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Comparison of Variational and Fractional Variational Iteration Methods in Solving Time-Fractional Differential Equations

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Abstract

This paper concerned with study of vartional iteration method(VIM) and fractional variational iteration method (FVIM) to solve time fractional differential equations (FDE's) . So FVIM is more effective than VIM in solving the FDE˞s.

Keywords: PDE˞s; FDE˞s; fractional calculus; Riemann-Liouville; Lagrange multiplier.

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1. Introduction

The class of fraction calculus is one of the most convenient classes of (FDEs) which viewed as generalized differential equation [1,2] . In the sense that, much of the theory and , hence , applications of differential equation can be extended smoothly to (FDEs) with the same flavor and spirit of the realm of differential equation. (FDEs) have been proved to be a valuable tool in modeling many phenomena in the fields of physics, chemistry, engineering, aerodynamics, electrodynamics of complex medium, polymer rheology, and so forth [17-20,7,8,22,26,28] . The (VIM), which was first proposed by He et al. [9-14] and has been shown to be very efficient for handling a wide class of physical problems. As early as 1998, the variational iteration method was shown to be an effective tool for factional calculus [15]; hereafter, the method has been routinely used to solve various (FDEs). [23,16,3-5,21,29] for many years ,the (VIM and FVIM) is relatively new and effective approaches to find the approximate solution of PDEs, because they provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear (PDEs) without linearization or discretization.

2. Preliminaries and Notations

In this section,we describe some necessary definitions and mathematical preliminaries of the fractional calculus theory.

Definition 1, [25]: A real function $h(t)$, $t > 0$, is said to be in the space C_μ , $\mu \in R$, if there exists a real number $p > \mu$, such that $h(t) = t^p h_1(t)$, where $h_1(t) \in C(0,\infty)$, and it is said to be in the space C_μ^n if and only if $h^{(n)} \in C_\mu$, $n \in N$.

Definition 2 ,[24]. Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$ is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \text{ , t >0} \quad \text{.......(1)}$$

$J^0 f(t) = f(t)$

$\Gamma(\alpha)$ is the well-known gamma function. Some properties of the operator J^α can be found in [6] .We give in the following for $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$,

2. $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$, ...(2)

3. $J^\alpha t^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with FDEs . Therefore , we will introduce a modified fractional differential operator D_x^α proposed by Caputo [27].

Definition 3, [25]. The fractional derivative of $f(x)$ in the Caputosense is defined as

$$(D_x^\alpha f)(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^x \dfrac{f^m(\xi)}{(x-\xi)^{\alpha-m+1}} d\xi, (\alpha > 0, m-1 < \alpha < m) \\ \dfrac{\partial^m f(x)}{\partial x^m}, \alpha = m \end{cases} \quad \text{.......(3)}$$

where $f : R \to R$, $x \to f(x)$ denotes a continuous (but not necessarily differentiable) function. Some useful formulas and results of modified Riemann- Liouville derivative , which we need here , are listed as follows:

$D_x^\alpha c = 0$, $\alpha > 0$, c = constant,

$D_x^\alpha [cf(x)] = c D_x^\alpha f(x)$, $\alpha>0$, c=constant, ...(4)

$$D_x^{\alpha} x^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \beta > \alpha > 0$$

$$D_x^{\alpha} [f(x)\, g(x)] = [D_x^{\alpha} f(x)]\, g(x) + f(x)\, [D_x^{\alpha} g(x)],$$

$$D_x^{\alpha} [f(x(t))] = f_x'(x) x^{(\alpha)}(t).$$

3. Variational Iteration Method ,[25].

In this section, the (VIM) is introduced. Here a description of the method of fractional differential initial value problem:

$$D_t^{\alpha} u(x,t) + N[u(x,t)] + L[u(x,t)] = g(x,t), t > 0 \qquad \text{………. (5)}$$

Where L is the linear operater , N is nonlinear operator in x , t ,and D^{α} is modified Riemann-Liouville derivative of order α, subject to the initial conditions.

