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PRINCIPALLY QUASI INJECTIVE SYSTEM OVER MONOID

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ABSTRACT :

In this paper , principally quasi injective system has been introduced and studied , which is a generalization of quasi injective system . We obtain a characterizations of PQ-injective systems , conditions on which, subsystems inherit the property of PQ-injectivity , and conditions have been considered to versus PQ-injective system with class of injectivity. Finally , the relationship between maximal reversible subsystem of system and maximal left ideal of the endomorphism monoid of the system has been studied .

Keywords

Quasi-injective system; Reversible system; Principally quasi injective system .

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Pure Mathematics

1- INTRODUCTION AND PRELIMINARIES:

It is well-known that the theory of monoids and systems is a generalization of the theory of rings and modules , which has a number of direct applications in theoretic Computer science,Dynamicsystems,Theory of differential equations and Functional analysis,…etc. Throughout this paper , let S be a monoid. A unitary right S-system M over S which denoted by M_s is a non-empty set with a function f : M $\times$ S → M such that f(m,s)↦ ms and the following properties hold : (1) m•1= m . (2) m(st) = (ms)t for all m ∈ M and s,t ∈ S . An element θ ∈ M_s is called **fixed** of M_s if θs = θ for all s ∈ S . An S-system M_s is **centered** if it has a fixed element θ necessary unique such that θs = θ for all s ∈ S and m 0 = θ for all m ∈ M_s , where 0 is the zero element of S and θ is the zero of M . A**subsystem** N of an S-system M_s , is a non-empty subset of M such that xs∈ N for all x∈N and s ∈ S which is denoted by $N \hookrightarrow M_s$. Let g be a function from an S-system A_s into an S-system B_s , then g will be called an S-homomorphism , if for any a ∈ A_s and s ∈ S , we have g(as) =g(a)s . An S-system A_s is called **injective** if for each monomorphism α :C_s → B_s and each S-homomorphism β:C_s →A_s ,there exists an S-homomorphism σ : B_s→A_s such that σα = β [2] . An S-system A_s is **weakly injective** if it is injective relative to all embeddings of right ideals into S_s ([7],p.205) . An S-system is called **principally weakly injective** if for any S-homomorphism from principal right ideal of S_s into M_s can be extended to S-homomorphism from S_s into M_s.In other words,anS-system M_s is called **principally weakly injective** if it is injective relative to embeddings of all principal right ideals into S_s(If this is the case ,we right **PW-injective system**)([7],p200) . A subsystem N of M_s is called **large** (or **essential**) in M_s if and only if any homomorphism f :M_s → H_s , where H_s is any S-system with restriction to N is one to one,then f is itself one to one [9].In this case we say that M_s is essential extension of N .In [9], Berthiaume showed that every S-system has a maximal essential extension which is injective and it is unique up to S-isomorphism over M_s . A non-zerosubsystem N of M_s is **intersection large** if for all non-zero subsystem A of M_s , A ∩ N ≠ θ ,and will denoted by N is ∩-large in M_s . In [4] , Feller and Gantos proved that every large subsystem of M_s is ∩-large,but the converse is not true in general . An equivalence relation ρ on a right S-system M_s is a **congruence** relation iff a ρ b implies that as ρ bs for all a,b∈ M_s and s ∈ S [1].The congruence ψ_M is called **singular** on M_s and it is defined by a ψ_M b if and only if ax = bx for all x in some ∩-large right ideal of S [2] . A **right annihilator** of an S-system M_s is denoted by $\gamma_s(T)$ where T is a subset of M_s and it is equal to the set {(s,t) ∈ S$\times$S │as = at for all a ∈ T } and if K is a subset of M$\times$M ,then $\gamma_s(K) = \{ s \in S \mid as = bs \text{ for all } (a,b) \in K \}$ and **aleft annihilator** of an S-system is dented by $\ell_M(H)$ where H is a subset of S and it is equal to the set { (m,n) ∈ M $\times$ M │mx = nx for all x ∈ H } but if J is a subset of S$\times$S , then $\ell_M(J) = \{ a \in M \mid am = an \text{ for all } (m,n) \in J \}$[6] . A non-zero S-system Mover a monoid S is called **reversible** (**∩-reversible**) iff every non-zero subsystem of M_s is lagre (∩-large) , it is clear that every nonzero reversible system is ∩-reversible system, but the converse is not true in general and they are coincide when ψ_M = i . An element s ∈ S is called **left** (**right**) **cancellable** if sr = st (rs = ts) for r,t∈ S implies r = t and cancellable if s is left and right cancellable.The semigroup S is called **cancellative** if all elements of S are cancellable ([7],P.30).In [9] , Berthiaume showed that injective system implies weakly injective , but the converse is not true in general . Berthiaume's counter example was a semilattice considered as an S-system over itself . In[3], Hinkle showed that when $\psi_M = i_M$, the identity congruence on M_s ,then the notions of injective and weakly injective system are coincide and also the concepts of large and ∩-large are the same . An S-system A_s is called **quasi injective** if for any S-subsystem B of A_s and any S-homomorphism α :B →A_s , there exists S-homomorphism σ : A_s → A_s such that σ is an extension of α, that is σ o i = α where i is the inclusion mapping of B into A_s [5] .quasi injective S-systems have been studied by Lopez and Luedeman [1] . It is clear that every injective system is quasi injective but the converse is not true in general see [1] . An S-system A_s is called cyclic (or principal) system if it is generated by one element and is denoted by A_s=< u > where u ∈ A_s ,then A_s= uS ([7],P.63) . A right S-system B_s is **a retract** of a right S-system A_s if and only if there exists a subsystem W of A_s and epimorphism f : A_s → W such that B_s≅ W and f(x) = x for every x∈ W ([7],P.84).An S-monomorphism f :A_s→B_s is called **a retraction** if f is a left invertible ([7] ,P.84).An S-system M_s is called **θ-simple** system if it contains no subsystems other than M_s and one element subsystem and M_s is called **simple** if it contains no subsystems other than M_s itself ([7],p50).AnS-system M_s is called **completely reducible** if it is a disjoint union of θ-simple subsystems ([7],P.74). In this paper ,a generalization of quasi injective system namely principally quasi injective was introduced and characterization of this new class of systems was investigated . Also, we give under which a condition for principally quasi injective to be quasi injective system .In spite of there is no relation between PQ-injective system and PW-injective , but they are coincide on the system S_s . A relationship between a maximal reversible subsystem of an S-system M_s and maximal left ideal of the endomorphism monoid of M_s was studied .

2- PRINCIPALLY QUASI INJECTIVE SYSTEMS:

(2-1) Definition : An S-system M_s is called **principally quasi injective** if every S-homomorphism from a principal subsystem of M_s to M_s extends to an S-endomorphism of M_s(If this is the case,we write M_s is **PQ-injectivesystem**).

(2-2) Remarks and examples :

1- Every quasi injective (and hence injective) S-system is PQ-injective .

