fluid slips or slides over the entire length of the plaque, then $L = L_p$. The results (3.12, 3.13) are consistent with those obtained in Rubin and Atkinson [16] for $M = 0$ and respectively $L = 0$.

Without loss of generality, also, for $M = 0$ and respectively $L = 0$, equation (3.10) reduces to

$$\delta(x) = 5.836\sqrt{\frac{\nu x}{u_\infty}} = 5.836x\,\mathrm{Re}_x^{-\frac{1}{2}}. \tag{3.14}$$

Consequently, the local stress on the plaque for negligible magnetic and fluid sliding factor has the expression:

$$\tau_w = 0.343\frac{\rho u_\infty^2}{\sqrt{\mathrm{Re}_x}}. \tag{3.15}$$

It is of interest to compare these results (3.14, 3.15) with previous estimates. For a boundary layer flow of a Newtonian fluid over a flat plate (i.e. the Blasius problem), Helmy et al. [17] obtained

$$\delta(x) = 5.77\sqrt{\frac{x}{a}}, \tag{3.16a}$$

$$\tau_w = 0.346\sqrt{\frac{a^3}{x}}, \tag{3.16b}$$

whereas the result obtained by Blasius as an exact solution for the local stress is

$$\tau_w = 0.333\sqrt{\frac{a^3}{x}}. \tag{3.17}$$

Thus, the results (3.14, 3.15) have excellent agreements with the results of Helmy et al. [17], equation (3.16) and the Blasius exact solution (3.17), provided the problem is scaled appropriately.

Bhattacharyya et al. [6] defined respectively, velocity slip factor and velocity slip parameter as

$$L_1 = L(\mathrm{Re}_x)^{1/2}, \tag{3.18a}$$

$$\delta_L = \frac{Lu_\infty}{\nu}. \tag{3.18b}$$

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Consequently, by the definition of the velocity slip factor (3.7), we estimate that

$$\delta(x) \approx \left(\mathrm{Re}_x\right)^{1/2} = \sqrt{\frac{xu_\infty}{\nu}} = \frac{u_\infty}{\nu}\sqrt{\frac{\nu x}{u_\infty}}\,. \tag{3.19}$$

Hence, the slip velocity parameter, δ_L is related to L by the estimate

$$\delta_L \approx 5.836L\,. \tag{3.20}$$

This implies that the constant coefficient 5.836 defines approximately the ratio of the ambient velocity to the kinematic viscosity.

4. Analyses of Results

The integral method of Von Karman type is adapted to the problem of MHD dynamic boundary layer flow on a plane plaque. The results are analyzed herein. For graphical representations, where necessary, properties of water at 40^0C with $\rho = 992.04kgm^{-3}, \mu = 6.556\times 10^{-4}kgsm^{-1}$ with $u_\infty = 0.5ms^{-1}$ are used in the computations. As for the magnetic parameter, M and L, typical values are used and are so indicated.