$$u^{(k)}(x,0) = c_k(x), k = 0,1,2,\ldots,m-1, m-1 < \alpha < m. \qquad \text{……….(6)}$$

According to He's variational iteration method [9-14] from(2) , we can construct a correction functional as follows:

$$u_{n+1}(t) = u_{n+1}(t) + \int_0^t \lambda(\tau)\{L(u_n(\tau)) + N(\tilde{u}_n(\tau)) - g(\tau)\}, n \geq 0 \qquad \text{……….(7)}$$

where λ is a general Lagrange multiplier which can be optimally identified via variational theory and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$, then several approximations $u_n(t)$, $n \geq 0$ follow immediately. Consequently, the exact solution may be obtained as:

$$u(t) = \lim_{n\to\infty} u_n(t). \qquad \text{……….(8)}$$

4. Fractional Variational Iteration Method,[25]:

We can construct a correction functional for (5) as follows:

$$u_{k+1}(x,t) = u_{k+1}(x,t) + \int_0^t \lambda(t,\tau)\left(D_t^{\alpha} u_k(x,\tau) + N[\tilde{u}_k(x,\tau)] + L[\tilde{u}_k(x,\tau)] - g(x,\tau)\right) d\tau, \qquad \text{………….(9)}$$

Where $\tilde{u}(x,t)$ is a restricted variation .Taking Laplace transform to both sides of (9) as

$$\bar{u}_{k+1}(x,t) = \bar{u}_k(x,t) + \bar{L}\left[\int_0^t \lambda(t,\tau)\left(D_t^{\alpha} u_k(x,\tau) + N[\tilde{u}_k(x,\tau)] + L[\tilde{u}_k(x,\tau)] - g(x,\tau)\right) d\tau\right], \qquad \text{……….(10)}$$

Where $\bar{u}_k(x,t)$ is Laplace transform of $u_k(x,t)$ with respect to t and L is operator of Laplace transform . By assuming that the Lagrange multiplier has the form as

$\lambda(t,\tau) = \lambda(t-\tau)$, so that $\bar{L}[J_\tau^{\alpha}\lambda(D_\tau^{\alpha} u_k(x,\tau) + N[\tilde{u}_k(x,\tau)] + L[\tilde{u}_k(x,\tau)] - g(x,\tau))].$

Is the convolution of the function $\lambda(t)$ and $D_t^{\alpha} u_k(x,t) + N[\tilde{u}_k(x,t)] + L[\tilde{u}_k(x,t)] - g(x,t).$

Because $\tilde{u}(x,t)$ is a restricted variation, we have

$$\delta\bar{L}[J_t^{\alpha}\lambda(N[\tilde{u}_k(x,t)] + L[\tilde{u}_k(x,t)] - g(x,t))] = 0 \qquad \text{………(11)}$$

Taking the variation derivative δ on the both sides of (10) ,we can derive

$$\delta\bar{u}_{k+1}(x,t) = \delta\bar{u}_k(x,t) + \delta\bar{L}[J_t^{\alpha}\lambda(N[\tilde{u}_k(x,t)] + L[\tilde{u}_k(x,t)] - g(x,t))] = 0$$

$$= (1 + \bar{\lambda}(s)s^{\alpha})\delta\bar{u}_k(x,s). \qquad \text{…………(12)}$$

If setting the coefficient of $\delta\bar{u}_k(x,s)$ to zero, we can get

$$\bar{\lambda}(s) = -\frac{1}{s^{\alpha}} \quad . \qquad \text{...........(13)}$$

And the Lagrange multiplier can be identified by using the invers Laplace transform

$$\lambda(t,\tau) = -\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} = \frac{(-1)(\tau-t)^{\alpha-1}}{\Gamma(\alpha)} \qquad \text{..............(14)}$$

Substituting (14) into (10) and using the definition of Riemann-Liouville fractional integral operator , we get the iteration formula as follows:

$$u_{k+1}(x,t) = u_k(x,t) - J_t^{\alpha}(D_t^{\alpha}u_k(x,t) + N[u_k(x,t)] + L[u_k(x,t)] - g(x,t)) \qquad \text{..... (15)}$$

4. Applications and Results

In this section, we will solve one example for performing comparative studies. The exact solution of the example is known for special case $\alpha = 1$ and has been solved by using VIM, FVIM.