2-The converse of (1) is not true in general , for example , let S be amonoidsuch that S = {a,b,c,e} ,with a,b be left zero of S and ca = cb = cc = a and e be the identity element .Then consider S as an S-system over itself .It is clear that every subset of S is subsystem of S_s . Since every homomorphism from right principal subsystem (aS = {a} orbS = {b} or cS = {a,c}) can be trivially extended to S-endomorphism of S_s , so S_s is PQ-injective system, but when we take N = {a,b} be subsystem of S_s and f be S-homomorphism defined by : $f(x) = \begin{cases} b & \text{if } x = a \\ a & \text{if } x = b \end{cases}$,then this S-homomorphism cannot be extended to S-homomorphism g : S_s → S_s . If not, that is there exists S-homomorphism g:S_s→S_s such that g(x) = f(x)

$\forall x \in N$, which is just the trivial S-homomorphism since other extension is not S-homomorphism . Then , b = f(a) = g(a) = a which implies that b = a , and this is a contradiction .

3- Recall that a subsystem N of an S-system M_s is direct summand iff there exists a subsystem W of M_s such that $M = W \oplus N$ that is $M = W \cup N$ and $W \cap N = \theta$. Now, let N be subsystem of an S-system M_s .Then N is a retract iff N is a direct summand .

Proof :⇒) Let N be a retract of M_s ,so there exists a subsystem W of M_s and epimorphism $f:M_s \to W$ such that f(x) = xfor each x belong to W and there is an S-isomorphism $g:W \to N$,then $h(=gof):M_s \to N$ is epimorphism ,so $hoi_N = I_N$ and N is a direct summand .

⇐) Let N be a direct summand of M_s , so there exists a subsystem W of M_s with $N \cup W = M$ and $N \cap W = \theta$. Now, we claim that $\alpha(= \pi_2 o j_1)$: $N \to W$ be S-isomorphism, where π_2 is the projection map of M_s into W and j_1 is the injection map of N into M_s. For $n \in N$, we have , $\pi_2 o j_1(n)$ = o , it is clear that α is S-isomorphism . Then , N is a retract of M_s . Also ,

W = 0 and M_s = N .

4- If every principal subsystem of an S-system M_s is a retract [that is for each $x \in M_s$, there is a subsystem K_s of M_s with $xS \dot{\cup} K_s = M_s$, this equality in fact is equivalently to $xS \cup K_s = M_s$ and $xS \cap K_s = \theta$] .Then M_s is PQ-injective. In particular let S ={1,z} with $z^2 = 1$. Consider S be an S-system over itself , then since every subsystem of S_s is a retractofS_s ,so $S_s$is PQ-injective.

The property of principal quasi-injectivity on systems is not closed under subsystems ,for example :let S={a,b,c,e} be a monoid with ca =cb=cc=a and a,b be left zero of S and e be the identity element.Consider S be S-system over itself .Then ,since any S-homomorphism from right principal subsystem of S_s (which is equal to aS or bS or cS)can be trivially extended to S-endomorphism of S_s,so$S_s$is PQ injective system , but the subsystem N={a,b,c} of $S_s$is not PO-injective system . If not , so for a right principal subsystem aS of N and f beS-homomorphism fromaSintoS$_s$ defined by f(x)=b ,where x∈aS , can be extendedto S-homomorphism $g:S_s \to S_s$ such that f(x)=g(x),∀x(≠0)∈ aS . Then, b = f (a) = g (a) = a . So b=a which is a contradiction .

Now, we give a condition for an S-subsystem N of PQ-injective system M_s to be PQ-injective . First ,we need the following concept :

A subsystem N of a right S-system M_s is called **fully invariant** if $f(N) \subseteq N$ for every endomorphism f of M_s and M_s is called **duo** if every subsystem of M_s is fully invariant [10],for example , let S = (Z,•) , then consider S as an S-system over itself , then S_s duo system [10] .This concept is generalization of right **duo semigroup**[10] ,such that a semigroup for which every right ideal is two sided ideal is called duo semigroup . If $M_s = M_1 \dot{\cup} M_2$ (means $M_s = M_1 \cup M_2$ and $M_1 \cap M_2 = \theta_M$) , then for every i ∈ I={1,2} , M_i is fully invariant subsystem of $M_s$iffHom(M_1,M_2) = 0 for all distinct i ∈ I={1,2} .

(2-3) Lemma:

1- Every fully invariant subsystem of PQ-injective system is PQ-injective.

2- Retract of PQ-injective system is PQ-injective .

Proof :1- It is clear from the definition .

2-Let Nbe a retract of a PQ-injective system M_s. By (2-2)(3) , N is direct summand . Consider the diagram(1) ,where A be principal subsystem of N ,and i_1,i_2 be the inclusion maps of A into N and N into M_s respectively . Let f be S-homomorphism of A into N, and ϕ , π be the injection and projection map respectively . Since M_s is PQ-injective system , so there exists S-homomorphism $g : M_s \to M_s$ such that $gi_2i_1 = \phi f$. Define an S-homomorphism $h : N \to N$ by $h = \pi o g o i_2$.

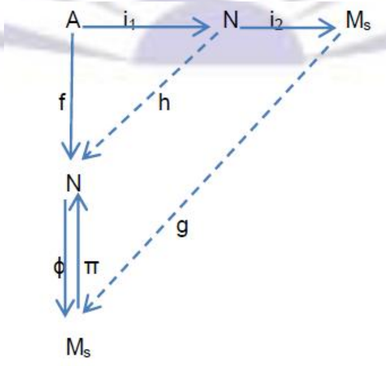

Diagram (1)

Thus $hoi_1 = \pi o g o i_2 o i_1$ this implies $hoi_1 = \pi o g o i_2 o i_1 = \pi o \phi o f = I_N o f = f$. Thus $hoi_1 = f$, and N be PQ-injective system .

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It is clear that there is no relation between PQ-injective systems and PW- injective in general for example, let S be the monoid$\{a_1,b_1,c_1,1\}$ and A={1,a,b} is a set with a,b are left zero and $a_1^2 = a$, $b_1^2 = b$, $c_1^2 = c$ and $a_1c_1=c_1b_1=c_1$, $b_1c_1 = b_1$, $c_1a_1=b_1a_1= a_1b_1= a_1$, then A_s is an S-system (such that $A_s=\{a,b,a_1,b_1,c_1,1\}$) ,so A_s is PQ-injective system whence any S-homomorphism from principal subsystem (aS ={a} ,bS = {b} ,$a_1S= c_1S=\{a_1,c_1\}$ or $b_1S=\{a_1,b_1\}$) of A_s can be trivially extended to S-endomorphism of A_s , but A_s is not PW-injective system . If not , so for the principal ideal $a_1S = \{a_1,c_1\}$ of S and the S-homomorphism f which is defined by $f = \begin{Bmatrix} a_1 & \text{if } x = a_1 \\ b_1 & \text{if } x = c_1 \end{Bmatrix}$, then this S-homomorphism can be extended to S-homomorphism from S into A_s where the only extension of f is trivially extension g :S → A_s , this means f(x) = g(x) , $\forall x \in aS$ this implies that $b_1 = f(c_1) \neq g(c_1) = c_1$ which is a contradiction . But, the concepts of PQ-injective and PW-injective are equivalent on monoid .

