ISSN 2347-1921

The role of the martingales in the stochastic models

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Abstract:

The stochastic model is one of the most efficient models in the stock price modeling. The martingales have the important role in this the stochastic models.The martingale theory is used for calculating the probability of the bankruptcy in the stochastic model of stock market and the stochastic model of insurance risk. In this here, we will provide an introduction to the martingale from applied point of view in the stochastic differential equations.

Key words: Martingale; Dividends modeling; Stochastic differential equations; Ito integral.

Council for Innovative Research

Peer Review Research Publishing System

Journal: JOURNAL OF ADVANCES IN MATHEMATICS

Vol .10, No 1

www.cirjam.com , editorjam@gmail.com

ISSN 2347-1921

1. INTRODUCTION

Randomness is a basic type of object uncertainty and a random variable is a function of sample space to set real number. A stochastic process is set of random variable. A differential equation that contains a random component is known as a stochastic differential equation (SDE).The solution that is a random process.

Stochastic and deterministic differential equations are fundamentals for the modeling in science, engineering and mathematical finance. As the computational power increases, it becomes feasible to use more accurate differential equation models and solve more demanding problems. The model can be stochastic by two reasons: if calibration of data implies this, as in financial mathematical, or, if fundamental microscopic laws generate stochastic behavior when coarse-grained, as in molecular dynamics for chemistry, material science and biology.

Fluctuations in statistical mechanics are usually modeled by adding a stochastic term to the deterministic differential equation. By doing this one obtains what is called a SDEs, and the term stochastic called noise [1]. Then, a SDE is a differential equation in which one or more of the terms is a stochastic process, and resulting in a solution which is itself a stochastic process. Every unwanted signal that add to the information called noise. Noise in dynamical system is usually considered a nuisance. Noise have the most important role in the SDE [2].

Bachelier developed the first theory of Brownian motion and used it to model the price of a stock in time in 1900. After the work of Bachelier, Black-Scholes and Merton in 1973, proved a pricing formula for call options, under accepted assumptions[4].

The theory of martingales plays a very important and useful role in the study of stochastic processes, stochastic analysis and mathematical finance. The basic martingale theory and many of its applications developed by Levy, Ville and Doob[5].

We consider some further applications of martingale pricing to the problems in financial engineering.

For attention, in the two next section, we describe the stochastic calculus, the martingale. In part 4, the paper ends with some the application of the martingales in the stochastic models.

2. Stochastic calculus

In many physical applications one has to deal with random quantities that depend on a parameter. This phenomenon is termed Brownian motion. Each coordinate of the Brownian particle is a random variable that depends on a parameter. A stochastic process $X_t(\omega)$, is a family of random variables $\{X_t(\omega): t \in T, \omega \in \Omega\}$ depending upon the parameter t and defined on the probability space (Ω, £, P) (£, is a σ- algebra of subsets of Ω and P is probability measure defined on all elements of £)[1-3].

Stochastic calculus is a branch of mathematics that operates on stochastic processes. It allows a consistent theory of integration to be defined for integrals of stochastic processes with respect to stochastic processes. The best-known stochastic process to which stochastic calculus is applied the Wiener process.

The main part of stochastic calculus is the Ito calculus and Stratonovich. Ito calculus extends the methods of calculus to stochastic processes such as Brownian motion. We go back to the definition of an integral:

$$\int_0^T f(t)dt = \lim_{n\to\infty} \sum_{j=1}^{n} f(\tau_j)(t_{j+1} - t_j) \qquad (2.1)$$

where τ_j is in the interval [t_j, t_{j+1}]. More generally have Riemann-Stieltjes integral:

$$\int_0^T f(t)dt = \lim_{n\to\infty} \sum_{j=1}^{n} f(\tau_j)(g(t_{j+1}) - g(t_j)). \qquad (2.2)$$

For a smooth measure g(t), limit converges to a unique value regardless where τ_j, taken in interval [t_j, t_{j+1}].