<u>Example (1)</u> :

consider the initial value problem of the fractional differential equation :

$$D_t^{\alpha}u(x,t) + \frac{1}{2}(u^2)_x(x,t) - u_{xx}(x,t) = x + xt^2 - 2t^2, t > 0, x \in R, 0 < \alpha \le 1.$$

................(16)

With initial condition

$$u(x,0) = 0 \qquad \text{…….....(17)}$$

By using the (VIM) is given by:

$$u_{k+1}(x,t) = u_k(x,t) - \int_0^t \left(\frac{\partial^{\alpha}}{\partial \xi^{\alpha}}u_k(x,\xi) + \frac{1}{2}\frac{\partial}{\partial x}(u_k(x,\xi))^2 - \frac{\partial^2}{\partial x^2}(u_k(x,\xi) - (x + x\xi^2 - 2\xi^2)d\xi\right.$$

.. ……..…..(18)

$$u_0(x,t) = 0$$

$$u_1(x,t) = x\left(t + \frac{t^3}{3}\right) - 2\frac{t^3}{3}$$

$$u_2(x,t) = x\left(2t + \frac{t^3}{3} - 2\frac{t^5}{15} - \frac{t^7}{63} - \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + 2\frac{t^{4-\alpha}}{\Gamma(5-\alpha)} - 4\frac{t^3}{3} + 2\frac{t^5}{15} + 2\frac{t^7}{63}\right)$$

$$u_3(x,t) = x\left(3t - 2\frac{t^3}{3} - 2\frac{t^5}{15} - 18\frac{t^7}{63} - 8\frac{t^{11}}{2475} - 19\frac{t^{15}}{3969} - 3\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{t^{(3-\alpha)^2}}{(\Gamma(4-\alpha))^2}\right)$$
$$-20\frac{t^{(5-\alpha)^2}}{(\Gamma(6-\alpha))^2} + 16\frac{t^{4-\alpha}}{3\Gamma(5-\alpha)} + 238\frac{t^{6-\alpha}}{15\Gamma(7-\alpha)} + 5038\frac{t^{8-\alpha}}{63\Gamma(9-\alpha)} + \frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)}$$
$$-2\frac{t^{5-2\alpha}}{\Gamma(6-2\alpha)} - 6\frac{t^3}{3} + 2\frac{t^5}{15} + 2\frac{t^7}{63}$$