The following lemma is a generalization of lemma (1-1) in [13] :

(2-4)Lemma : Given an S-system M_s with T= $End_s(M_s)$,the endomorphism monoid of M_s . The following statements are equivalent :

1- M_s is PQ-injective ,

2- $\ell_M(\gamma_s(m)) = Tm$, $\forall$ m ∈ M_s .

3- If $\gamma_s(m) \subseteq \gamma_s(n)$, then $Tn \subseteq Tm$, ∀ m,n ∈ M_s ,

4- If S-homomorphisms α,β :mS→M_s are given with β is monomorphism , there exists σ∈ T such that σoβ = α .

Proof :(1→2) Let αm ∈ Tm .For each s,t∈ S with ms = mt ,we have α(ms)=α(mt) , so $\alpha m \in \ell_M(\gamma_s(m))$. Thus $Tm \subseteq \ell_M(\gamma_s(m))$. Conversely, if $n \in \ell_M(\gamma_s(m))$, then define σ :mS→M_s by σ(ms) = ns , for s∈ S . If ms = mt ,for s,t ∈ S, then $(s,t) \in \gamma_s(m) \subseteq \gamma_s(n)$, hence ns = nt ,so this shows that σ is well-defined, it is an easy matter to see that σ is an S-homomorphism . By (1) ,σ can extended to $\bar{\sigma} \in T$. So n = σ(m) = $\bar{\sigma}$(m) ∈ Tm . Thus $\ell_M(\gamma_s(m)) \subseteq Tm$ and hence $\ell_M(\gamma_s(m)) = Tm$.

(2→3) If $\gamma_s(m) \subseteq \gamma_s(n)$, then$n \in Tn = \ell_M(\gamma_s(n)) \subseteq \ell_M(\gamma_s(m)) = Tm$, so n ∈ Tm and hence $Tn \subseteq Tm$.

(3→4) Let $(s,t) \in \gamma_s(\beta(m))$ for s,t ∈ S . Then β(ms) = β(mt) . Since β is monomorphism , then ms = mt and α(m)s = α(m)t , hence $(s,t) \in \gamma_s(\alpha(m))$. Then $\gamma_s(\beta(m)) \subseteq \gamma_s(\alpha(m))$. By(3) , αm∈ Tβ(m) . So there is σ∈ T such that α(m) = σβ(m) and hence α = σβ .

(4→1) Take β : mS→M_s to be the inclusion homomorphism in (4) .

(2-5) Corollary : The following statements are equivalent for a monoid S :

1- S is PQ(PW)-injective ,

2- $\ell_s(\gamma_s(a)) = Sm$, ∀ a ∈ S .

3- If $\gamma_s(a) \subseteq \gamma_s(b)$, then $Sb \subseteq Sa$, ∀ a,b ∈ S,

4- If S-homomorphisms α,β :aS→S_s are given with β is monomorphism , there exists σ∈ T such that σ o β = α .

Next , we give a generalization of lemm(1.2) in [13] :

(2-6) Lemma: LetM_s be a PQ-injective with T= $End_s(M_s)$. If α∈T and m∈M_s , then :$\ell_T(\ker(\alpha) \cap (mS \times mS)) = T\alpha \cup \ell_T(mS \times mS)$.

Proof : Let $\beta \in \ell_T[\ker(\alpha) \cap (mS \times mS)]$. We claim that $\gamma_s(\alpha m) = \gamma_s(\beta m)$, for each s,t ∈ S, if $(s,t) \in \gamma_s(\alpha m)$, then αms = αmt , this implies that (ms,mt) ∈ ker(α)∩ (mS×mS) , so βms = βmt and hence $(s,t) \in \gamma_s(\beta m)$. By lemma(2-4) , we have $T\beta m \subseteq T\alpha m$,in particular βm ∈ Tαm ,say βm = σαm for some σ∈ T . Thus $\beta \in T\alpha \cup \ell_T(mS \times mS)$.This shows that $\ell_T(\ker(\alpha) \cap (mS \times mS)) \subseteq T\alpha \cup \ell_T(mS \times mS)$. Conversely, let $\beta \in T\alpha \cup \ell_T(mS \times mS)$, then β =σα for some σ ∈ T or β(ms) = β(mt) for all s, t ∈ S and m ∈ M_s . For each (ms,mt) ∈ ker(α)∩ (mS×mS) , if β =σα , then α(ms) = α(mt) and hence σα(ms) = σα(mt) , so β(ms) = β(mt) . Thus $\beta \in \ell_T[\ker(\alpha) \cap (mS \times mS)]$. If β(ms) = β(mt) , then $\beta \in \ell_T(mS \times mS)$ and hence$\beta \in \ell_T(\ker(\alpha) \cap (mS \times mS))$ .Thus$T\alpha \cup \ell_T(mS \times mS) \subseteq \ell_T(\ker(\alpha) \cap (mS \times mS))$.

If $M_s \simeq S_s \simeq T$ ([7],P.65) , then the condition in lemma(2-6) gives , when S is a PW-injective .

(2-7) Corollary: If S is a right PQ(PW)-injective , then each s, t ∈ S , we have :$\ell_s(\gamma_s(s) \cap (tS \times tS)) = Ss \cup \ell_s(tS \times tS)$

In [13] , define principally self-generator module which motivate us to define principally self-generator system as follows :

(2-8) Definition: An S-system M_s is **principally self-generator** if every x ∈ M_s , there is an S-homomorphism

f : M_s → xS such that x = f(x_1) for x_1∈ M_s .

In the following proposition we discuss the converse of lemma (2-6) :

(2-9)Proposition: LetM_s be a principal and principally self-generator and T=$End_s(M_s)$ .Then the following statements are equivalent :

1- M_s is PQ-injective .

2- $\ell_T(\ker(\alpha) \cap (mS \times mS)) = T\alpha \cup \ell_T(mS \times mS)$, ∀ m ∈ M_s , α ∈ T .

3- $\ell_T(\ker(\alpha)) = T\alpha$, ∀ α ∈ T .

4- $\ker(\alpha) \subseteq \ker(\beta)$ implies that $\beta \in T\alpha$, ∀ α , β ∈ T .

Proof :(1→2) This follows from (2-6) .

(2→3) If $M_s = m_0S$, and take $m = m_0$ in (2) , we have :

$\ell_T(\ker(\alpha) \cap (M_s \times M_s)) = T\alpha \cup \ell_T(M_s \times M_s)$, so $\ell_T(\ker(\alpha)) = T\alpha$.