The Ito and Stratonovich calculus follows the same rules as for the regular Riemann-Stieltjes calculus. If our choose is lower end point, of the partition [t_j, t_{j+1}], we have the case Ito integral, but if we choose midpoint $\frac{t_{j+1}+t_j}{2}$, we got Stratonovich case.

$t_0 = 0 \le t_1 \le t_2 \le \ldots \le t_n =$ T be a partition of the interval [0, T] and $\delta = \max(t_j - t_{j-1})$. The Ito integral $\int_0^T h(t, X_t))\, \mathrm{d}W_t$ is defined as the limit in the quadratic mean.

$$\int_0^T h(t,X_t)dW_t = \lim_{\delta n\to 0}\sum_{j=1}^{n} h(\tau_{i-1}\ X_{\tau_{i-1}})(W_{t_i} - W_{t_{i-1}}). \qquad (2.3)$$

If the integrand h is jointly measurable and

$$\int_0^T E(|h(s,X_s|^2)\ ds < \infty, \qquad (2.4)$$

The stochastic integral in (2.1) is defined as the limit in probability. The Stratonovich integral is defined by

$$\int_0^T h(t,X_t)odW_t = \lim_{\delta n\to 0}\sum_{i=1}^{n} \mathrm{h}(\tau_{i-1}, \frac{X_{\tau_{i-1}} + X_{\tau_i}}{2})(W_{t_i} - W_{t_{i-1}}), \qquad (2.5)$$

(where the symbol o, is employed).

In addition to the conditions on the existence of the Ito integral,it is required for the existence of the Stratonovich integral in (2.3) that the h(t, X_t) function be continuous in t and have continuous partial derivatives (see [1-7]). Moreover

$$\int_0^T h(t,X_t)odW_t = \int_0^T h(t,X_t)dW_t + \frac{1}{2}\int_0^T g(t,X_t)\frac{\partial h}{\partial x}(t,X_t)\mathrm{dt} \qquad (2.6)$$

or, equivalently [1]

$$h(t,X_t)odW_t = h(t,X_t)dW_t + \frac{1}{2}g(t,X_t)\frac{\partial \mathrm{h}}{\partial \mathrm{x}}(t,X_t)dt. \qquad (2.7)$$

Consider a SDE,

$$dX_t = \mathrm{f}(t,X_t)dt + g(t,X_t)\,dW_t, \qquad (2.8)$$

where f is an n-vector valued function, g is an $n \times p$ matrix valued function, W_t is an p-dimennsional Brownian motion process or Wiener process, and the solution X_t of the stochastic differential equation (2.6), is meant a process X_t for all t, in some interval [0,T].

We also assume that the distribution of X_0 is known and independent of W_t. There is an explicit several-dimensional formula which experesses the Stratonovich interpretation of (2.6)

$$dX_t = \tilde{f}(t,X_t)dt + g(t,X_t)odW_t, \qquad (2.9)$$

where:

$$\tilde{f}(t,X_t) = f(t,X_t) + \frac{1}{2}\sum_{j=1}^{p}\sum_{k=1}^{n}\frac{\partial g_{ij}}{\partial x_j}g_{kj}, \qquad (2.10)$$

(see Oksendal (2000)).

$$(I)\int_0^t W_s\,dW_s = \frac{1}{2}[\mathrm{W}_t^2 - \mathrm{W}_0^2 - t] \qquad (2.11)$$

while

$$(S)\int_0^t W_s\,odW_s = \frac{1}{2}[\mathrm{W}_t^2 - \mathrm{W}_0^2] \qquad (2.12)$$

Ito integral and Stratonovich integral have applications in the SDEs. In this paper, we consider SDE of Ito kind. A SDE is given by

$$X_t' = f(t,X_t) + g(t,X_t)\,\varepsilon_t,\ X_0 = x_0, \quad t\geq 0, \qquad (2.13)$$

where f is the deterministic part. $g\varepsilon_t$ is the stochastic part, and ε_t denotes a generalized stochastic process [1-2].

