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Oscillation of third order impulsive differential equations with delay

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ABSTRACT

This paper deals with the oscillation of third order impulsive differential equations with delay. The results of this paper improve and extend some results for the differential equations without impulses. Some examples are given to illustrate the main results.

Keywords: Oscillation; Third-order; Impulse; Differential equations.

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1 INTRODUCTION

This paper concerned with the oscillatory and asymptotic behavior of third order impulsive differential equation of the form

$$\begin{cases} \left[a(t)(b(t)(x(t)+p(t)x(t-\tau))')'\right]' + q(t)x(t-\sigma) = 0, t \geq t_0 > 0, t \neq t_k; \\ x(t_k^+) = a_k x(t_k), \quad x'(t_k^+) = b_k x'(t_k) \\ x''(t_k^+) = c_k x''(t_k), k = 1,2,\ldots. \end{cases} \tag{1.1}$$

where τ and σ are nonnegative constants with $\sigma > \tau, \{t_k\}$ is a sequence of impulsive moments which satisfies $0 \leq t_0 < t_1 < \ldots < t_k < \ldots$ with $\lim_{k\to\infty} t_k = \infty$ and $t_{k+1} - t_k > \tau$. Throughout this paper, we will assume that the following assumptions are satisfied:

(H1) a, b and p are positive continuously differentiable functions with $0 \leq p(\mathrm{t}) \leq p < 1$;

(H2) $q \in C([t_0,\infty),[0,\infty))$ and $q(t)$ is not identically zero on any ray of the form $[t^*,\infty)$ for all $t^* \geq t_0$;

(H3) a_k, b_k, c_k are positive constants.

Let $J \subset \mathbb{R}$ be an interval. We define $PC^1(J,\mathbb{R}) = \{x : J \to \mathbb{R} : x(t)$ is differentiable for $t \geq 0$ and $t \neq t_k$, $x'(t_k^-)$ and $x'(t_k^+)$ exist and $x'(t_k^-) = x'(t_k)\}$.

By a solution of equation (1.1), we mean a real function $x(t)$ such that $x, x', x'' \in PC^1(J,\mathbb{R})$ which satisfies equation (1.1). Our attention is restricted to those solutions $x(t)$ of equation (1.1) which exist on half line $[t_0,\infty)$ and satisfy $\sup\{|x(t)| : t \geq T_x\} > 0$ for all $T_x \geq t_0$. It will be assumed that equation (1.1) has solutions which are nontrivial for large t. Such a solution of equation (1.1) is said to be non-oscillatory if it is eventually positive or eventually negative, otherwise it is oscillatory.

It is well known that there is a drastic difference in the behavior of solutions between differential equations with impulses and those without impulses. Some differential equations are non-oscillatory, but they may become oscillatory if some proper impulse controls are added to them , see [2].

In recent years, the oscillation theory and asymptotic behavior of impulsive differential equations and their applications have been and still receiving intensive attention. But to the best of our knowledge, it seems that little has been done for oscillation of third order impulsive differential equations[10].

Our aim in this paper is to establish some new sufficient conditions which ensure that solutions of equation (1.1) are oscillatory or converge to zero as t tends to ∞. In particular, we extend the results in [9, 7] to the impulsive differential equation (1.1).

In this paper, we shall study the behavior of solutions of equation (1) under the following three cases:

$$\int_{t_0}^{\infty} \frac{ds}{a(s)} = \infty, \quad \int_{t_0}^{\infty} \frac{ds}{b(s)} = \infty; \tag{1.2}$$

$$\int_{t_0}^{\infty} \frac{ds}{a(s)} < \infty, \quad \int_{t_0}^{\infty} \frac{ds}{b(s)} = \infty; \tag{1.3}$$

$$\int_{t_0}^{\infty} \frac{ds}{a(s)} < \infty, \quad \int_{t_0}^{\infty} \frac{ds}{b(s)} < \infty. \tag{1.4}$$

In the following, all functional inequalities considered are assumed to hold eventually, that is, they are satisfied for all sufficiently large t.

2 Main results

In this section, we present the main results. We write $z(t) = x(t) + p(t)x(t-\tau)$. Furthermore, assume that $a_k \le 1, b_k \ge 1$ and $c_k \le 1$. First we begin with a useful lemma, which is borrowed from [6].

Lemma 2.1 *Suppose*

(i) the sequence $\{t_k\}_{k\in N}$ satisfies $0 \le t_0 < t_1 < \ldots < t_k < \ldots$ with $\lim_{k\to\infty} t_k = \infty$;

(ii) $m, m' : \mathbb{R}_+ \to \mathbb{R}$ are right continuous on $\mathbb{R}_+ \setminus \{t_k : k \in \mathbb{N}\}$, there exist the lateral limits $m(t_k^-), m'(t_k^-), m(t_k^+)$ and $m'(t_k^+)$ with $m(t_k^-) = m(t_k), k = 1,2,3\ldots$;

