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Properties of Derivations on KU-ALGEBRAS

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Abstract :

In this paper the notion of (ℓ,r) (or (r,ℓ)) -derivations and t-derivation of a KU-algebra are introduced, and some related properties are investigated. Also, we consider regular derivations and the *D*-invariant on ideals of *KU*-algebras .We also characterized $Ker\,D$ by derivations.

Keywords. *KU*-algebras , (ℓ,r) (or (r,ℓ)) - derivations of *KU*-algebras ; t-derivation of a KU- algebras ; *D*-invariant on ideals of *KU*-algebras.

Subject Classification

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1. Introduction.

As it is well known, BCK and BCI-algebras are two classes of algebras of logic. They were introduced by Imai and Iseki [9,10,11] and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper sub class of the BCI-algebras.The class of all *BCK*-algebras is a quasivariety. Is´eki posed an interesting problem (solved by Wro´nski [19]) whether the class of *BCK*-algebras is a variety. In connection with this problem, Komori [14] introduced a notion of *BCC*-algebras, and Dudek [6] redefined the notion of *BCC*-algebras by using a dual form of the ordinary definition in the sense of Komori. Dudek and Zhang [7] introduced a new notion of ideals in *BCC*-algebras and described connections between such ideals and congruences . C.Prabpayak and U.Leerawat ([17], [18]) introduced a new algebraic structure which is called KU - algebra . They gave the concept of homomorphisms of KU- algebras and investigated some related properties. Several authors [2,3 ,4 ,5,8,13] have studied derivations in rings and near rings. Jun and Xin [12] applied the notion of derivations in ring and near-ring theory to *BCI*-algebras, and they also introduced a new concept called a regular derivation in *BCI* -algebras. They investigated some of its properties, defined a *d* -derivation ideal and gave conditions for an ideal to be *d*-derivation. Later, Hamza and Al-Shehri [1], defined a left derivation in *BCI*-algebras and investigated a regular left derivation. Zhan and Liu [20] studied f-derivations in BCI-algebras and proved some results. G. Muhiuddin and Al-roqi [15 , 16] introduced the notion of (α,β)-derivation in a BCI-algebra and investigated related properties. They provided a condition for a (α,β)- derivation to be regular. They also introduced the concepts of a $d_{(\alpha,\beta)}$- invariant (α,β) -derivation and α-ideal, and then they investigated their relations. Furthermore, they obtained some results on regular (α,β) - derivations. Moreover, they studied the notion of *t*-derivations on BCI-algebras and obtained some of its related properties. Further, they characterized the notion of *p*-semisimple BCI-algebra *X* by using the notion of *t*-derivation. In this paper we introduce the notions of (ℓ,r) or((r,ℓ)) -derivation and t-derivation of a KU-algebra and some related properties are explored.

2. *Preliminaries.*

In this section, we recall some basic definitions and results that are needed for our work.

Definition 2.1. [17]. Let X be a set with a binary operation $*$ and a constant 0.

($X,*,0$) is called KU-algebra if the following axioms hold : $\forall x,y,z\in X$:

$KU_1)$ $(x*y)*[(y*z)*(x*z)]=0$

$KU_2)$ $x*0=0$

$KU_3)$ $0*x=x$

$KU_4)$ $if\ x*y=0=y*x\ implies\ x=y$

Define a binary relation $\leq$ by : $x\leq y \Leftrightarrow y*x=0$, we can prove that $(X,\leq)$ $is\, a\ partially ordered set$.

By the binary relation$\leq$, we can write the previous axioms in another form as follows:

$(K\grave{U}_1)\,(y*z)*(x*z)\leq x*y$

$(K\grave{U}_2)\,0\leq x$

$(K\grave{U}_3)\,x\leq y\Leftrightarrow y*x=0$

$(K\grave{U}_4)\,if\ x\leq y\, and\ y\leq x\Rightarrow x=y$

Example 2.2. Let X = $\{0,1,2,3,4\}$ be a set in which the operation $*$ is defined as follows.:

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	3
3	0	0	2	0	2
4	0	0	0	0	0

Using the algorithms in Appendix A, we can prove that (X, *, 0) is a KU-algebra

Corollary 2.3 [18]. In KU-algebra the following identities are true for all $x, y, z \in X$:

(i) $z * z = 0$

(ii) $z * (x * z) = 0$

(iii) If $x \leq y$ $impliesthat\, y * z \leq x * z$

(v) $z * (y * x) = y * (z * x)$

(vi) $y * [(y * x) * x] = 0$

Definition 2.4 [17]. A subset S of KU-algebra X is called sub algebra of X if $x * y \in S$, whenever $x, y \in S$

Definition 2.5 [17,18]. Anon empty subset A of KU-algebra X is called ideal of X if it is satisfied the following conditions:

(i) $0 \in A$

(ii) $y * z \in A \ , y \in A \quad implies \quad z \in A \qquad \forall y, z \in X$

Definition 2.6 . For elements x and y of KU-algebra ($X, *, 0$), we denote $x \wedge y = (x * y) * y$.

Proposition .2.7. Let ($X, *, 0$) be a KU-algebra then the following identities are true for all $x, y, z \in X$:

(i) $(x * y) * (x * z) \leq y * z$

(ii) $If\ x \leq y\, then\, z * x \leq z * y$

(iii) $z * (x * y) \leq (z * x) * (z * y)$

(v) $x \wedge y \leq x, y$

Proof. Since $\overbrace{(y * z) * [(x * y) * (x * z)] = (x * y) * [(y * z) * (x * z)] = 0}^{From\ corollary\ 2.3(v)\ \ and\ \ definition 2.1(KU_1)}$.

Then $(x * y) * (x * z) \leq y * z$.

(ii) Since $\overbrace{(z * y) * (z * x) \leq y * x}^{\text{from proposition 2.7(i)}}$, then we have $\overbrace{(z * y) * (z * x) \leq 0}^{x \leq y \Leftrightarrow y * x = 0}$

But $\overbrace{0 \le (z*y)*(z*x)}^{\text{(from definition 2.1}KU^{l}_2}$, hence $\overbrace{(z*y)*(z*x)=0}^{\text{from definition 2.1}KU^{l}_4}$, and therefore $z*x \le z*y$.

(iii) Since $[(z*x)*(z*y)]*[z*(x*y)]=$

$$=[(z*x)*(z*y)]*[\overbrace{x*(z*y)]}^{\text{from corollary 2.3(v)}} \le \overbrace{x*(z*x)}^{\text{from definition 2.(}KU^{l}_1\text{)) and corollary 2.3(ii))}} = 0 .$$

Then $[(z*x)*(z*y)]*[z*(x*y)] \le 0$ .We have $0 \le \overbrace{[(z*x)*(z*y)]*[z*(x*y)]}^{\text{from definition 2.1(}KU^{l}_2\text{)}}$, hence $[(z*x)*(z*y)]*[z*(x*y)]$=0. Therefore $z*(x*y) \le (z*x)*(z*y)$.

(v) Since $x*[(x*y)*y]=(x*y)*(x*y)$ (from corollary 2.3 (v))

=0 . Then $(x*y)*y \le x$. i.e $x \wedge y \le x$

Since $y*[(x*y)*y]=(x*y)*(y*y)$ (from corollary 2.3 (v))

= $(x*y)*0=0$. **Then** $x \wedge y \le y$.

Proposition 2.8 . Let A be an ideal of KU-algebra X . Then A is sub algebra of X

Proof . Let $x, y \in A$ and $y*(x*y) \in A$. Since A is ideal of X and $y \in A$,then $x*y \in A$ .Therefore A is KU-sub algebra of X .

3. The derivations on KU-algebra.

Throughout this article, *X* will denote a KU-algebra unless otherwise mentioned.

Definition 3.1. Let X be a KU-algebra. A map $d: X \to X$ is a left –right derivation (briefly, (ℓ, r) -derivation) of X if it satisfies the identity

$$d(x*y)=(d(x)*y) \wedge (x*d(y)) \ \forall x, y \in X$$

If d satisfies the identity

$$d(x*y)=(x*d(y)) \wedge (d(x)*y) \ \forall x, y \in X$$

then d is a right-left derivation (briefly, (r, ℓ) -derivation) of X . Moreover, if d is both (ℓ, r) *and* $(r, \ell)-$*derivation then* d is called a derivation of X .