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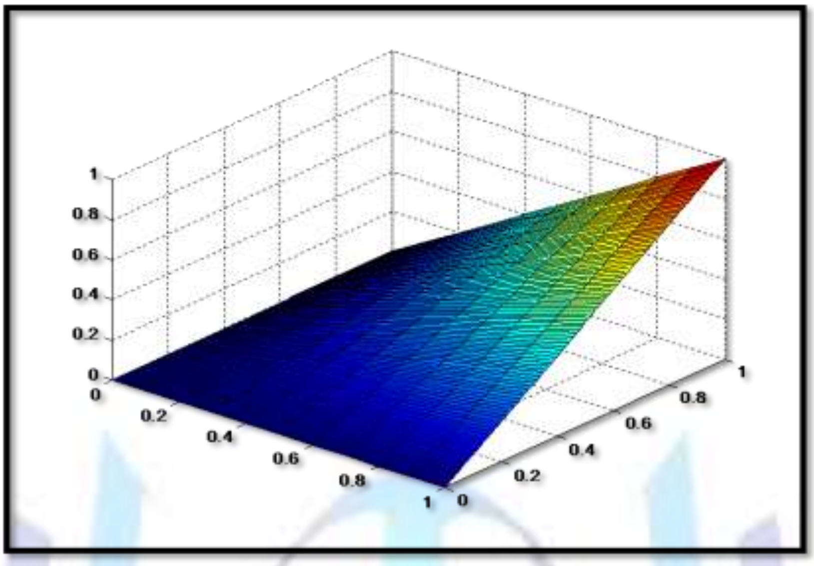
$$
\begin{aligned}
u_4(x,t) &= 4xt - 10x\frac{t^3}{3} - 2x\frac{t^5}{15} - 22x\frac{t^7}{63} - 12x\frac{t^{11}}{2475} - 9739x\frac{t^{15}}{3969} - 6x\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + 4x\frac{t^{4-\alpha}}{\Gamma(5-\alpha)} \\
&+ 16x\frac{t^{6-\alpha}}{\Gamma(7-\alpha)} + 1440x\frac{t^{8-\alpha}}{\Gamma(9-\alpha)} + 129024x\frac{t^{12-\alpha}}{\Gamma(13-\alpha)} + 2x\frac{t^{16-\alpha}}{\Gamma(17-\alpha)} + 3x\frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} \\
&- x\frac{(t^{3-\alpha})^2}{(\Gamma(4-\alpha))^2} - 18x\frac{(t^{3-\alpha})^2}{(\Gamma(4-\alpha))^2} - 2x\frac{(t^{4-\alpha})^2}{(\Gamma(5-\alpha))^4} + x\frac{(t^{4-2\alpha})^2}{(\Gamma(5-2\alpha))^2} - 128x\frac{t^{23}}{140889375} \\
&- 722x\frac{t^{31}}{488341791} + 64\frac{t^{4-\alpha}}{3\Gamma(5-\alpha)} - 2\frac{t^{6-\alpha}}{15\Gamma(7-\alpha)} - 5042\frac{t^{8-\alpha}}{63\Gamma(9-\alpha)} - 20\frac{(t^{5-\alpha})^2}{(\Gamma(6-\alpha))^2} \\
&+ \frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} - 22\frac{t^{5-2\alpha}}{3\Gamma(6-2\alpha)} - 238\frac{t^{7-2\alpha}}{15\Gamma(8-2\alpha)} - 5038\frac{t^{9-2\alpha}}{63\Gamma(10-2\alpha)} + 20\frac{(t^{6-2\alpha})^2}{(\Gamma(7-2\alpha))^2} \\
&- \frac{t^{4-3\alpha}}{\Gamma(5-3\alpha)} + 2\frac{t^{6-3}}{\Gamma(7-3\alpha)} - 8\frac{t^3}{3} + 2\frac{t^5}{15} + 2\frac{t^7}{63}
\end{aligned}
$$

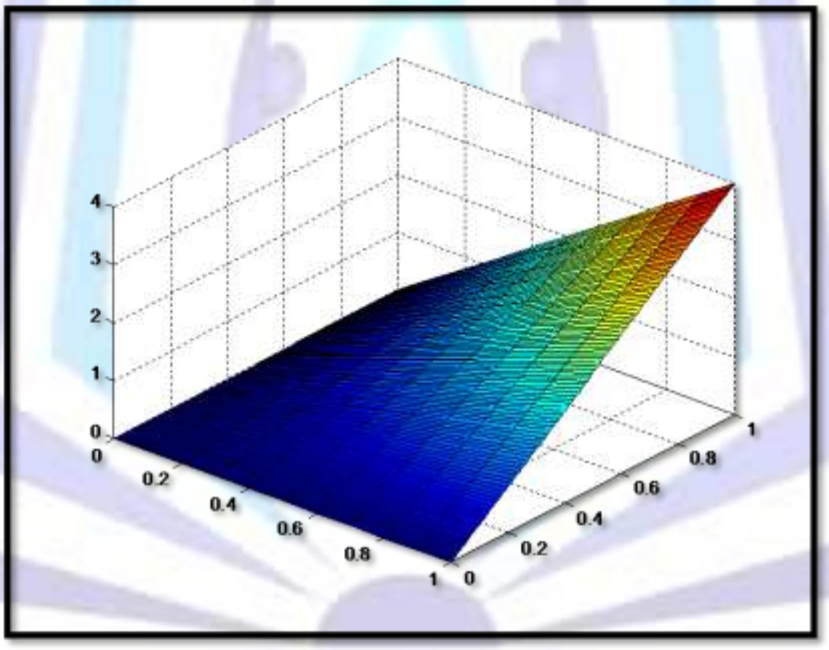
Table (1) . Numerical values by using VIM when α = 0.9