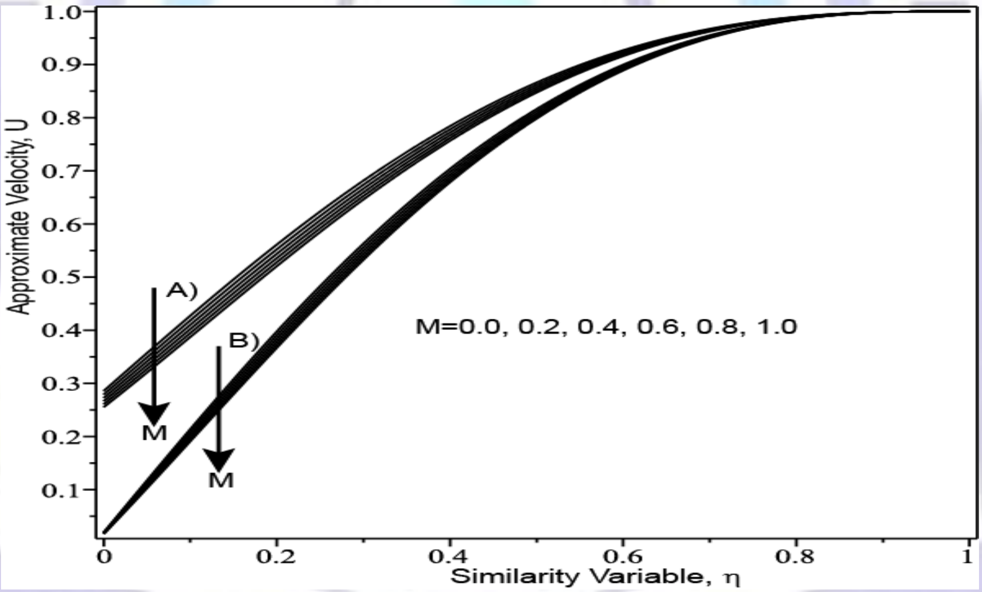

Figure 2: Approximate Velocity Profile versus similarity variable for A) $L = 0.01$, **B)** $L = 0.2$

Figure 2 depicts the approximate velocity profile, equation (3.5) at various values of magnetic parameter. It is observed that increase in magnetic parameter tends to reduce the similarity variable boundary layer η with the slip velocity or fluid sliding parameter enhancing the reduction. Consequently, increase in magnetic parameter increases the boundary layer thickness $\delta(x)$.

Another important feature of the equation (3.5) is that it satisfies the conditions (3.4). In particular, the slip velocity or fluid sliding condition from equation (3.5) is simply the result:

$$U\big|_{\eta=0} = L\frac{\partial U}{\partial \eta}\bigg|_{\eta=0} = \frac{2L}{2L+1+\dfrac{M}{6}}\,. \tag{4.1}$$

For applications of varying degree of magnetic field, such as an electromagnet in which the magnetic field is produced by electric current, the strength of the magnetic field whether large or small is directly proportional to the applied electric current.

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To this end, the magnetic field disappears when the current is turned off. This implies that it is important to evaluate asymptotic results valid for large or small M via the equation (4.1). Therefore, for large M, we obtain

$$U\big|_{\eta=0} \approx \frac{12L}{M} - \frac{72L(1+2L)}{M^2} + \frac{432L(1+2L)^2}{M^3} - \frac{2592L(1+2L)^3}{M^4} + \frac{15552L(1+2L)^4}{M^5} - \cdots \quad (4.2)$$