(3→4) By (3) we have $T\beta = \ell_T(\ker(\beta)) \subseteq \ell_T(\ker(\alpha)) = T\alpha$, so β ∈ Tα .

(4→1) Let σ : mS → M_s be an S-homomorphism where m ∈ M_s . Since M_s is principal self-generator,there is α ∈ T such that $m=\alpha(m_0)$, again there is β∈T such that $\sigma(m) = \beta(m_0)$. We claim that $\ker(\alpha) \subseteq \ker(\beta)$. For if (k,h)∈ $\ker(\alpha)$, write $k = m_0s$, $h = m_0t$,s,t∈ S . Then,if $\beta(k) = \beta(m_0s) = \sigma(m)s = \sigma[\alpha(m_0)s] = \sigma[\alpha(m_0)t] = \sigma(m)t = \beta(m_0t) = \beta(h)$. Thus(k,h) ∈ ker(β) .By (4) , there is λ ∈ T, such that β = λα , and $\lambda(m) = \lambda(\alpha(m_0)) = \beta(m_0) = \sigma(m)$. This implies that λ is an extension of σ and hence M_s is PQ-injective .

In the following proposition we state a characterization of PQ-injective system which give a corresponding between principal subsystems of M_s and principal subsystems of $_TM$, where T = $End_s(M_s)$. But first ,we need the following concept :

(2-10) Definition([7],P.218) : An S-system $M_s$is called torsion free if as = bs implies a = b ,∀ a,b ∈ M_s where s is a right cancellable element of S .

(2-11) Proposition : Let M_s be a PQ-injective system , and torsion free system over right cancellativemonoid with T= $End_s(M_s)$ and m,n∈ M_s ,then :

1- If nS is an image of mS , then Tn embeds in Tm .

2- If mS embeds in nS , then Tm is an image of Tn .

3- If mS≅ nS , then Tn ≅ Tm .

Proof :

(1) Let f : mS → nS be S-epimorphism , so f(m) ∈ nS , so there exists s ∈ S such that f(m) = ns . Define α :Tn→ M_s by α(gn) = (gn)s = g(ns) = g(f(m)) , ∀ g ∈ T . Consider the diagram (2), where i_1 , i_2 be the inclusion maps of mS , nS respectively . Since M_s is PQ-injective system , so there exists S-homomorphism $\bar{f}$: M_s → M_s extends f (i.e.$\bar{f} \circ i_1 = i_2 \circ f$) , then :

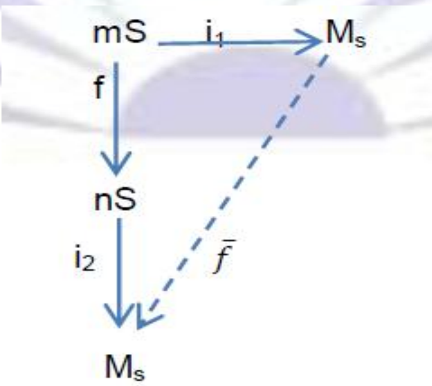

Diagram (2)

α (gn) = gn = g (f(m)) = g ($\bar{f}$ (m)) ∈ Tm , so α : Tn → Tm . Now,∀ β , g ∈ T, gn∈T, then we have α(β(gn))=β(gn)s =β(g(ns))= β(g(f(m)))= β(α(gn))= βα(gn) . Thus α is T-homomorphism . Let $g_1n = g_2n$ where g_1 , g_2∈ T then $g_1ns_1 = g_2ns_1$, such that s_1∈ S . This implies $(g_1,g_2) \in \gamma_s(ns_1)$ and$(g_1,g_2) \in \gamma_s(\bar{f}(m))$.Thus,$g_1(\bar{f}(m)) = g_2(\bar{f}(m))$ and $\alpha(g_1n)=\alpha(g_2n)$,so α is well-defined.Since f isepimorphism ,so there exists b ∈ S such that n = f (mb) , let (g_1n,g_2n) ∈ ker α .Then$\alpha(g_1n)=\alpha(g_2n)$ which implies $g_1n = g_2n$, then $g_1(f(mb))= g_2(f(mb))$,since $g_i(f(mb))\in M_s$(where i =1,2) and M_s is torsion free over right cancellativemonoid , so from $g_1[f(m)]b = g_2[f(m)]b$, we obtain $g_1f(m) = g_2f(m)$. Since $\bar{f}$ extends f , so $g_1\bar{f}(m) = g_2\bar{f}(m)$ which implies $g_1n = g_2n$. Thus α is T-monomorphism .

(2) Let f : mS → nS be S-monomrphism . Consider the same diagram above , since M_s is PQ-injective system , so there exists S-homomorphism$\bar{f}$: $M_s \to M_s$ such that $\bar{f} \circ i_1 = i_2 \circ f$. Define α : Tn → Tm by α(gn) = g(fm) = g ($\bar{f}$m) , such that g, $\bar{f}$ ∈ T . α is well-defined and T-homomorphism as in (1) . We claim that $\gamma_s(fm) \subseteq \gamma_s(m)$, let $(s,t) \in \gamma_s(fm)$ which implies f (ms) = f (mt) . Since f is S-monomorphism , so ms = mt , then $(s,t) \in \gamma_s(m)$, so by lemma (2-4) $Tm \subseteq Tfm$. For βm ∈ Tm , so there exists g ∈ T such that βm = gf(m) = g $\bar{f}$ (m) = α (gn) . Then ,βm = α (gn) , thus α is T-epimorphism .

(3) By (1) and (2) , if f : mS → nS be S-isomorphism , then α : Tn → Tm is T-isomorphism .

(2-12)Lemma: Let S be a monoid and M_s be an S-system .Then an S-system $M_s$is simple iffM_s = xS for each x ∈ M_s.

Proof :⇒) Let x(≠0) ∈ M_s , so xS subsystem of M_s and on the other hand $M_s$is simple , hence M_s generated by x and x = x • 1 ∈ xS which implies M_s is subsystem of xS . Thus , M_s= xS.

⇐) Let N be a non-zero subsystem of M_s and n(≠0) ∈ N , then nS = M_s , but nS subsystem of N, hence M_s = N.

Let N be simple subsystem of an S-system M_s , then **$Soc_N(M_s)$** represent **homogeneous component** of Soc(M_s) containing N . Thus , we denote $Soc_N(M_s) := \cup\{X \subseteq M_s \mid X \cong N\}$. Next we characterize PQ-injective system which represent a generalization of proposition (1.3) in [13] :

(2-13) Proposition : Let M_s be a PQ-injective system with T=$End_s(M_s)$, then :

1- If N is a simple subsystem of M_s , then $Soc_N(M_s)$ = TN .

2- If nS is a simple S-system , n ϵM_s , then Tn is a simple T-system .

3- Soc (M_s) ⊆ Soc($_TM$) .