T	X	$u_{i,n}$	Exact solution	Absolute error
0.01	0	0	0	0
	0.1	0.0009993	0.001	13E-7
	0.2	0.0019994	0.002	6E-7
	0.3	0.0029996	0.003	5E-7
	0.4	0.0039997	0.004	53E-7
	0.5	0.0049995	0.005	5E-7
	0.6	0.0059995	0.006	46E-7
	0.7	0.0069996	0.007	43E-7
	0.8	0.0079996	0.008	4E-7
	0.9	0.0089996	0.009	33E-7
	1	0.0099996	0.01	3E-7
0.02	0	0	0	0
	0.1	0.0019949	0.002	50E-7
	0.2	0.0039951	0.004	48E-7
	0.3	0.0059954	0.006	45E-7
	0.4	0.0079957	0.008	42E-7
	0.5	0.009996	0.01	4E-6
	0.6	0.0119962	0.012	37E-7
	0.7	0.0139965	0.014	34E-7
	0.8	0.0159968	0.016	32E-7
	0.9	0.0179971	0.018	29E-7
	1	0.0199973	0.02	26E-7

Table (1) shows the approximate solution for (16) obtained for value α using VIM . the values of α = 1 is the only case for which know the exact solution u(x , t) = x t .



Figure(1): Exact solution of example (1)



Figure(2): Approximate solution of example (1) by VIM

Now, to solve problem (16) by (FVIM) ,we can obtain the following approximation:

$$u_{k+1}(x,t)=u_k(x,0)+J^{\alpha}(x+xt^2-2t^2)-J^{\alpha}(\frac{1}{2}(u_k{}^2)_x-(u_k)_{xx}) \qquad \text{.......(19)}$$

we get:

$$u_0(x,t)=0$$

$$u_1(x,t)=x\,(\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{2t^{2+\alpha}}{\Gamma(3+\alpha)})-\frac{4t^{2+\alpha}}{\Gamma(3+\alpha)}$$

$$u_3(x,t)=x\,(\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}-\frac{t^{3+\alpha}}{(\Gamma(1+\alpha))^2(\Gamma(3\alpha+1))}-\frac{4t^{4+3\alpha}}{(\Gamma(3+\alpha))^2(\Gamma(3\alpha+5))}$$

$$-\frac{4t^{2+\alpha}}{\Gamma(3+\alpha)}$$

$$u_3(x,t)=x\,(\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}-\frac{2t^{3\alpha}}{(\Gamma(1+\alpha))^2(\Gamma(3\alpha+1))}-\frac{8t^{4+3\alpha}}{(\Gamma(3+\alpha))^2(\Gamma(3\alpha+5)}$$

$$+\frac{t^{7\alpha}}{(\Gamma(1+\alpha))^4(\Gamma(3\alpha+1))^2(\Gamma(7\alpha+1)}-\frac{16t^{8+7\alpha}}{(\Gamma(3+\alpha))^4(\Gamma(3\alpha+5))^2(\Gamma(9+7\alpha)})$$

$$-\frac{4t^{2+\alpha}}{\Gamma(3+\alpha)}$$

$$u_4(x,t)=x(\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}-\frac{3t^{3\alpha}}{(\Gamma(1+\alpha))^2(\Gamma(3\alpha+1))}$$

$$-\frac{12t^{4+3\alpha}}{(\Gamma(3+\alpha))^2(\Gamma(3\alpha+5)}+\frac{3t^{7\alpha}}{(\Gamma(1+\alpha))^4(\Gamma(3\alpha+1))^2(\Gamma(7\alpha+1)}$$