Definition 3.2. A derivation of KU-algebra is said to be regular if $d(0)=0$.

Lemma 3.3. A derivation d of KU-algebra X is regular.

Proof. If d is (ℓ, r) - derivation of X , $d(0)=d(x*0)=(d(x)*0) \wedge (x*d(0))$

$$=0 \wedge (x*d(0)) \quad \text{(from definition 2.1 (}KU_2\text{))}$$

$$=[0*(x*d(0))]*(x*d(0))$$

$$= \overbrace{(x*d(0))*(x*d(0))}^{\text{from definition 2.1(KU}_3\text{),from corollary 2.3(i).}} = 0 .$$

If d is (r, ℓ) -derivation of X , $d(0)=(x*d(0)) \wedge (d(x)*0)$

$$=(x*d(0)) \wedge 0 \quad \text{(Definition 2.1 (}KU_2\text{))}$$

$$=[(x*d(0))*0]*0=0*0=0 .$$

Example 3.4. Let $X = \{0,1,2,3,4\}$ be a set in which the operation $*$ is defined as follows:

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	2	4
2	0	0	0	1	4
3	0	0	0	0	4
4	0	1	1	1	0

Using the algorithms in Appendix A, we can prove that (X, *, 0) is a KU-algebra. Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 0 & if\ x = 0,1,2,3 \\ 4 & if\ x = 4 \end{cases}$$

Then it is easy to show that d is both a (ℓ, r) and (r, ℓ)-derivation of X.

Proposition 3.5. Let d be a self map of KU-algebra X, then

(I) if d is (ℓ, r)-derivation of X, then $d(x) = x \wedge d(x)\ \forall x \in X$.

(II) If d is (r, ℓ)-derivation of X, then $d(x) = d(x) \wedge x\ \forall x \in X$.

Proof.

(I) Let d is (ℓ, r) - derivation of X then,

$$d(x) = d(0 * x) = (d(0) * x) \wedge (0 * d(x)) = (0 * x) \wedge d(x) = x \wedge d(x)$$

(II) If d is (r, ℓ)-derivation of X then ,

$$d(x) = d(0 * x) = (0 * d(x)) \wedge (d(0) * x) = d(x) \wedge (0 * x) = d(x) \wedge x.$$

Proposition 3.6. Let X be a KU-algebra with partial order $\leq$, and let d be a derivation of X. Then the following hold $\forall\, x, y \in X$:

(i) $d(x) \leq x$.

(ii) $d(x * y) \leq d(x) * y$.

(iii) $d(x * y) \leq x * d(y)$.

(v) $d(x * d(x)) = 0$.

(vi) $d^{-1}(0) = \{x \in X \mid d(x) = 0\}$ is a sub algebra of X.

Proof. (i)Since $\overbrace{x * d(x) = x * ((d(x) * x) * x)}^{\text{Proposition 3.5(II), corollary 2.3(ii)}} = 0$, we have $d(x) \leq x$.

(ii) Since $\overbrace{d(x) * y) * d(x * y)} = \overbrace{d(x) * y) * [(d(x) * y) \wedge (x * d(y))]}^{(\ell, r)-\text{derivation}} = 0$. Similarly, if $d\ is (r, \ell)$ derivation on X, then $d(x * y) \leq d(x) * y$

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(iii) If $d \; is (r,\ell)(or(\ell,r))$ derivation on X we can prove $(x*d(y))*d(x*y)=0$ and hence $d(x*y)\le x*d(y)$.

(v) If d is (ℓ,r) -derivation of X, then $d(x*d(x))=(d(x)*d(x))\wedge(x*d(d(x))$

$= 0\wedge(x*d(d(x))$ =0 (Corollary 2.3 (i))

If d is (r,ℓ) -derivation of X ,then $d(x*d(x))=(x*d(d(x))\wedge(d(x)*d(x))$

$=(x*d(d(x)))\wedge 0$ =0 .