An example of generalized stochastic processes is white noise. For a generalized stochastic process, derivatives of any order can be defined. Suppose that W_t is ageneralized version of a Wiener process which is used to model the motion of stock prices.

A Wiener process is a time continuous process whit the property W_t ~N(0, t), (0≤ t≤ T), usually it is differentiable almost nowhere. White noise $\varepsilon_t = \frac{dW_t}{dt}$ [1].

If we replace $\varepsilon_t dt$ by dW_t in equation (2.13), an Ito SDE can be rewritten

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t, \qquad (2.14)$$

Where f and g are drift and diffusion term, respectively, and X_t is a solution which we try to find based on the integrating factor [2].

A martingale is the mathematical description of o fair game. The first time, in the 18th century, a martingale meant a gambling strategy.

The most important model in the financial mathematic is the model black-Scholes (1973). The standard Black-Scholes analysis does not allow for dividends payments.

3. Martingales

For a family $\{X_{1,\dots}, X_n\}$ of random variables, denote by σ($X_{1,\dots}, X_n$) the smallest σ- field containing the events of the form {ω, a<$X_k(\omega) < b$}, k=1,…,n for all a, b. σ($X_{1,\dots}, X_n$) is called the σ- field generated by $X_{1,\dots}, X_n$. Random variables determined by σ($X_{1,\dots}, X_n$) are functions of $X_{1,\dots}, X_n$.

A stochastic process X={$X_n : n \geq 0$} is called a martingale(MG) if E($|X_n|) < \infty$, $n \geq 0$ and E ($X_{n+1} \mid X_0, X_1, \dots, X_n) = X_n \quad n \geq 0$.

So,

$$E(E(X_{n+1} \mid X_0, \dots, X_n)) = E(X_n).$$

This yields E(X_{n+1}) =E (X_n) and we conclude that

$$E(X_n) = E(X_0) \qquad n \geq 0 \quad \text{for any MG.}$$

Now, let $G_n = \sigma(X_1, \dots, X_n)$ and $Z_n = X_1 + X_2 + \dots + X_n$ n=1,2,….

$$E(S_{n+1} \mid G_n) = E(S_n + X_{n+1} \mid G_n)$$

$$= E(S_n | G_n) + E(X_{n+1} \mid G_n)$$

$$= S_n + E(X_{n+1})$$

Whit suppose, E(X_i) $= 0$ for all i, then S_n ($S_0 = 0$) is a martingale.

Any integrable stochastic process $(X_t)_{t\in R^+}$ whit centered and independent is a martingale:

$$E(X_t|F_s) = E(X_t - X_s + X_s \,|\, F_s)$$
$$= E(X_t - X_s|F_s) + E(X_s|F_s)$$
$$= E(X_t - X_s) + X_s$$
$$= X_s \qquad 0 \le s \le t$$

In particular, the standard Brownian motion $(B_{t)t\in R^+}$ is a martingale because it has centered, in dependent increments.

The discounted asset price

$$X_t = X_0 \; e^{(\mu - r)t + \sigma B_t - \frac{\sigma^2 t}{2}}$$

Is a martingale when $\mu - r.$

Indeed we have

$$E\;(X_t|F_s) = E(X_0 \; e^{\sigma B_t - \frac{\sigma^2 t}{2}} |F_s)$$
$$= X_0 \; e^{-\frac{\sigma^2 t}{2}} \; E(e^{\sigma B_t} \,|\, F_s)$$
$$= X_0 \; e^{-\frac{\sigma^2 t}{2}} \; E(e^{\sigma(B_t - B_s) + \sigma B_s} \,|\, F_S)$$
$$= X_0 \, e^{\frac{-\sigma^2 t}{2} + \sigma B_s} \; E(e^{\sigma(B_t - B_S)} \,|\, F_S)$$
$$= X_0 \, e^{\frac{-\sigma^2 t}{2} + \sigma B_s} E\,(e^{6(B_t - B_s)})$$
$$= X_0 \, e^{\frac{-\sigma^2 t}{2} + \sigma B_s} \; e^{\frac{\sigma^2 (t-s)}{2}}$$
$$= X_0 \; e^{\sigma B_s - \frac{\sigma^2 s}{2}}$$
$$= X_s \qquad 0 \le s \le t$$