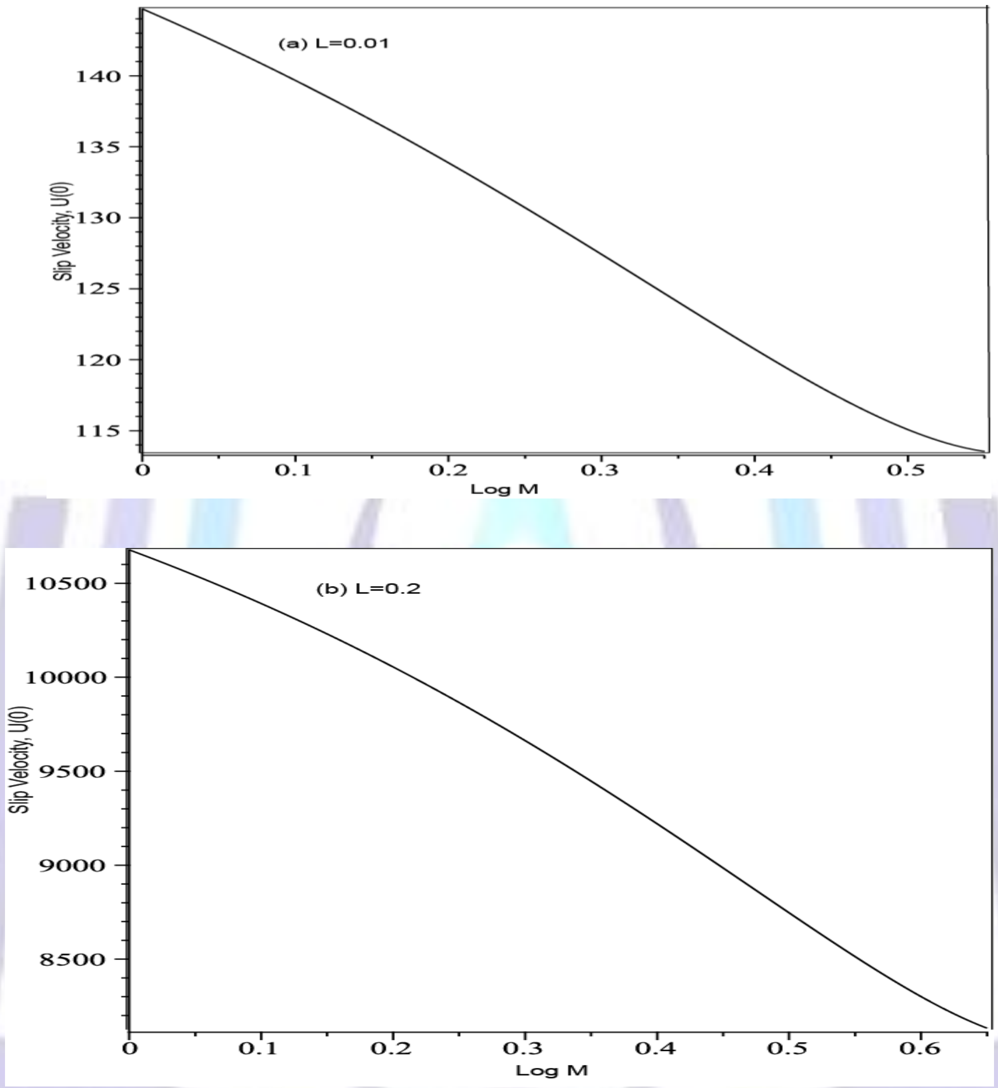

Figure 3: Plots of Sliding Velocity versus M for Asymptotic result of large M : (a) $L = 0.01$, (b) $L = 0.2$

Figure 3 depicts plots of the sliding velocity of the asymptotic result (4.2) for large M ; the abscissa is a logarithmic axis for M to give due emphasis to very large values of M in order to demonstrate the accuracy of the asymptotic result. It is observed that the slip factor plays an enhancing role for the slip velocity.

On the other hand, for small M, we obtain

$$U\big|_{\eta=0} \approx \frac{2L}{(1+2L)} - \frac{1}{3}\frac{LM}{(1+2L)^2} + \frac{1}{18}\frac{LM^2}{(1+2L)^3} - \frac{1}{108}\frac{LM^3}{(1+2L)^4} + \frac{1}{648}\frac{LM^4}{(1+2L)^5} - \frac{1}{3888}\frac{LM^5}{(1+2L)^5} + \cdots \quad (4.3)$$

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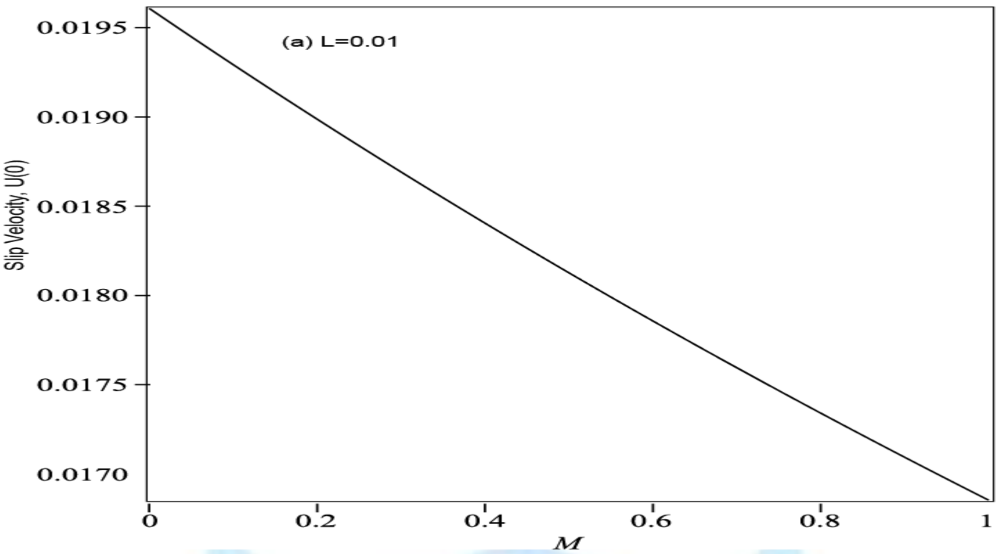