Therefore $d(x*d(x))=0$

(vi) Since d is regular, we have $d^{-1}(0)\ne\Phi$ .Let $x,y\in d^{-1}(0),\ then\ \ d(x)=d(y)=0$,

$d(x*y)=(x*d(y))\wedge(d(x)*y)=(x*0)\wedge(0*y)=0\wedge y=(0*y)*y=0.$ We get $x*y\in d^{-1}(0)$.

Hence $d^{-1}(0)$ is KU-sub algebra of X .

Definition 3.7 . Let d a derivation of KU-algebra X **.** An ideal A of X is said to be d-invariant if $d(A)\subseteq A$ **,** where $d(A)=\{d(x)\ \ ,x\in A\}$**.**

Proposition 3.8. Let X be a KU-algebra. Then

(i) If $x\le y$ $,\ then$ $d(x)\le y$

(ii) If $y\le x$ $,\ then$ $d((y*z)*(x*z))=0$

(iii) If I is an ideal of X , then every ideal I of X is d-invariant .

i.e $d(I)\subseteq I$ where $d(I)=\{\ d(x)\ ,x\in I\ \}$.

Proof. (i) Let $x\le y$, then (from corollary 2.3(iii)) we have $y*d(x)\le x*d(x)$.

Since $\overbrace{0\le y*d(x)}^{\text{from definition 2.1}}$, $\overbrace{y*d(x)\le 0}^{\text{Proposition 3.6(i)}}$, we have $\overbrace{y*d(x)=0}^{KU_4}$. Hence $d(x)\le y$.

(ii) Since $\overbrace{(y*z)*(x*z)\le x*y}^{\text{from definition 2.1}}$, we have $\overbrace{d((y*z)*(x*z))\le x*y}^{\text{from Proposition 3.8(i)}}$, hence

$\overbrace{d((y*z)*(x*z))\le 0}^{y\le x}$, but $\overbrace{0\le d((y*z)*(x*z)}^{\text{from definition 2.1 }(KU_2^{\backslash})}$. We have $d((y*z)*(x*z))=0$

(iii) Let $y\in d(I)$ such that $y=d(x)$ for some $x\in I$. Since I is an ideal of X , $d(x)\le x$, $x*d(x)=0\in I\ \ and\, x\in I$. Then $y\in I$, which implies that $d(I)\subseteq I$. Therefore the ideal I is d-invariant.

Remark3.9. Let X be a KU-algebra ,then in general $x*(y*z)\ne(x*y)*(x*z)\ \ \forall x,y,z\in X$.

Proof: Since in example 3.4 if $x=4,y=1,z=3$.Then, $x*(y*z)\ne(x*y)*(x*z)$.

Lemma3.10. let X be a KU-algebra and let if d is (r,ℓ) -derivation of X .

Then $d(x*y)\le d(x)*d(y)$.

Proof. Since $(d(x)*d(y))*d(x*y)=(d(x)*d(y))*[(x*d(y))\wedge(d(x)*y)]$

$=(d(x)*d(y))*[((x*d(y)*(d(x)*y))*(d(x)*y)]$

$=[(x*d(y))*(d(x)*y)]*[(d(x)*d(y))*(d(x)*y)]$ Corollary 2.3 (v)

$\le(d(x)*d(y))*(x*d(y))\le x*d(x)=0$ (($KU^{l}1$),Proposition 3.6 (i)).

But $0\le(d(x)*d(y))*d(x*y)$, then $(d(x)*d(y))*d(x*y)=0$, which implies that $d(x*y)\le d(x)*d(y)$.

Theorem 3.11. Let X be a KU-algebra and d be a derivation of X . if $y\in\ker(d)$ and $x\in X$, then $x\wedge y\in\ker(d)$

Proof . Since $d(x\wedge y)=d((x*y)*y)=(d(x*y)*y)\wedge((x*y)*d(y))$

$=(d(x*y)*y)\wedge((x*y)*0) = 0$, then $x\wedge y\in\ker(d)$.