(iii) for $k = 1,2,3,\ldots$ and $t \ge t_0$, we have

$$m'(t) \le p(t)m(t) + q(t), t \ne t_k, \tag{2.1}$$

$$m(t_k^+) \le \alpha_k m(t_k) + \beta_k \tag{2.2}$$

where $p, q \in C(\mathbb{R}_+, \mathbb{R}), \alpha_k$ and β_k are real constants with $\alpha_k \ge 0$. Then the following inequality holds

$$m(t) \le m(t_0) \prod_{t_0<t_k<t} \alpha_k \exp\left(\int_{t_0}^{t} p(s)ds\right) + \int_{t_0}^{t} \prod_{s<t_k<t} \alpha_k \exp\left(\int_{s}^{t} p(u)du\right) q(s)ds$$

$$+ \sum_{t_0<t_k<t} [\prod_{t_k<t_j<t} \alpha_j \exp(\int_{t_k}^{t} p(s)ds)]\beta_k, t \ge t_0. \tag{2.3}$$

Theorem 2.1 *Assume that (1.2) holds. If there exists a function* $\rho \in C^1([t_0,\infty), (0,\infty))$ *for all sufficiently large* $t \ge t_3 \ge t_2 \ge t_1 \ge t_0$, *one has*

$$\lim_{t\to\infty} \int_{t_3}^{t} \prod_{t_3<t_k<s} \frac{1}{b_k} (\rho(s)q(s)(1-p(s-\sigma)) \frac{\int_{t_2}^{s-\sigma} \frac{\int_{t_1}^{v} \frac{1}{a(u)} du}{b(v)} dv}{\int_{t_1}^{s} \frac{1}{a(u)} du} - \frac{a(s)(\rho'(s))^2}{4\rho(s)}) ds = \infty, \tag{2..4}$$

$$\lim_{t\to\infty} \int_{t_3}^{t} [\frac{1}{b(v)} \int_{t_2}^{v} \prod_{t_1<t_k<v} b_k \frac{1}{a(u)} (\int_{t_1}^{u} \prod_{t_1<t_k<s} \frac{1}{b_k} L_1 q(s)ds - L_2) du] dv] = \infty, \tag{2.5}$$

where L_1 and L_2 are positive constants, then every solution $x(t)$ of equation (1.1) is either oscillatory or satisfying $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may suppose that $x(t)>0, x(t-\tau)>0,$ and $x(t-\sigma)>0$ for all $t\geq t_1\geq t_0$. For $t\neq t_k$ from (1.2), there exists $t\geq t_1\geq t_0$ such that the following two cases arise:

(1) $z(t)>0, z'(t)>0, (b(t)z'(t))'>0, [a(t)(b(t)z'(t))']'\leq 0;$

(2) $z(t)>0, z'(t)<0, (b(t)z'(t))'>0, [a(t)(b(t)z'(t))']'\leq 0$

for all $t\geq t_2\geq t_1$. Assume that case(1) holds. For $t\neq t_k$ define a function ω by

$$\omega(t)=-\rho(t)\frac{a(t)(b(t)z'(t))'}{b(t)z'(t)}, t\geq t_2. \tag{2.6}$$

Then $\omega(t_k^+)<0, k=1,2,\ldots and$ $\omega(t)<0,$ for $t\geq t_2$. Differentiating (2.6), we have

$$\omega'(t)=-\rho'(t)\frac{a(t)(b(t)z'(t))'}{b(t)z'(t)}-\rho(t)\frac{(a(t)(b(t)z'(t))')'}{b(t)z'(t)}+\rho(t)\frac{a(t)(b(t)z'(t))'(b(t)z'(t))'}{(b(t)z'(t))^2}. \tag{2.7}$$

Since $z'(t)>0,$ we have

$$x(t)\geq(1-p(t))z(t), t\neq t_k, and\ t\geq t_2. \tag{2.8}$$

It follows from equation (1.1), (2.7) and (2.8) that

$$\omega'(t)\geq-\frac{\rho'(t)}{\rho(t)}\omega(t)+\rho(t)q(t)(1-p(t-\sigma))\frac{z(t-\sigma)}{b(t)z'(t)}+\frac{\omega^2(t)}{\rho(t)a(t)}.$$

or

$$\omega'(t)\geq-\frac{\rho'(t)}{\rho(t)}\omega(t)+\rho(t)q(t)(1-p(t-\sigma))\frac{z(t-\sigma)}{b(t-\sigma)z'(t-\sigma)}\frac{b(t-\sigma)z'(t-\sigma)}{b(t)z'(t)}+\frac{\omega^2(t)}{\rho(t)a(t)}, t\neq t_k, t\geq t_2. \tag{2.9}$$

Now,

$$b(t)z'(t)\geq\int_{t_1}^{t}\frac{a(s)(b(s)z'(s))'}{a(s)}ds\geq a(t)(b(t)z'(t))'\int_{t_1}^{t}\frac{1}{a(s)}ds. \tag{2.10}$$

Thus

$$\left(\frac{b(t)z'(t)}{\int_{t_1}^{t}\frac{1}{a(s)}ds}\right)'\leq 0. \tag{2.11}$$