This fact can also be recovered from proposition:

$$E(X_t \,|\, F_s) = E(E(F \,|\, F_t) \,|\, F_s) = E(F \,|\, F_s) = X_s \qquad 0 \le s \le t$$

Since X_t satisfies the equation

$$dX_t = \sigma X_t \, dB_t$$

i.e. it can be written as the Brownian Stochastic integral

$$X_t = X_0 + \sigma \int_0^t X_u \, dB_{u,} \qquad t \in R^+.$$

4. Applications of the Martingale

4.1: The option-pricing theory of Black and Scholes(1973) is perhaps the most important development in the theory of financial economics. Then the value S_t of a portfolio is to form:

$$dS_t = (\mu + \frac{\sigma^2}{2} - r) S_t \, dt + \sigma S_t \, dW_t , \qquad (3.1)$$

where, $\{W_t\}_{t\geq 0}$ denotes the standardized Wiener process, μ and σ^2 denote respectively, the mean and variance of $\{S_t\}$ per unit time and S_t is the price of a stock at time t, t≥ 0.

We suppose that the dividends are paid continuously at the rate $\delta = \frac{\sigma^2}{2}$, and that all dividends are reinvested in the stock.

Gerber and Shlu in [1] showed that $e^{-rt}S_t$ is martingales under the measure Q defined by $\frac{dQ}{dP} = e^{-\gamma W_T - \frac{1}{2}\gamma^2 T}$ by this $\gamma = \frac{\mu + \delta - r}{\sigma}$.

4.2: Let a bevariate geometric Brownian motion of the form

$$dS_1(t) = S_1(t)\{\mu_1 \, dt + \sigma_{11} \, dW_1(t) + \sigma_{12} \, dW_2(t)\}$$

$$dS_2(t) = S_2(t)\{\mu_2 dt + \sigma_{21} dW_1(t) + \sigma_{22} dW_2(t)\}$$

where $W_1(t)$ and $W_2(t)$ are independent Brownian motions, and μ_i and σ_{ij}, $i, j = 1, 2.$

$e^{-rt} \, S_1(t)$ and $e^{-rt} \, S_2(t)$ are both martingales under measure Q defined by this

$$\gamma_t = \frac{\mu_{1t} - r_t}{\sigma_{1t}} = \frac{\mu_{2t} - r_t}{\sigma_{2t}}.$$

4.3: A discrete time risk model for the surplus U_n of an insurance company at the end of year *n, n=1, 2, …* is given by

$$U_n = U_0 + cn - \sum_{k=1}^{n} X_k ,$$

where, *c* is the total annual premium, X_k is the total (aggregate) claim in year *k*.

The time of ruin *T* is the first time when the surplus becomes negative, *T*=min $\{n : U_n \prec 0\}$, with *T*=∞ if $U_n \geq 0$ for all *n*.

Assume that $\{X_k , \, k = 1, 2, \cdots\}$ are independent identity distribution (i.i.d) random variables, and there exists a constant $R \succ 0$ such that

$$E(e^{-R(c-X_1)}) = 1.$$

We know that $M_n = e^{-RU_n}$ is a martingale. So, for all *n*, $P_x(T \leq n) \leq e^{-Rx}$, where $U_0 = x$ the initial funds, and the ruin probability

$$P_x(T \prec \infty) \leq e^{-Rx}.$$

5. Conclusion

The first, we defined the stochastic calculus and the martingale. Then we described some applications of martingales in the financial engineering. We saw, as a result, how the problems of financial mathematical and martingales are very closely related.

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