Definition 3.12 . Let X be a KU-algebra and d be a derivation of X .

Denote $Fix_d(X)=\{x\in X: d(x)=x\}$.

Proposition 3.13. Let X be a KU-algebra and d be a derivation of X .Then $Fix_d(X)$ is a sub algebra of X .

Proof . Let $x,y\in Fix_d(X)$,we get $d(x)=x, d(y)=y$ **,** and so

$d(x*y)=(d(x)*y)\wedge(x*d(y))=(x*y)\wedge(x*y)=(x*y)$. Hence $x*y\in Fix_d(X)$.

Proposition 3.14. Let X be a KU-algebra and d be a derivation of X .

If $x,y\in Fix_d(X)$, then $x\wedge y\in Fix_d(X).$

Proof. Let $x,y\in Fix_d(X)$, we get $d(x)=x, d(y)=y$. From Proposition 3.14 we have $d(x*y)=x*y$ and hence $d(x\wedge y)=d((x*y)*y)=(d(x*y)*y)\wedge((x*y)*d(y))$

$=((x*y)*y)\wedge((x*y)*y)$

$=(x*y)*y=x\wedge y$. Therefore $x\wedge y\in Fix_d(X).$

Proposition 3.15. Let X be a KU-algebra. Then $d_n(d_{n-1}(...(d_2(d_1(x)))....))\le x\ \forall n\in N,$ where $d_1, d_2, ...d_n$ are derivations of X .

Proof. For $n=1$. $d_1(x)=d_1(0*x)=(d(0)*x)\wedge(0*d(x))=x\wedge d(x)\le x$.

Let $n\in N$ and assume that $d_n(d_{n-1}(...(d_2(d_1(x)))....))\le x.$ For simplicity,

Let $D_n=d_n(d_{n-1}(...(d_2(d_1(x)))....))$.. Then $d_{n+1}(D_n)=d_{n+1}(0*D_n)$

$=(d_{n+1}(0)*D_n)\wedge(0*d_{n+1}(D_n))$

$=D_n\wedge d_{n+1}(D_n)\le D_n\le x$.

4. t-Derivations ON KU-ALGEBRAS.

The following definitions introduces the notion of t -derivation of KU-algebra X .

Definition 4.1. Let X be a KU-algebra. Then for any $t \in X$, we define a self map $d_t : X \to X$ by $d_t(x) = t * x \quad \forall x \in X$.

Definition 4.2. Let X be a KU-algebra. Then for any $t \in X$, a self map $d_t : X \to X$ is called a $t-(r,l)$ (ℓ, r) -derivation of X if it satisfies the condition $d_t(x * y) = (d_t(x) * y) \wedge (x * d_t(y)) \quad \forall x, y \in X$

Definition 4.3. Let X be a KU-algebra. Then for any $t \in X$, a self map $d_t : X \to X$ is called a t- (r, ℓ) -derivation of X if it satisfies the condition

$$d_t(x * y) = (x * d_t(y)) \wedge (d_t(x) * y) \quad \forall x, y \in X .$$

Definition 4.4. Let X be a KU-algebra. Then for any $t \in X$, a self map $d_t : X \to X$ is called a t -derivation on X if d_t is both a $t-(\ell, r)$ or $t-(r,l)$ derivation on X .

Example 4.5. Consider the KU-algebra ($X, *, 0$) in example (3.4). Define the mapping d_t as follows: When $t = 0, d_t(x) = x \quad \forall x \in X.$

When $t = 1, d_t(0) = d_t(1) = 0, d_t(2) = d_t(3) = 2, d_t(4) = 4.$

When $t = 2, d_((0) = d_t(1) = d_t(2) = 0, d_t(3) = 1, d_t(4) = 4$.

When $t = 3, d_t(0) = d_t(1) = d_t(2) = d_t(3) = 0, d_t(4) = 4.$

When $t = 4$, $d_t(0) = d_t(4) = 0, d_t(1) = d_t(2) = d_t(3) = 1.$

$\forall t \in X,\ d_t$ is a t -derivation of X .

