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Blow-up of Solution for Initial Boundary Value Problem of Reaction Diffusion Equations

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ABSTRACT

In this paper, the blow-up of solution for the initial boundary value problem of a class of reaction diffusion equations with multiple nonlinearities is studied. We prove, under suitable conditions on memory and nonlinearities term and for negative or positive initial energy, a global nonexistence theorem.

Keywords：

Reaction diffusion equations；blow-up of solution；multiple nonlinearities；positive initial energy；initial boundary；value problem.

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1 INTRODUCTION

In this paper, we study the blow-up of solution for the initial boundary value problem of the following reaction-diffusion equations with multiple nonlinearities

$$u_t - \Delta u + |u|^{m-2} u_t = f_1(u,v), \quad (1.1)$$

$$v_t - \Delta v + |v|^{m-2} v_t = f_2(u,v), \quad (1.2)$$

$$u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0, \quad (1.3)$$

$$u(x,0) = u_0(x), \; v(x,0) = v_0(x), \; x \in \Omega, \quad (1.4)$$

where Ω is bounded domain in $R^n (\mathrm{n} \geq 1)$ with smooth boundary $\partial\Omega$ so that the divergence theorem can be applied, Δ denotes the Laplace operator, $m > 1$, and the two functions $f_1(u,v)$, $f_2(u,v)$ will be given in the later.

Reaction-diffusion equations like equation (1.1) appear in population dynamics, chemical reactions, heat transfer like, for instance, the description of turbulent filtration in porous media, the theory of non-Newtonian fluids perturbed by nonlinear terms and forced by rather irregular period in time excitations (see [1, 2, 3] and the references given therein). It is well known that when the nonlinear diffusion term $|u|^{m-2} u_t (|v|^{m-2} v_t)$ is absent, the nonlinear reaction term $f_i(u,v)$ drives the solution of (1.1)-(1.3) to blow up in finite time. In addition if the reaction term is removed from the equation, then the diffusion term is known to yield existence of global solution [2]. There are a lot of works to discuss the Cauchy problem for the following equations (see [3-6] and the references given therein) on bounded domain or on whole R^n

$$u_t - \Delta u = f(u,v), \quad (1.5)$$

$$v_t - \Delta v = g(u,v). \quad (1.6)$$

They have gave the existence and uniqueness of local solution, positive of the solution, existence of global solution and blow-up of solution. There are more works for a single equation (1.5) with $f(u,v) = f(u)$, here we omit it. Recently, there are some papers to study this class equations, for example, in [7], the authors got the existence and uniqueness of solution and asymptotic for the equations

$$u_{it} - \nabla(a_i D_i(u_i)\nabla u_i) - b_i \bullet \nabla(D_i(u_i)\nabla u_i) = f_i(t,x,u_1,...,u_n), \quad i = 1,2,...,\mathrm{n}$$

with Dirichlet boundary condition; the authors in [8] discussed the existence and uniqueness of solution and blowup for the following equations with local source term and Dirichlet boundary condition

$$u_t = v^p(\Delta u + au(x_0,t)), \quad v_t = u^q(\Delta v + bv(x_0,t));$$

the result about finite-time blowup was obtained in [9] for a supercritical quasilinear parabolic-parabolic Keller-Segel system

$$u_t = \nabla(\phi(u)\nabla u) - \nabla(\psi(u)\nabla u), \quad v_t = \Delta v + u - v.$$