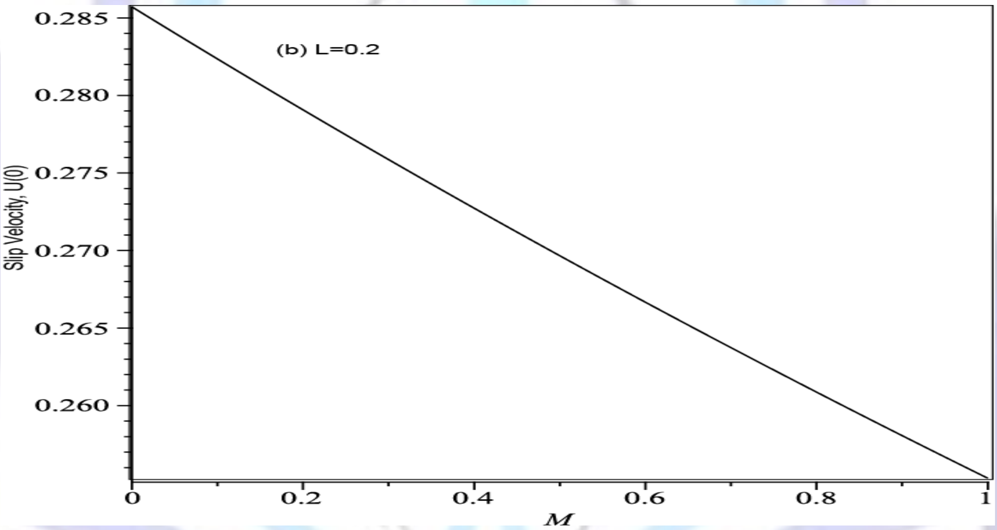

Figure 4: Plots of Sliding Velocity versus M for Asymptotic result of small M : (a) $L = 0.01$, (b) $L = 0.2$.

Figure 4 portrays the asymptotic result (4.3) of small M for the slip velocity. It is seen that respectively $U(0) \approx 0.017$ for $L = 0.01$ and $U(0) \approx 0.255$ for $L = 0.2$ as $M \approx 0$, with the slip factor playing its enhancing role on the slip velocity.

It is important to state at this point that in order to doubly check the accuracy and credibility of the asymptotic results (4.2, 4.3), numerical computations should have been advanced. For the purpose of brevity, numerical computations are not considered in this work.

The functional relationship of the magnetic-slip factor depended boundary layer thickness given in equation (3.10) is plotted in Figure 5. It is observed that the effect of an increased magnetic parameter is to increase the boundary layer thickness as a result of the increased retarding force. This effect is analogous to flow against a positive pressure gradient.