Proof :

(1) Let $N_1 \subseteq Soc_N(M_s)$, and f : N → N_1 be an S-isomorphism , where N_1⊆ M_s . If N = nS , then$\gamma_s(n) = \gamma_s(f(n))$, so Tn = Tf(n) [by lemma(2-4)] . Thus $f(n) \in Tn \subseteq TN$. Hence, if α is an extension of f to T ,we have $N_1 = f(nS) = \alpha(nS) \subseteq TN$. Thus $Soc_N(M_s) \subseteq TN$. The other inclusion always holds [that is$TN \subseteq Soc_N(M_s)$,since for α ∈ TN ,we have α : N → N be identity map and since N ≅ N and N be subsystem of M_s , so $\alpha(N) = N \subseteq Soc_N(M_s)$, then $TN \subseteq Soc_N(M_s)$] . Therefore $Soc_N(M_s)$ = TN .

(2) Let 0≠αn ∈ Tn . Then α : nS → α(nS) is an S-isomorphism by hypothesis , so let σ :α (nS) → nS be the inverse . If $\bar{\sigma}$ ∈ T extends σ, then $\bar{\sigma}$(α(n)) = σ(α(n)) = n ∈ Tαn .

(3) This implies by (2) .

In[14], Zhang define (m,1)-quasi injective module which motivate us to formulate this concept for an S-system such that we need in the next proposition :

(2-14) Definition : An S-system M_s is called **(m,1)-quasi injective** if for each S-homomorphism from an principal subsystem of M_S^m to M_s can be extended to an S-homomorphism from M_S^m to M_s where m is a fixed positive integer . Note that ,M_s is **(m,1)-quasi injective** iffM_s is **(n,1)-quasi injective**for alln ≤ m .

(2-15)Proposition: LetM_s be (m,1)-quasi system with W = $Hom(M_S^m, M_s)$ and let m_1 , m_2 , …, m_n denote elements of M_s . Then :

1-If$Wm_1 \oplus Wm_2 \oplus ... \oplus Wm_n$ is direct ,then any S-homomorphism $\alpha: m_1S \dot\cup m_1S \dot\cup ... \dot\cup m_nS \to M_s$ has an extension in W .

2-If$m_1S\oplus m_2S\oplus ... \oplus m_nS$is direct, then $W(m_1, m_2, ..., m_n) = Wm_1 \dot\cup Wm_2 \dot\cup ... \dot\cup Wm_n$.

Proof : (1) Let α_i and β denote the restriction of α to m_iS and $(m_1,m_2,…,m_n)S$ respectively ,that is $\alpha_i(=\alpha|_{miS})$: $m_iS \to M_s$ and β : $(m_1, m_2, …,m_n)S \to M_s$. Let $\bar{\alpha}_i$ and $\bar{\beta}$be an extension of α_i and β respectively to M_S^m (since $M_s$is (m,1)-quasi injective system) . For each$x \in m_1S \dot\cup m_2S \dot\cup ... \dot\cup m_nS$, there exists unique j ∈I={1,2,…,n} such that x = m_js_j ,$\bar{\beta}(x) = \bar{\beta}(m_js_j) = \beta(m_j)s_j = \alpha(m_js_j) = \alpha(x)$ .This shows that $\bar{\beta}$ is an extension of α .

(2) Let $x \in Wm_1 \dot\cup Wm_2 \dot\cup ... \dot\cup Wm_n$, so x = $\alpha_i(m_i)$ [where $\alpha_i(=\alpha|_{miS})$: $m_iS \to M_s$,α∈T] .Define an S-homomorphism β:$(m_1,m_2,…,m_n)S \to M_s$ by $\beta((m_1,m_2,…,m_n)s) = \alpha_i(m_i)s = m_is$,such that s ∈ S . Now , let $(m_1,m_2,…,m_n)s = (m_1,m_2,…,m_n)t$, such that s,t ϵ S ,this implies $(m_1s,m_2s,…,m_ns) = (m_1t ,m_2t,…,m_nt)$,then $m_is = m_it$and $\alpha_i(m_i)s = \alpha_i(m_i)t$ .Thus $\beta[(m_1,m_2,…,m_n)S] =\beta[(m_1,m_2,…,m_n)t]$, and β is well-defined . Since M_s is (m,1)-quasi-injective system , so there exists an S-homomorphism $\sigma: M_S^m \to M_s$ extends β . Thus , for m_j∈ M_s such that j ∈I={1,2,…,n},this implies $m_j= \beta_j(m_j)= \beta(m_1,m_2,…,m_n) \in W(m_1, m_2, ..., m_n)$. Hence ,$Wm_1 \dot\cup Wm_2 \dot\cup ... \dot\cup Wm_n \subseteq W(m_1, m_2, ..., m_n)$. The reverse inclusion always holds .

From above proposition when m = 1 [that isM_s is PQ-injective system] , we have the following corollary :

(2-16)Corollary: LetM_s be a PQ-injective system with T= $End_s(M_s)$, and $m_1,m_2,\ldots,m_n$ denote elements of M_s ,then:

1- If $Tm_1 \oplus Tm_2 \oplus \ldots \oplus Tm_n$ is direct , then any S-homomorphism $\alpha: m_1S \dot{\cup} m_1S \dot{\cup} \ldots \dot{\cup} m_nS \rightarrow M_s$ has an extension in T .

2- If $m_1S \oplus m_2S \oplus \ldots \oplus m_nS$ is direct ,then $T(m_1, m_2, \ldots, m_n)=Tm_1 \dot{\cup} Tm_2 \dot{\cup} \ldots \dot{\cup} Tm_n$.

Recall that an S-system M_s is **finitely generatedweakly injective** if for any S-homomorphism from finitely generated right ideal of S_s into M_s can be extended to S-homomorphism from S_s into M_s [If this is the case, we write FGW-injective system]([7],P.204) .Then , its is clear that there is no relation between PQ-injective system and FGW-injective , but they are equivalent on monoid S , so corollary (2-16) will be in the following form :

(2-17)Corollary: Let S be a FGW-injective system and let $a_1,a_2,\ldots,a_n$ denote elements of S .Then:

1- If $Sa_1 \oplus Sa_2 \oplus \ldots \oplus Sa_n$ is direct , then any S-homomorphism $\alpha : a_1S \dot{\cup} a_1S \dot{\cup} \ldots \dot{\cup} a_nS \rightarrow S_s$ has an extension in S .

2- If $a_1S \oplus a_2S \oplus \ldots \oplus a_nS$ is direct ,then $S(a_1, a_2, \ldots, a_n) = Sa_1 \dot{\cup} Sa_2 \dot{\cup} \ldots \dot{\cup} Sa_n$.

(2-18)Proposition: LetM_s be (m,1)-quasi injective system with$W = Hom(M_s^m, M_s)$,and let $A,B_1,B_2 , \ldots , B_n$ be an S-subsystems of M_s . If $\oplus_{i=1}^n B_i$ is direct , then$A \cap (\oplus_{i=1}^n B_i) = \oplus_{i=1}^n (A \cap B_i)$.