$$-\frac{32t^{8+7\alpha}}{(\Gamma(3+\alpha))^4(\Gamma(3\alpha+5))^2(\Gamma(7\alpha+9)}-\frac{t^{14\alpha}}{(\Gamma(1+\alpha))^8(\Gamma(3\alpha+1)(\Gamma(7\alpha+1)(\Gamma(14\alpha+1)}$$

$$-\frac{256t^{16+14\alpha}}{(\Gamma(3+\alpha))^8(\Gamma(3\alpha+5)^4(\Gamma(7\alpha+9)^2(\Gamma(14\alpha+17)}-\frac{4t^{2+\alpha}}{\Gamma(3+\alpha)}$$

Table –(2) Numerical Solution by using FVIM when α = 0.9

T	X	$u_{i,n}$	Exact solution	Absolute error
0.01	0	0	0	0
	0.1	0.0015839	0.001	5838E-7
	0.2	0.0021688	0.002	11688E-7
	0.3	0.0037537	0.003	7537E-7
	0.4	0.0043397	0.004	3397E-7
	0.5	0.0059236	0.005	9236E-7
	0.6	0.0065086	0.006	5086E-7
	0.7	0.0110938	0.007	40938E-7
	0.8	0.0126785	0.008	46785E-7
	0.9	0.0142634	0.009	52634E-7
	1	0.0158484	0.01	58484E-7
0.02	0	0	0	0
	0.1	0.0029524	0.002	9524E-7
	0.2	0.0059102	0.004	19102E-7

	0.3	0.0088680	0.006	2868E-7
	0.4	0.0118257	0.008	38257E-7
	0.5	0.0147835	0.01	47835E-7
	0.6	0.0177413	0.012	57413E-7
	0.7	0.0206991	0.014	66991E-7
	0.8	0.0236569	0.016	76569E-7
	0.9	0.0266147	0.018	86147E-7
	1	0.0295724	0.02	95724E-7

Table (2) shows the approximate solution for (16) obtained for value α using methods FVIM. The values of α = 1 is the only case for which know the exact solution u(x , t) = x t.

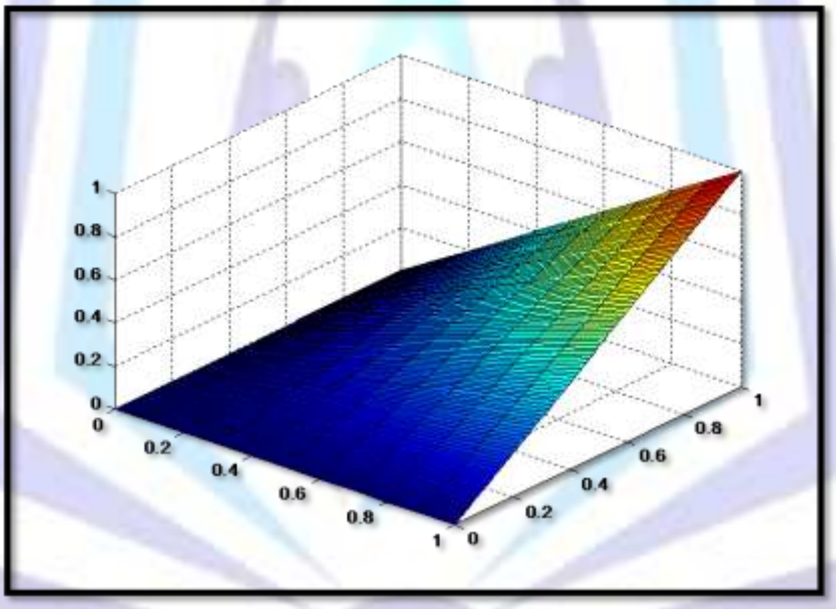
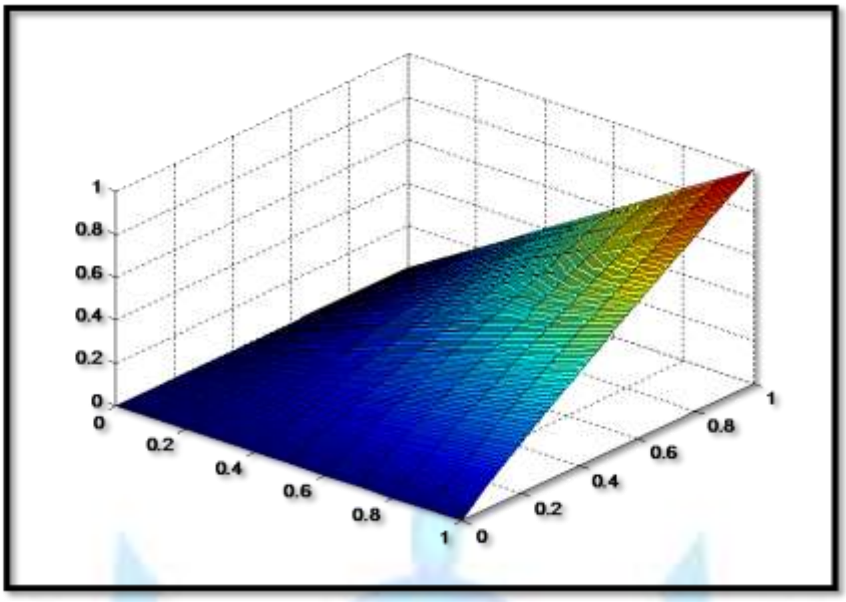