Remark 4.6. In a KU- algebra , $x \wedge y = (x * y) * y \quad \forall x, y \in X$. By using Corollary 2.7(vi) , if d_t is a $t-(\ell, r)$ derivation of X , then $d_t(x * y) \le d_t(x) * y.$

Remark 4.7. We observe that by using Corollary 2.7(vi) , if d_t is a $t-(r, \ell)$ derivation of X , then $d_t(x * y) \le x * d_t(y).$

Definition 4.8. A self map d_t on KU- algebra X is said to be $t-$ regular if $d_t(0) = 0.$

Corollary 4.9. $t-$ Derivation on KU-algebras is $t-$ regular .

Proof. If d_t is a $t-(\ell, r)$ derivation on X . Then

$$d_t(0) = d_t(x * 0) = (d_t(x) * 0) \wedge (x * d_t(0)) = 0 \wedge (x * d_t(0)) = 0.$$

Similarly, if d_t is a $t-(r, \ell)$ derivation on X , then $d_t(0) = 0$.

Proposition 4.10.

(I) If d_t is $t-(r, \ell)$ derivation of X . Then $d_t(x) = d_t(x) \wedge x \quad \forall x \in X.$

(II) If d_t is $t-(\ell, r)$ derivation of X . Then $d_t(x) = x \wedge d_t(x) \quad \forall x \in X.$

Proof.

(I) Let d_t is $t-(r, \ell)$ derivation on X . Then

$$d_t(x) = d_t(0*x) = (0*d_t(x)) \wedge (d_t(0)*x)$$

$= d_t(x) \wedge (0*x)$ as d_t is t-regular

$= d_t(x) \wedge x$.

(II) Let d_t is $t-(\ell, r)$ derivation on X . Then it is easily to prove that $d_t(x) = x \wedge d_t(x) \quad \forall x \in X.$

Proposition 4.11. Let d_t is $t-$ derivation of KU-algebra X . Then the following holds:

If $x \le y$ implies $d_t(x) \le d_t(x) \quad \forall x, y \in X.$ Then we call d_t an *isotone* derivation

Proof. Since $x \le y$,(from corollary 2.7(ii)), we have $t*x \le t*y$, therefore

$d_t(x) \le d_t(y).$

Definition 4.12. Let X be a KU-algebra and $a \in X$. Define a function on X by

$d_a(x) = a \wedge x \ \ \forall x \in X.$ Then we can see that d_a is a derivation on X . We refer to such derivations as principle .

Proposition 4.13. Every principle derivation of X is an *isotone* derivation of X .

Proof. Let d_a be a principle derivation of X . For any $x, y \in X$ and $x \le y$ we have $a \wedge x \le a \wedge y$, hence $d_a(x) \le d_a(y)$.

Theorem 4.14. Let X be KU-algebra and d_t be a t-derivation on X . If $x \le y$, $d_t(x*y) = d_t(x)*d_t(y)$ .Then $d_t(x) = d_t(y) \wedge d_t(x) \quad \forall x, y \in X$.

Proof.

$$d_t(x) = d_t(0*x) = \overbrace{d_t((y*x)*x)}^{x \le y \Leftrightarrow y*x=0} = d_t(y*x)*d_t(x) = (d_t(y)*d_t(x))*d_t(x) = d_t(y) \wedge d_t(x).$$

Lemma 4.15. Let X be a KU-algebra , if d_t is $t-(r,\ell)$ derivation of X ,

then $d_t(x*y) \le d_t(x)*d_t(y)$.

Proof. Since $(d_t(x)*d_t(y))*d_t(x*y) = (d_t(x)*d_t(y))*[(x*d_t(y)) \wedge (d_t(x)*y)]$

$$= (d_t(x)*d_t(y))*[((x*d_t(y))*(d_t(x)*y))*(d_t(x)*y)]$$

$$= \overbrace{[(x*d_t(y))*(d_t(x)*y)]*[(d_t(x)*d_t(y))*(d_t(x)*y)]}^{\text{from corollary 2.3(v)}} \le \overbrace{(d_t(x)*d_t(y))*(x*d_t(y)}^{\text{From definition 2.1}(KU^{1})} \le x*d_t(x) = 0.$$

So we have $(d_t(x)*d_t(y))*d_t(x*y) = 0.$ Therefore $d_t(x*y) \le d_t(x)*d_t(y)$.