Proof : Let $x \in \oplus_{i=1}^n (A \cap B_i)$, then there exists j ∈ I = { 1,2,…,n} , such that $x \in A \cap B_j$which implies $x \in A$ and $x \in B_j$ for some j ∈ I , so $x \in A \cap (\oplus_{i=1}^n B_i)$. Then,$\oplus_{i=1}^n (A \cap B_i) \subseteq A \cap (\oplus_{i=1}^n B_i)$. Conversely , let $a \in A \cap (\oplus_{i=1}^n B_i)$which implies that a ∈ A and $a \in \oplus_{i=1}^n B_i$ .so there exists j ∈ I such that $a \in B_j$ .Let $\pi_j: \oplus_{i=1}^n b_iS \rightarrow b_jS$be the projection ,then take

$\alpha(= \pi_j |_{b_jS}): b_jS \rightarrow b_jS$. Consider the diagram(3) ,where i_1,i_2 be the inclusion maps of b_iS and b_jS respectively . Since M_s is (m,1)-quasi injective system , so by (1) of proposition (2-15), α can be extended to S-homomorphism $\beta : M_s^m \rightarrow M_s$ [that is there exists β ∈ W] , so β extends π_j.

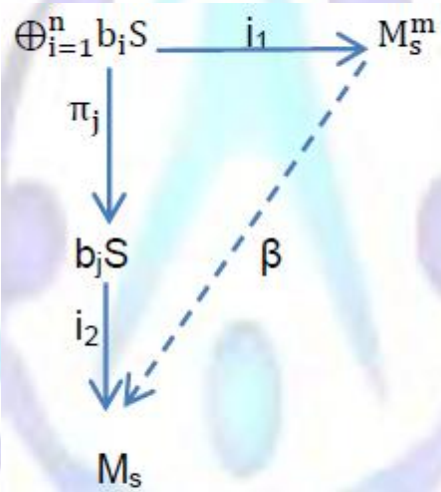

Diagram (3)

Thus for $a \in b_jS$, we have : $b_j = \pi_j(a) = \beta(a) = \alpha(a)$. Thus , $a \in \oplus_{i=1}^n (A \cap B_i)$ and $A \cap (\oplus_{i=1}^n B_i) \subseteq \oplus_{i=1}^n (A \cap B_i)$.

From above proposition when m = 1,[that isM_s is PQ-injective system] , we have the following corollary :

(2-19) Corollary : Let M_s be PQ-injective system with T= $End_s(M_s)$,and let $A,B_1,B_2 , \ldots , B_n$ be an S-subsystems of M_s . If $\oplus_{i=1}^n B_i$ is direct , then $A \cap (\oplus_{i=1}^n B_i) = \oplus_{i=1}^n (A \cap B_i)$.

Now we give equivalent condition for PQ-injective system to be quasi injective, before this we need the following concept .In [8] , define a fully stable module which modified in [10] for an S-system as follows :

(2-20) Definition[10]: A subsystem N of an S-system M_s is called stable if f(N) ⊆ N for each S-homomorphism

f :N →M_s .An S-system M_s is called fully stable if each subsystem of M_s is stable .

(2-21) Example :

(1) Every simple system is fully stable .

(2) Let the semigroup S is equal to the set of all integers with multiplication . Now, consider Z as an S-system over itself . Then Z is not fully stable , for define α :$2Z_z \rightarrow Z_z$ by α(2n) = 3n , ∀ n∈ Z . It is clear that α is S-homomorphism , but

α (2Z) ⊈ 2Z .

(3) Consider the set S^xS and let S be system on (S^xS) on the right by the componentwise multiplication, that is (x ,y)s = (xs ,ys) for x ,y ,s ∈ S . Then S×S with this action is a right S-system . Let N = {(s,0) ∈ S^xS | s ∈ S} be a subsystem of M_s [=$(S^xS)_s$] . Define α : N →M_s by α [(s,0)] = (0,s) ∀(s,0) ∈ N , where s ≠ 0 . It is clear that α is S-homomorphism ,but since α[(s,0)]=(0,s) ∉ N [in particular α [(1,0)] = (0,1) ∉ N] . Hence α (N) ⊈ N . This implies that N is not stable .But M_s[=$(S^xS)_s$] ,is PQ-injective system whence any S-homomorphism from principal subsystem ((a,0)S or (a,b)S or (0,b)S) of M_s can be trivially extended to S-endomorphism of M_s .

(4) An S-system M_s over commutative monoidis fully stable iff each principal subsystem of M_s is stable , that is for each principal subsystem N of M_s and S-homomorphism α : N → M_s , there exists an element s ∈S , such that α(n) = ns for all n ∈ N .For each s ∈ S , define λ_s: M_s→M_s by λ_s(m) = ms , m ∈ M_s . Thus in the above case α(n) = ns = λ_s • n for each n ∈N . From all above , it is clear that every fully stable system is PQ-injective , but the converse is not true in general see example (2-21)(3). Hence , we need the following concept to be the converse is true :

An S-system M_s is **multiplication** if each subsystem of M_s is of the form MI , for some right ideal I of S . This is equivalent to saying that every principal subsystem is of this form [11] . For example , Z_z is multiplication system . Every multiplication S-system is duo .

The following proposition is a special case of theorem (1.18) in [11] :

(2-22) Proposition: Let S be commutative monoidandMs be a multiplication S-system . Then M_s is fully stable iffM_s is PQ-injective .

Proof :⇒)It is clear .

⇐) Let α : mS → M_s be S-homomorphism , where m∈ M_s . Then , since M_s is PQ-injective system , so α extends to S-homomorphism β : M_s → M_s . Now, there exists an ideal I of S , such that mS = MI . Hence α(mS) = β(MI) = β (M)I ⊆ MI

= mS .

Now, since every cyclic(principal) system is multiplication [For ,if N is a subsystem of a cyclic S-system M_s =mS and

x ∈N then,x∈ M_s so x = ms where s belong to ideal of S and m belong to M_s. Hence , N = MI] .Then we have the following corollary :

(2-23) Corollary :A cyclic(principal) S-system M_s is fully stable iffM_s is PQ-injective .

Now, we give a sufficient conditions for the PQ-injective system to be quasi injective .First ,the following lemma will be useful to give a complete answer when PQ-injective is quasi injective :

(2-24) Lemma[11] : Over a monoid S , the following statement hold , a right S-system M_s is duo iff for each endomorphism f of M_s and for each element a of M_s ,f(a) = as for some s ∈ S. In particular , if S is commutative and M_s is duo right S-system , then End(M_s) is a commutative monoid .

The following proposition give an important result about PQ-injective to be quasi injective :

(2-25) Proposition: LetM_s be a multiplication S-system . If M_s is PQ-injective , then M_s is quasi injective system .