Equations (1.1)-(1.2) can also be as a special case to describe the flow of a gas through a porous medium in a turbulent regime or the spread of biological

$$b_1(u_1)_t - \Delta u_1 + f_1(u_1,u_2) = 0, \; b_2(u_2)_t - \Delta u_2 + f_2(u_1,u_2) = 0,$$

if we take $b_i(u_i) = u_i + |u_i|^{m-2} u_i$. To the best of our knowledge, this class nonlinear parabolic equations have not been well studied. The author of [10,11,12] took the above equations as a dynamical systems and studied their attractor. The above equation can also describe an electric breakdown in crystalline semiconductors with allowance for the linear dissipation of bound- and free-charge sources [15]. We should also point out that Polat [16] established a blow up result for the solution with vanishing initial energy of the following initial boundary value problem

$$u_t - u_{xx} + |u|^{m-2} u_t = |u|^{p-2} u.$$

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In this paper, we derive the blowup properties of problem (1.1)-(1.4) with negative and positive initial energy, respectively, by modifying the concavity method. The main difficult of our proof to handle with the term $|u|^{m-2}u_t(|v|^{m-2}v_t)$ for a desired differential inequalities. The result is differential with the early works. This article is organized as follows. Section 2 is concerned with some notations and statement of assumptions. In Section 3, we give and prove the main result.

2 PRELIMINARY

In this section, we will give some notations and statement of assumptions for $p, m, \mathrm{f}_i(u,v)$. We denote $L^p(\Omega)$ by L^p, $H_0^1(\Omega)$ by H_0^1, the usual Soblev space[17]. The norm and inner of $L^p(\Omega)$ are denoted by $\|.\|_p = \|.\|_{L^p(\Omega)}$ and $(u,v)=\int_\Omega u(x)v(x)dx$, respectively. Especially, $\|.\|=\|.\|_{L^2(\Omega)}$ for $p=2$.

Now, we assume that:

There exists a function $F(u,v)$, such that $f_1(u,v)=\partial F/\partial u,\ f_2(u,v)=\partial F/\partial v$ and there exists positive constants k_1, k_2 such that

$$k_1(|u|^p+|v|^p)\le F(u,v)\le k_2(|u|^p+|v|^p),\ uf_1(u,v)+vf_2(u,v)=(p+1)F(u,v). \tag{2.1}$$

Furthermore, p and m satisfies

$$2<m<p\le\frac{n+2}{n-2} \text{ if } n>2 \text{ and } 2<m<p<+\infty \text{ if } n=1,2.$$

For the main result, we introduce two functions

$$E(t)=\frac{1}{2}\|\nabla u\|^2+\frac{1}{2}\|\nabla v\|^2-\int_\Omega F(u,v)dx, \tag{2.2}$$

$$E(0)=\frac{1}{2}\|\nabla u_0\|^2+\frac{1}{2}\|\nabla v_0\|^2-\int_\Omega F(u_0,v_0)dx, \tag{2.3}$$

where $u\in H_0^1(\Omega)$.

In this paper, we always assume that the problem (1.1)-(1.4) exist a local solution u and the solution u satisfies

$$u,v\in C(0,T;H_0^1)\cap C^1(0,T;L^2),\ |u|^{m-2}u_t, |v|^{m-2}v_t\in L^2((0,T)\times\Omega)$$

$u(0)=u_0$, $v(0)=v_0$ and for any $t\in[0,T)$ and $\varphi\in C(0,T;H_0^1)$ there are

$$\int_0^t\int_\Omega [\nabla u(s)\nabla\varphi(s)+\varphi(s)u_t(s)+|u|^{m-2}u_t\varphi-f_1(u,v)\varphi]dxds=0, \tag{2.4}$$

$$\int_0^t\int_\Omega [\nabla v(s)\nabla\varphi(s)+\varphi(s)v_t(s)+|u|^{m-2}u_t\varphi-f_1(u,v)\varphi]dxds=0. \tag{2.5}$$

Multiplying Equation (1.1),(1.2) by u_t, v_t, respectively, then integrating over Ω and taking their sum, we have,

$$\frac{d}{dt}E(t)=-\|u_t\|^2-\|v_t\|^2-\int_\Omega|u|^{m-2}u_t^2dx-\int_\Omega|v|^{m-2}v_t^2dx<0. \tag{2.6}$$

3 MAIN RESULTS

In this section, we consider the blow-up condition for the system (1.1)-(1.4) and prove the main result. Our technique is more complex than that in [13,14] for a desired differential inequalities because of the presence of the nonlinear term $|u|^{m-2}u_t$ and $|v|^{m-2}v_t$.