Theorem 4.16. Let X be KU-algebra and d_t be a t-derivation on X . Then the following hold :

(I) $d_t(x) \le x.$

(II) $d_t(x*y) \le x*d_t(y)$.

(III) $d_t(x*y) \le d_t(x)*y.$

(IV) $\ker(d_t)=\{x\in X : d_t(x)=0\}$ is a sub algebra of X .

Proof.

(I) Since $x*d_t(x)=x*[x\wedge d_t(x)]$ from proposition 4.10 (II)

$$=x*[(x*d_t(x))*d_t(x)]=\overbrace{(x*d_t(x))*(x*d_t(x))}^{\text{from Corollary 2.3(i)}}=0.$$

Therefore $d_t(x)\leq x.$

(II) If d_t is a $t-(\ell,r)$ derivation on X . Then

$$(x*d_t(y))*d_t(x*y)=(x*d_t(y))*[(d_t(x)*y)\wedge(x*d_t(y))]$$

$$=(x*d_t(y))*\overbrace{[((d_t(x)*y)*(x*d_t(y)))*(x*d_t(y)]}^{\text{from corollary 2.3(ii).}}=0.$$

Therefore $d_t(x*y)\leq x*d_t(y)$.

If d_t is a $t-(r,\ell)$ derivation on X .Then it is easily to prove that

$(x*d_t(y))*d_t(x*y)=0$. Then $d_t(x*y)\leq x*d_t(y)$.

(III) If d_t is a $t-(r,\ell)$ derivation on X ,we have

$$(d_t(x)*y)*d(x*y)=(d_t(x)*y)*[(x*d_t(y))\wedge(d_t(x)*y)]$$

$$=\overbrace{(d_t(x)*y)*[((x*d_t(y))*(d_t(x)*y))*(d_t(x)*y)]}^{\text{from corollary 2.3(ii)}}=0,$$

then $d_t(x*y)\leq d_t(x)*y.$

If d_t is a $t-(\ell,r)$ derivation on X .Then it is easily to prove that $(d_t(x)*y)*d(x*y)=0$ and hence $d_t(x*y)\leq d_t(x)*y.$

(IV) Since d_t is $t-$regular, $d_t(0)=0$ and $0\in Ker(d_t)$, which implies that $Ker(d_t)$ is a non-empty set. Let $x,y\in Ker(d_t)$, we have $d_t(x)=d_t(y)=0.$ Now

$$d_t(x*y)=(d_t(x)*y)\wedge(x*d_t(y))=(0*y)\wedge(x*0)=0.$$

Then $d_t(x*y)=0$ and therefore $(x*y)\in Ker(d_t)$, which implies that $Ker(d_t)$ is a sub-algebra of X .

Definition 4.17. Let X be a KU-algebra and let d_t, $d_{t\backslash}$ be two self maps of X . Then we define $d_t\circ d_{t\backslash}$: $X\to X$ by $(d_t\circ d_{t\backslash})(x)=d_t(d_{t\backslash}(x))\quad\forall x\in X.$

Proposition 4.18. Let X be a KU-algebra and let d_t, $d_{t\backslash}$ be a $t-(\ell,r)$ derivations of X . Then $(d_t\circ d_{t\backslash})$ is a $t-(\ell,r)$ derivation of X .

Proof. Let X be a KU-algebra and let d_t, $d_{t\backslash}$ be a $t-(\ell,r)$ derivations of X . Then $\forall x,y\in X$. We have

$(d_t\circ d_{t\backslash})(x*y)=d_t(d_{t\backslash}(x*y))$

$$\leq d_t(d_{t\backslash}(x) * y) \quad (d_{t\backslash} \text{ is a } t-(\ell,r) \text{ derivation of } X)$$

$$\leq (d_t(d_{t\backslash}(x))) * y \quad (d_{t\backslash} \text{ is a } t-(\ell,r) \text{ derivation of } X)$$

$$= (d_t \circ d_{t\backslash})(x) * y$$

Then $d_t \circ d_{t\backslash}(x * y) \leq (d_t \circ d_{t\backslash})(x) * y$, therefore $(d_t \circ d_{t\backslash})$ is a $t-(\ell,r)$ derivation of X .