Proof : Assume that M_s is PQ-injective and multiplicationsystem . Let N be S-subsystem of M_s and f be S-homomorphism from N into M_s .Since M_s is multiplication system, so N=MIfor some right ideal I of S . Since, every multiplication is duo, so by lemma (2-24) and since M_s is PQ-injective , so for each endomorphism g of M_s and each element a of M_s , g(a) = as for some s ∈ S. Now , for each n∈ N and s∈S ,we have ns∈N (since N=MI), thus

ns = f(n) = g(n) which means that g is extension of f and M_s is quasi injective .

3- REVERSIBLE SUBSYSTEMS:

In this section,It is shown that in duo and PQ-injective system there is a one to one corresponding between maximal left ideal of the endomorphism monoid of M_s and maximal ∩- reversible subsystem of M_s .

Recall that a proper **subsystem N** of an S-system M_s is called **maximal** if for each subsystem K of M_s with

$N \subseteq K \subseteq M_s$ implies either K = N or K = M_s . At the same time ,**a non-zero subsystem N** of centered S-system M_s over semigroup with zero is called **intersection large**if∀m (≠θ) ∈ M_s , there exists s ∈ S , such ms(≠θ) ∈ N ,that ismS ∩ N ≠θ for anym (≠θ) ∈ M_s. **Asubsystem N** of an S-system M_s is called **closed** if it has no proper ∩-large in M_s that is the only solution of $N \hookrightarrow^{\cap l} L \;{}_{\neq}\!\hookrightarrow M_s$ is N = L .

(3-1)Remark: LetM_s be an S-system with T=$End_s(M_s)$. Let Nbe a non-zero subsystem of M_s ,and let P,Q be a subsystems of M_s. If N ∩-large subsystem of P and Q respectively , then N is ∩-large subsystem of $P \cup Q$.

(3-2)Lemma: Every non-zero subsystem N of centered S-system $M_s$oversemigroup with zero has maximal intersection large in M_s called closure of Nin M_s .

Proof : Let $\mathbb{C}$ = { B is proper subsystem of M_s | B ∩ N = θ } , we ordered $\mathbb{C}$ by inclusion , it is clear that $\mathbb{C} \neq \emptyset$. Let $\bar{\mathbb{C}}$ = { B_α |α ∈ I } be any chain of $\mathbb{C}$. Then clH $= \cup B_\alpha$ is an upper bound of $\bar{\mathbb{C}}$ in $\mathbb{C}$. According to Zorn's lemma , $\mathbb{C}$ has maximal element W(say) . Uniqness of W implies by its maximality of W . Thus, $\mathbb{C}$ has unique maximal element Wsay, which is called closure of N in M_s .

(3-3)Lemma: Let M_s be an S-system and suppose a non-zero subsystem N of M_s has closure P in M_s . Then P contains every ∩-large extension of N and so P is the unique closure of N in M_s .

Proof : Assume that N is ∩-large subsystem of Q in M_s .Since N is ∩-large subsystem of P in M_s , so by remark(3-1) N is ∩-large subsystem of $P \cup Q$. Since $N \subseteq P$, so it follows that P is ∩-large subsystem of $P \cup Q$. Since P is closed , so $P = P \cup Q$. Hence $Q \subseteq P$.

(3-4)Lemma: Every non-zero ∩-reversible subsystem N of centered S-system M_s has maximal ∩-reversible extension in M_s .

Proof : Let $\mathbb{C}$ = { B be subsystem of M_s |$N \subseteq B$, B is reversible } we ordered $\mathbb{C}$ by inclusion . It is clear that $\mathbb{C} \neq \emptyset$. Let

$\bar{\mathbb{C}}$ = { D_α |α ∈ I } be any chain in $\mathbb{C}$. Now , we claim $D = \cup_{\alpha \in I} D_\alpha$ is ∩-reversible subsystem of M_s . Let W be a non-zero subsystem in D , so for some α ∈ I , we have W subsystem of D_α and since D_α ∩-reversible , hence W is ∩-large subsystem of D_α and so W∩ D_α is ∩-large subsystem of D_α .It is enough to prove W∩ D_α ≠ θ,if not that is W∩D_α= θ and since W is ∩-large subsystem of D_α, so,D_α=θ and this is a contradiction .Thus ,W is ∩-large in D .Therefore D is ∩-reversible and D ∈ $\mathbb{C}$. So , by Zorn's lemma $\mathbb{C}$ has maximal element A (say) which is a maximal reversible extension of N in M_s .

(3-5) Corollary : Every ∩-reversible closed subsystem N of an S-system M_s is maximal ∩-reversible subsystem of M_s [and hence maximal ∩-reversible extensions of each of its non-zero subsystem] .

Let M_s be an S-system. Let N be ∩-reversible subsystem of M_s ,then define:A_N={α ∈$End_s(M_s)$|kerα∩(N×N)≠i_N} .An element x ∈ M_s is called **reversible** if xS is a non-zero ∩-reversible subsystem of M_s .In the following ,we list properties of A_N .

(3-6)Properties of A_N: Let N and nS be ∩- reversible subsystems of an S-system M_s. Then A_N has the following properties with $\psi_M = i_M$:

1- $A_N = A_{nS}$, ∀n (≠0) ∈ N .

2- A_N is a left ideal of T .

3- $\ell_T(nS \times nS) \subseteq A_{nS} \neq T$, ∀ reversible elements n ∈ M_s .

4- $A_N = A_P$, where P is any maximal ∩-reversible system containing N.

Proof : Since $\psi_M = i_M$, so ∩-reversible subsystem of M_s is reversible subsystem , then :

1- Let α ∈ A_N .This implies that α ∈ T and ker α ∩ (N × N) ≠ i_N . Since α|$_N$ is not one-to-one ,so α|$_{nS}$ is not one-to-one .Thus,kerα ∩ (nS × nS) ≠ i_{nS} and α ∈ A_{nS} . Hence $A_N \subseteq A_{nS}$ …(1) . Conversely , let α ∈ A_{nS} .This implies that α ∈ T and kerα ∩ (nS × nS) ≠ i_{nS}. Since n ∈ N and nS large (essential) in N, so α|$_N$ is not one-to-one ,then ker α ∩ (N × N) ≠ i_N which implies that α ∈ A_N . So , $A_{nS} \subseteq A_N$ …(2) .From (1) and (2) ,we have $A_N = A_{nS}$, ∀ n (≠0) ∈ N .

2- Let α ∈ A_N and β ∈ T . For α ∈ A_N , we have ker α ∩ (N × N) ≠ i_N, so∃ (x,y) ∈ ker α ∩(N × N) .This implies that x,y∈ N and α(x) = α(y) with x ≠y . Since β is well-defined ,so β(α(x)) = β(α(y)) .Then ,(βα)(x) =(βα)(y) with x ≠y .So

kerβα ∩ (N × N) ≠ i_N. Thus βα ∈ A_N and TA_N ⊆ A_N which implies that A_N is a left ideal of T .