Figure (3): Exact solution of example (1)



Figure(4): Approximate solution of example (1) by FVIM

5. Conclusion:

The VIM has been successfully employed to obtain the approximate solution of FDE's, FVIM is powerful and effective tools for solution of FDE΄s and give approximation of higher accuracy, reduces the computational workload by avoiding the evaluation of VIM . So FVIM is more effective than VIM in solving the FDE΄s. The compared results of (VIM and FVIM) to solve FDE with the exact solutions also give a good comparative between two method, using Programming MATLAB to get the results of this method **.**

References

[1] A. G. Nikitin, T. A. Barannyk, "Solitary waves and other solutions for nonlinear heat equations, " Cent. Eur. J. Math., 2005, vol.2,pp.840-858.

[2] A. Rafiq, M. Ahmed, S. Hussain, "A general approach to specific second order ordinary differential equations using homotopy perturbation method, "Phys. Lett. A, 2008, vol. 372, pp. 4973- 4976.

[3] D. D. Ganji and S. H. Hashemi Kachapi, "Analysis of nonlinear equations in fluids ," Progress in Nonlinear Science, vol. 2, pp. 1– 293, 2011 .

[4] D.D. Ganji and S.H . Hashemi Kachapi , "Analytical and numerical methods in engineering and applied sciences," Progress in Nonlinear Science, vol. 3, pp. 1–579, 2011.

[5] G. E. Drˇagˇanescu, "Application of a variational iteration method to linear and nonlinear viscoelastic models with fractional derivatives," Journal of , vol. 47, no. 8, p. 082902, 9, 2006.

[6] I. Podlubny, " Fractional Differential Equations, " Academic Press, San Diego, 1999.

[7] I. Petr´aˇs, "A note on the fractional-order Chua's system," Chaos, Solitons & Fractals, vol. 38, no. 1, pp. 140–147, 2008.

[8] I. Podlubny, " Fractional Differential Equations, "vol. 198 of Mathematics in Science and Engineering, Academic Pres, San Diego, Calif, USA, 1999.

[9] J. H. He, "Variational iteration method—a kind of non-linear analytical technique: some examples," International Journal of Non-Linear Mechanics, vol. 34 , no. 4, pp. 699–708, 1999.

[10] J.-H. He, "Variational iteration method for autonomous ordinary differential systems , " Applied Mathematics and Computation, vol. 114, no. 2-3, pp. 115–123,2000.

[11] J.-H. He, "Variational principles for some nonlinear partial differential equations with variable coefficients," Chaos, Solitons & Fractals, vol. 19, no. 4, pp. 847–851 , 2004.