Proposition 4.19. Let X be a KU-algebra and let d_t, $d_{t\backslash}$ be a $t-(r,\ell)$ derivation of X . Then $(d_t \circ d_{t\backslash})$ is also a $t-(r,\ell)$ derivation of X .

Proof. Let X be a KU-algebra and let d_t, $d_{t\backslash}$ be a $t-(r,\ell)$ derivation of X . Then $\forall x, y \in X$ We have $(d_t \circ d_{t\backslash})(x * y) = d_t(d_{t\backslash}(x * y))$

$$\leq d_t(x * (d_{t\backslash}(y)) \quad (\text{since } d_{t\backslash} \text{ is a } t-(r,\ell) \text{ derivation of } X)$$

$$\leq (x * (d_t(d_{t\backslash}(y)) \quad (\text{since } d_{t\backslash} \text{ is a } t-(r,\ell) \text{ derivation of } X)$$

$$= x * (d_t \circ d_{t\backslash})(y)$$

Then $d_t \circ d_{t\backslash}(x * y) \leq x * (d_t \circ d_{t\backslash})(y)$. Therefor $(d_t \circ d_{t\backslash})$ is a $t-(r,\ell)$ derivation of X .

Theorem 4.20. Let X be a KU-algebra and let d_t, $d_{t\backslash}$ be $t-$derivation of X . Then $(d_t \circ d_{t\backslash})$ is also $t-$derivation of X .

Proof. Clear

Lemma 4.21. Let X be a KU-algebra and let d_t be $t-$derivation of X . Then

$$d(x) * d_t(x) = d_t(d(x) * x).$$

Proof . Since $d(x) * d_t(x) = d(x) * ((t * x)$

$$= \overbrace{t * (d(x) * x)}^{\text{from corollary 2.3(v)}} = d_t(d(x) * x).$$

Lemma 4.22. Let X be a KU-algebra . Then $t * d(x) \leq d_t(x)$

Proof . from proposition 3.6(i) , $d(x) \leq x$ and by using proposition 2.7(ii), we get

$$t * d(x) \leq t * x$$

that is $\quad t * d(x) \leq d_t(x).$

Conclusion.

Derivation is a very interesting and important area of research in the theory of algebraic structures in mathematics. In the present paper, The notion of (ℓ,r) (or (r,ℓ)) -derivations and t-derivation of a KU-algebra are introduced and investigated the useful properties of these types derivations in KU-algebras.

In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as BCH-algebra -Hilbert algebra -BF-algebra -J-algebra -WS-algebra -CI-algebra- SU-algebra -BCL-algebra -BP-algebra -Coxeter algebra -BO-algebra and so forth.

The main purpose of our future work is to investigate the fuzzy derivations ideals in KU-algebras, which may have a lot of applications in different branches of theoretical physics and computer science.

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Appendix A.

Algorithm for KU-algebras

Input (X : set, $*$: binary operation)

Output (" X is a KU-algebra or not")

Begin

If $X = \phi$ then go to (1.);

EndIf

If $0 \notin X$ then go to (1.);

EndIf

Stop: =false;

$i := 1$;

While $i \le |X|$ and not (Stop) do

If $x_i * x_i \ne 0$ then

Stop: = true;

EndIf

$j := 1$

While $j \le |X|$ and not (Stop) do

If $((y_j * x_i) * x_i) \ne 0$ then

Stop: = true;

EndIf

EndIf

$k := 1$

While $k \le |X|$ and not (Stop) do

If $(x_i * y_j) * ((y_j * z_k) * (x_i * z_k)) \ne 0$ then

Stop: = true;

EndIf

EndIf While

EndIf While

EndIf While

If Stop then

(1.) Output (" X is not a KU-algebra")

Else

Output (" X is a KU-algebra")

EndIf

End

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