Now, we give firstly the blow-up condition for negative initial energy.

Theorem 1 Suppose that the assumption about m, p hold and $u_0, v_0 \in H_0^1(\Omega)$,and u, v are a local solution of the system (1.1)-(1.4), $E(0) < 0$ is sufficient negative. Then the solution of the system (1.1)-(1.4) blows up in finite time.

Proof Let

$$H(t) = \tfrac{1}{2}\|u\|^2 + \tfrac{1}{2}\|v\|^2 + \frac{1}{m}\|u\|_m^m + \frac{1}{m}\|v\|_m^m. \qquad (3.1)$$

Noting that

$$m\int_\Omega |u|^{m-2} uu_t dx = \frac{d}{dt}\|u\|_m^m,\ m\int_\Omega |v|^{m-2} vv_t dx = \frac{d}{dt}\|v\|_m^m, \qquad (3.2)$$

then using (1.1) and (1.2), we get from (3.2) and (2.1), we arrive

$$H'(t) = \int_\Omega uu_t dx + \int_\Omega |u|^{m-2} uu_t dx + \int_\Omega vv_t dx + \int_\Omega |v|^{m-2} vv_t dx$$

$$= -\int_\Omega [\nabla u \nabla u + |u|^{m-2} uu_t - f_1(u,v)u]dx + \int_\Omega |u|^{m-2} uu_t dx$$

$$-\int_\Omega [\nabla v \nabla v + |v|^{m-2} vv_t - f_2(u,v)v]dx + \int_\Omega |v|^{m-2} vv_t dx$$

$$= -\|\nabla u\|^2 - \|\nabla v\|^2 + (p+1)\int_\Omega F(u,v)dx. \qquad (3.3)$$

From the expression of $E(t)$

$$\|\nabla u\|^2 + \|\nabla v\|^2 = 2[E(t) + \int_\Omega F(u,v)dx],$$

and then the equation (3.3) can be rewritten

$$H'(t) = (p-2)\int_\Omega F(u,v)dx + 2(-E(t)). \qquad (3.4)$$

Since $E(0) < 0$ we have $-E(t) \ge -E(0) > 0$ by (2.6), then we deduce by embedding theorem

$$H'(t) \ge (p-2)\int_\Omega F(u,v)dx \ge Ck_1(\|u\|_p^p + \|v\|_p^p). \qquad (3.5)$$

By the assumption that $E(t) \le E(0) < 0$ is sufficient negative and from the expression of $E(t)$, we can assume that $\int_\Omega F(u,v)dx > 1$. Furthermore, by the condition (2.1) we can also assume that $\|u\|_p^2 \ge 1, \|v\|_p^2 \ge 1$. Noticing $p > m > 2$, by embedding theorem, we have

$$\|u\|_p^p \ge C(\|u\|^2)^{\frac{p}{2}} \ge C(\|u\|^2)^{\frac{p}{m}}$$

and

$$\|u\|_p^p \ge C(\|u\|_m^m)^{\frac{p}{m}},$$

Similar,

$$\|v\|_p^p \ge C(\|v\|^2)^{\frac{p}{2}} \ge C(\|v\|^2)^{\frac{p}{m}},\ \|v\|_p^p \ge C(\|v\|_m^m)^{\frac{p}{m}},$$

where and in the following $C > 0$ is a positive constant. By (3.5) we get

$$F'(t) \ge C(\|u\|_p^p + \|v\|_p^p) = \tfrac{C}{2}\|u\|_p^p + \tfrac{C}{2}\|v\|_p^p \ge C(\|u\|^2)^{\frac{p}{m}} + C(\|u\|_m^m)^{\frac{p}{m}} + C(\|v\|^2)^{\frac{p}{m}} + C(\|v\|_m^m)^{\frac{p}{m}}$$

$$\ge C(\|u\|^2 + \|u\|_m^m + \|v\|^2 + \|v\|_m^m)^{\frac{p}{m}} = CF^{\frac{p}{m}}(t),$$

where we have used the inequality

$$\left(a_1+a_2\right)^k \le 2^{k-1}\left(a_1^k+a_2^k\right),\ a_{1,}a_2 \ge 0\,, k>1.$$