3- Let $\alpha \in \ell_T(nS \times nS)$. Let s_1,s_2 ∈ S with ns_1≠ ns_2 ,then α(ns_1) = α(ns_2) with ns_1≠ ns_2, so α|$_{nS}$ is not (1-1) which implies that kerα ∩ (nS × nS) ≠ i_{nS} and α∈A_{nS} .Therefore $\ell_T(nS \times nS) \subseteq A_{nS}$.

4- Let α∈ A_N .This implies that α ∈ T and ker α ∩ (N × N) ≠ i_N . Since N is essential subsystem of P , then α|$_P$ is not (1-1) , so ker α ∩ (P× P) ≠ i_P and α ∈ A_P . Thus $A_N \subseteq A_P$ …(1). Conversely , let α ∈ A_P , so α ∈ T and ker α ∩ (P × P) ≠ i_P, then α|$_P$ is not one-to-one which implies that α|$_N$ is not one-to-one and ker α ∩ (N × N) ≠ i_N.Then α ∈ A_N . Hence ,

$A_P \subseteq A_N$ …(2) . From (1) and (2) , we have $A_N = A_P$.

(3-7)Proposition : Let M_s be a PQ-injective S-system. If n is a reversible element of M_s . Then A_{nS} is the unique maximal left ideal of T containing $\ell_T(nS \times nS)$.

Proof : A_{nS} is a left ideal of T by ((3-6)(2)) .By ((3-6)(3)) A_{nS} containing $\ell_T(nS \times nS)$. Let X be a left ideal of T which contains $\ell_T(nS \times nS)$ and X ≠T. If α ∈ X and $\alpha \notin A_{nS}$, then kerα ∩ (nS × nS) = i_{nS} , so by lemma (2-6) gives : $T = \ell_T(i_{nS}) = \ell_T[\ker\alpha \cap (nS \times nS)] = T\alpha \cup \ell_T(nS \times nS) \subseteq X$, Which implies a contradiction ,so $X \subseteq A_{nS}$ and then $X = A_{nS}$ .Thus , A_{nS} is maximal left ideal of T . Let W be another left ideal of T (≠T) which contains $\ell_T(nS \times nS)$ and A_{nS} ,that is A_{nS} is subsystem of W ,where W is a proper left ideal of T . By maximality of A_{nS} , we have A_{nS} = W . Therefore , A_{nS} is the unique maximal left ideal of T containing $\ell_T(nS \times nS)$.

The following proposition aimed at discovering first part of the one to one corresponding between maximal left ideal of the endomorphism monoid of M_s and maximal reversible subsystem of M_s :

(3-8)Proposition : Let M_s be a PQ-injective system and let P and N be fully invariant maximal ∩-reversible subsystems of M_s . Then $A_P = A_N$ iff P = N .

Proof :⇐)The necessary condition follows from ((3-6)(4)) .

⇒) Assume that $A_P = A_N$. If P ∩ N ≠ θ , then by lemma(3-2) , since P ∩ N be a non-zero subsystem of S-system M_s , there exists maximal intersection large in M_s [called closure of P ∩ N in M_s] . So , P and N be maximal ∩-reversible (closure) of P ∩ N in M_s. By lemma(3-3), the closure must be unique ,so P = N . If P ∩ N = θ .Then ,take p (≠0) ∈ P and n (≠0) ∈ N , and consider $f : nS \dot\cup pS \to M_s$defined by $\forall x \in nS \dot\cup pS,\ f(x) = \begin{cases} I_P & \text{if } x \in pS \\ 0 & \text{o.w} \end{cases}$. Since $Tp \oplus Tn$ is direct whence N and P is fully invariant . So by corollary(2-16)(1) , f extends to $\bar{f}$ in M_s .Then ,we have $\bar{f}$(p)= p and$\bar{f}$(n) =0 .Thus ,$\bar{f}_{|P}$is one-to-one which implies that ker$\bar{f}$ ∩ (P × P) = $i_P$and then$\bar{f} \notin A_P = A_N$,so$\bar{f} \notin A_N$. On the other hand $\bar{f}_{|N}$ is not one-to-one ,so

ker$\bar{f}$ ∩ (N × N) = N × N ≠ i_N.Thus,$\bar{f} \in A_N$ and this is a contradiction .

(*) An S-systemM_s satisfies condition (*) if $\gamma_M(A \times A) \neq \theta$for each maximal left ideal A of T.

The following theorem give a complete answer under which there is a one to one corresponding between maximal left ideal of endomorphism monoid of M_s and maximal reversible subsystem of M_s :

(3-9)Theorem : Let M_s be a PQ-injective and duo system such that M_s is satisfy condition (*), with $\psi_M = i_M$and every non-zero subsystem contains a reversible subsystem. Then , the map $\psi : H \to L$is bijective , where H ={ N_i | i ϵ I} denote the set of distinct maximalreversible subsystems of M_s and L ={ A | A is maximal left ideal of T and $\ell_T(nS \times nS) \subseteq A$ for each reversible element n in M_s } , (that is$\psi : N_i \longmapsto A_{Ni}$ is bijective) .

Proof : Assume that $A_{Ni} = A_{Nj}$ where i ≠ j . By proposition(3-8) ,we have N_i=N_j and then ψ is one-to-one .Now,assume that A is maximal left ideal of T . Since M_s is satisfy condition (*) , so $\gamma_M(A \times A) \neq \theta$. Since ψ_M=i_M , so reversible subsystem equivalent to ∩-reversible subsystem of M_s . By assumption ,there exists a non-zero reversible subsystem N of M_s such that $N \subseteq \gamma_M(A \times A)$. Thus , n (≠0) ∈ N is a reversible element of M_s .Then , by proposition(3-7) , A_n is the unique maximal left ideal of T containing $\ell_T(nS \times nS)$. By ((3-6)(1)) ,$A_N = A_{nS}$, where nS be a non-zero reversible subsystem of M_s. By lemma (3-4), there exists unique maximal reversible subsystem K of M_s which containing nS (and hence N) .By ((3-6)(4)) ,we have $A_{nS} = A_K$, where K is maximal reversible subsystem of Ms containing nS (and hence N) .Since the map ψ is one-to-one by the first part of the proof , so nS = K [that is N=K] which implies that nS (=N) is maximal reversible subsystem of M_s .Then ,ψ(nS) = A_{nS} (that is ψ(N) = A_N) . Let A be maximal left ideal of T. It is clear that A ⊆ A_{nS} . Let α ∈ A_{nS}which implies that α ∈T andkerα ∩(nS × nS) ≠ i_{nS} . Let V = ker α ∩ (nS × nS) ≠ i_{nS} . By lemma (2-6) , we have :

$\ell_T(V) = T\alpha \cup \ell_T(nS \times nS)$ which implies that $\alpha \in \ell_T(V)$.Then α ∈ A and A_n ⊆ A . Then A = A_n .

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