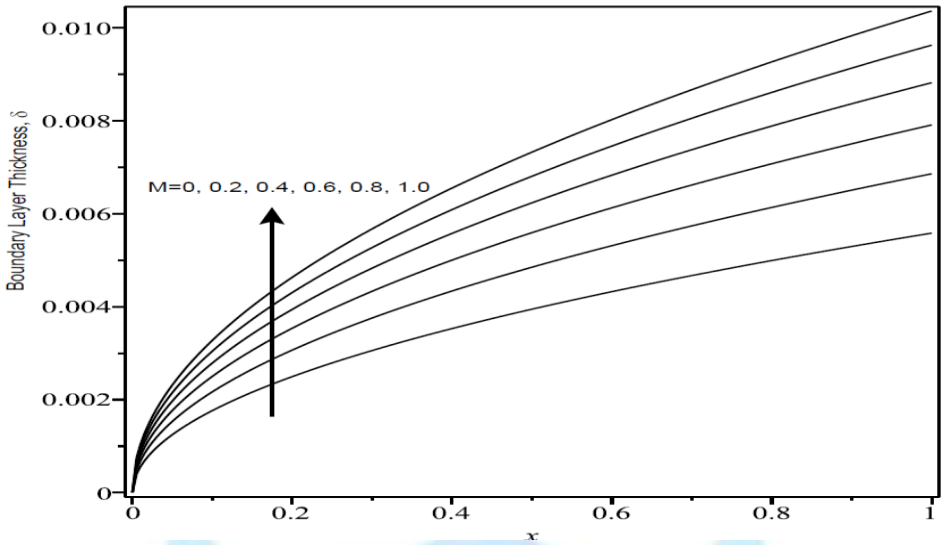

Figure 5: Boundary Layer Thickness, δ versus x for $L = 0.2$ at various values of magnetic parameter

Tables 1 and 2 contain values of the coefficient of the friction drag for various values of the magnetic parameter M and L, respectively. For $M = 0$, Table 1 indicates that increase in L from $L = 0.01$ to $L = 0.2$ decreases $\sqrt{\mathrm{Re}_L} C_{Df}$ from 1.3644 to 1.1768. It is observed generally that increase in the magnetic parameter has a retarding effect on the coefficient of the friction drag, while for $M \geq 0.2$, increase in L from $L = 0.01$ to $L = 0.2$ enhance the coefficient of the friction drag of the fluid on the plaque.

Table 1: Values of $\sqrt{\mathrm{Re}_L} C_{Df}$ for $L = 0.01\left(\delta_L = 0.06\right)$ and $L = 0.2\left(\delta_L = 1.17\right)$ at various values of M

M	$L = 0.01$	$L = 0.2$
	$\sqrt{\mathrm{Re}_L} C_{Df}$	$\sqrt{\mathrm{Re}_L} C_{Df}$
0.0	1.3644	1.1768
0.2	0.4640	0.9354
0.4	0.3295	0.7927
0.6	0.2648	0.6953
0.8	0.2246	0.6230
1.0	0.1964	0.5666

Table 2: Values of $\sqrt{\mathrm{Re}_L} C_{Df}$ for $M = 0.01$ and $M = 0.2$ at various values of L

L	$M = 0.01$	$M = 0.2$
	$\sqrt{\mathrm{Re}_L} C_{Df}$	$\sqrt{\mathrm{Re}_L} C_{Df}$
0.0	0.0000	0.0000
0.2	1.1605	0.9354
0.4	0.9811	0.8417
0.6	0.8406	0.7390
0.8	0.7325	0.6527
1.0	0.6480	0.5824

Table 2 is pertaining to the effect of the increase of L for $M = 0.01$ and $M = 0.2$, respectively. For $L = 0$, Table 2 indicates that $M = 0.01$ and $M = 0.2$, makes no contribution to $\sqrt{\mathrm{Re}_L} C_{Df}$. It is, however, seen that for $L \geq 0.2$, the $\sqrt{\mathrm{Re}_L} C_{Df}$ decreases, with the magnetic parameter, M playing a de-enhancing role, clearly manifesting in the columns respectively for $M = 0.01$ and $M = 0.20$ in the table. According to the estimate (3.20), the values of the slip factor δ_L used for Table 2 due to the values of L are $0.00, 1.17, 2.33, 3.50, 4.67$ and 5.84.

5. Concluding Remarks

The problem of MHD dynamic boundary layer sliding flow over a plane plaque is studied using von Karman integral boundary layer analysis method. Physical characteristics of the problem are derived which have excellent agreements to existing literatures. The integral boundary layer analysis is a key to validating similarity variables for numerical or analytical computations. Possible further study of the problem is the application of similarity variables to advance its solution via numerical or analytical results.

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