A simple integration of the inequality (3.6) over $\left(0,t\right)$ yields

$$F^{\frac{p-m}{m}}\left(t\right) \ge \frac{1}{F^{\frac{p-m}{m}}\left(0\right)-\frac{p-m}{m}Ct}$$

Therefore there exists a positive constant given by $t \to T = \dfrac{m}{C\left(p-m\right)F^{\frac{p-m}{m}}\left(0\right)}$ such that $F\left(t\right) \to \infty$ when $t \to T$.

This completes the proof.

Next, we discuss the blow-up condition for positive initial energy. For this purpose, we give the following lemma, which is introduced by Vitillaro[19] for studying wave equation. Denote

$$\lambda_1 = C_*^{-\frac{p}{p-2}},\quad E_1 = (\tfrac{1}{2}-\tfrac{1}{p})\lambda_1^2,$$

where C_* is a embedding constant.

Lemma 2[19] Suppose that u,v are local solution of problem (1.1)-(1.4) and the assumption about m,p and f_1, f_2 hold. If $E\left(0\right)<E_1$ and $\|u_0\|^2+\|v_0\|^2>\lambda_1^2$, then there exists a constant $\lambda_2>\lambda_1$ such that

$$\|u\|^2+\|v\|^2>\lambda_2^2, \|u\|_p^p+\|v\|_p^p \ge C\int_\Omega F(u,v)dx \ge C\lambda_2^{2p}.$$

By the above lemma, we can get the following blow-up condition for positive initial energy.

Theorem 3 Suppose that u,v are local solution of problem (1.1)-(1.4) and the assumption about m,p and f_1, f_2 hold. If $E\left(0\right)<E_1$ and $\|u_0\|^2+\|v_0\|^2>\lambda_1^2$, then the solution of the system (1.1)-(1.4) blows up in finite time.

Proof Let

$$H(t)=E_2-E\left(t\right), \qquad (3.6)$$

where $E(0)<E_2<E_1$. By the definition of $H(t)$ and (2.6), we have

$$H'(t)=-E'\left(t\right), \qquad (3.7)$$

Consequently,

$$H(0)=E_2-E\left(0\right)>0, \qquad (3.8)$$

It is clear that by (3.7) and (3.8),

$$0<H(0)=H\left(t\right). \qquad (3.9)$$

By (3.6), the expression of $E(t)$, and Lemma 2, we have

$$H\left(t\right)=E_2-(\frac{1}{2}\|\nabla u\|^2+\frac{1}{2}\|\nabla v\|^2-\int_\Omega F(u,v)dx)$$

$$\le E_2-\frac{1}{2}\lambda_2^2+\frac{1}{2}\int_\Omega F(u,v)dx$$

$$\le E_2-\frac{1}{2}\lambda_1^2+\frac{1}{2}\int_\Omega F(u,v)dx$$

$$\leq -\frac{1}{p}\lambda_1^2 + \frac{1}{2}\int_\Omega F(u,v)dx \leq C\int_\Omega F(u,v)dx . \quad (3.10)$$

Let us define the functional

$$L(t) = H(t) + \varepsilon(\|u\|^2 + \|v\|^2) , \quad (3.11)$$

for ε to be chosen later. By taking the time derivative of (3.11) , we have

$$L'(t) = H'(t) + \varepsilon(\int_\Omega uu_t dx + \int_\Omega vv_t dx) , \quad (3.12)$$

The reminder of the proof is similar to the proof of Theorem1 combined with the proof in [20], and then we get the result.

Remark 1.The condition (2.1) can assure that the local solution of problem (1.1)-(1.4) exist, and this condition can be found in [18].

Remark 2. The method in this paper can be used to more general equations.

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