[12] J.-H. He and X.-H. Wu, "Variational iteration method: new development and applications," Computers &Mathematics with Applications, vol. 54, no. 7-8, pp. 881– 894, 2007.

[13] J.-H. He, "Variational iteration method—some recent results and new interpretations, " Journal of Computational and Applied Mathematics,vol. 207, no. 1, pp. 3–17, 2007.

[14] J. H. He, G. C. Wu, and F. Austin, "The variational iterational method which should be follow," Nonlinear Science Letters A, vol. 1, no. 1, pp. 1–30, 2010.

[15] J.-H.He, "Approximate analytical solution for seepage flow with fractional derivatives in porous media," Computer Methods in Applied Mechanics and Engineering, vol. 167, no. 1-2, pp. 57–68, 1998.

[16] J. H. He, "Asymptotic methods for solitary solutions and compactions," Abstract and Applied Analysis, vol. 2012, Article ID 916793, 130 pages, 2012.

[17] R. P. Agarwal, B. de Andrade, and C. Cuevas, "Weighted pseudo-almost periodic solutions of a class of semilinear fractional differential equations," Nonlinear Analysis: Real World Applications, vol. 11, no. 5, pp. 3532–3554, 2010.

[18] R. P. Agarwal , V. Lakshmikantham , and J. J. Nieto, "On the concept of solution for fractional differential equations with uncertainty ," Nonlinear Analysis: Theory, Methods & Applications, vol.72, no. 6, pp. 2859–2862, 2010.

[19] R. Garrappa and M. Popolizio, "On the use of matrix functions for fractional partial differential equations," Mathematics and Computers in Simulation, vol. 81, no. 5, pp. 1045–1056, 2011.

[20] R. Hilfer, " Applications of Fractional Calculus in Physics, " World ScientificPublishing, River Edge, NJ, USA, 2000.

[21] R. Y. Molliq, M. S. M. Noorani, and I. Hashim, "Variational iteration method for fractional heat- and wave-like equations," No nlinear Analysis: Real World Applications, vol. 10, no. 3, pp. 1854–1869, 2009.

[22] S. G. Samko, A. A. Kilbas, and O. I. Marichev, " Fractional Integrals and Derivatives: Theory and Applications, " Gordon and Breach, Yverdon, Switzerland, 1993.

[23] S. Momani and Z. Odibat, "Analytical approach to linear fractional partial differential equations arising in fluid mechanics," Physics Letters A, vol. 355, no. 4-5, pp. 271–279, 2006.

[24] S. Mohamed ,S .K .Elagan ,Y .S. Hamed and M .Sayed, " Using complextransformation to get an exact solution for fractional generalized coupled MKDV and equation, " International Journal or Basic and Applied Sciences , vol.13, no .01 , 2013.

[25] S . Junqiang , Y . Fukang ,C . Xiaoqun and L . Fengshun , "Fractional Variational Iteration Method versus Adomian's Decomposition Method in Some Fractional Partial Differential Equations, " Journal of Applied Mathematics, vol. 2013 , Article ID 392567, 10 pages,2013.

[26] Y. Khan, N. Faraz, A. Yildirim, and Q. Wu, "Fractional variational iteration method for fractional initial-boundary value problems arising in the application of nonlinear science," Computers & Mathematics with Applications, vol. 62, no. 5, pp.2273–2278, 2011.

[27] Y. Luchko and R. Gorenflo, "An operational method for solving fractional differential equations with the Caputo derivatives," Acta Mathematica Vietnamica, vol. 24, no. 2, pp. 207–233, 1999.

[28] Z. Odibat, "On Legendre polynomial approximation with the VIM or HAM for numerical treatment of nonlinear fractional differential equations," Journal ofComputational and Applied Mathematics, vol. 235, no. 9, pp. 2956–2968, 2011.

[29] Z. M. Odibat and S. Momani, "Application of variational iteration method to nonlinear differential equations of fractional order," International Journal of Nonlinear Sciences and Numerical Simulation, vol. 7, no. 1, pp. 27–34, 2006.