Therefore,

$$z(t)=z(t_2)+\int_{t_2}^{t}\frac{b(s)z'(s)}{\int_{t_1}^{s}\frac{1}{a(u)}du}\frac{\int_{t_1}^{s}\frac{1}{a(u)}du}{b(s)}ds,$$

$$\geq\frac{b(t)z'(t)}{\int_{t_1}^{t}\frac{1}{a(u)}du}\int_{t_2}^{t}\frac{\int_{t_1}^{s}\frac{1}{a(u)}du}{b(s)}ds, t\geq t_2>t_1. \tag{2.12}$$

Using (2.11) and (2.12) in (2.9), we obtain

$$\omega'(t) \geq -\frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)q(t)(1-p(t-\sigma))\frac{\int_{t_2}^{t-\sigma}\frac{\int_{t_1}^{s}\frac{1}{a(u)}du}{b(s)}ds}{\int_{t_1}^{t-\sigma}\frac{1}{a(u)}du}\frac{\int_{t_1}^{t-\sigma}\frac{1}{a(u)}du}{\int_{t_1}^{t}\frac{1}{a(u)}du} + \frac{\omega^2(t)}{\rho(t)a(t)}$$

$$= -\frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)q(t)(1-p(t-\sigma))\frac{\int_{t_2}^{t-\sigma}\frac{\int_{t_1}^{s}\frac{1}{a(u)}du}{b(s)}ds}{\int_{t_1}^{t}\frac{1}{a(u)}du} + \frac{\omega^2(t)}{\rho(t)a(t)}$$

or

$$\omega'(t) \geq \rho(t)q(t)(1-p(t-\sigma))\frac{\int_{t_2}^{t-\sigma}\frac{\int_{t_1}^{s}\frac{1}{a(u)}du}{b(s)}ds}{\int_{t_1}^{t}\frac{1}{a(u)}du} - \frac{a(t)(\rho'(t))^2}{4\rho(t)}, t \neq t_k \ and \ t_2 \geq t_1. \tag{2.13}$$

Since $t_{k+1} - t_k > \tau$ for each $k \in \mathbb{N}$, we have

$$t_k < t_{k+1} - \tau < t_{k+1}. \tag{2.14}$$

Since x, x', x'' are continuous on $(t_k, t_{k+1}]$, we have from the inequality (2.14) that

$$z(t_k^+) = x(t_k^+) + p(t_k^+)x(t_k^+ - \tau)$$

$$= a_k x(t_k) + p(t_k)x(t_k - \tau)$$

$$\leq z(t_k), k = 1, 2, \ldots. \tag{2.15}$$

Now

$$z'(t_k^+) = x'(t_k^+) + p'(t_k^+)x(t_k^+ - \tau) + p(t_k^+)x'(t_k^+ - \tau)$$

$$= b_k x(t_k) + p'(t_k)x'(t_k - \tau) + p(t_k)x'(t_k - \tau)$$

$$\leq b_k z'(t_k), k = 1, 2, \ldots. \tag{2.16}$$

Similarly

$$z''(t_k^+) = x''(t_k^+) + p''(t_k^+)x(t_k^+ - \tau) + 2p'(t_k^+)x'(t_k^+ - \tau) + p(t_k^+)x''(t_k^+ - \tau)$$

$$= c_k x''(t_k) + p''(t_k)x(t_k - \tau) + 2p'(t_k)x'(t_k - \tau) + p(t_k)x''(t_k - \tau)$$

$$\leq z''(t_k), k = 1, 2, \ldots. \tag{2.17}$$

Now from (2.16) and (2.17), we have

$$\omega(t_k^+) = -\rho(t_k^+)a(t_k^+)\left(\frac{b(t_k^+)z''(t_k^+) + b'(t_k^+)z'(t_k^+)}{b(t_k^+)z'(t_k^+)}\right)$$

$$= -\rho(t_k^+)a(t_k^+)\left(\frac{z''(t_k^+)}{z'(t_k^+)} + \frac{b'(t_k^+)}{b(t_k^+)}\right)$$

$$\geq -\rho(t_k)a(t_k)\left(\frac{z''(t_k)}{b_k z'(t_k)} + \frac{b'(t_k)}{b(t_k)}\right)$$

$$\geq \frac{1}{b_k}\omega(t_k), k = 1,2,\ldots. \tag{2.18}$$

Using Lemma 2.1 in (2.13) and (2.18), we obtain

$$\omega(t) \geq \omega(t_3)\prod_{t_3<t_k<t}\frac{1}{b_k} + \int_{t_3}^{t}\prod_{s<t_k<t}\frac{1}{b_k}\quad(\rho(s)q(s)(1-p(s-\sigma))\frac{\int_{t_2}^{s-\sigma}\frac{\int_{t_1}^{v}\frac{1}{a(u)}du}{b(v)}dv}{\int_{t_1}^{s}\frac{1}{a(u)}du} - \frac{a(s)(\rho'(s))^2}{4\rho(s)})ds,$$

Taking limit as $t \to \infty$ and using (2.4) we get a contradiction with $\omega(t) < 0.$

Next assume that case (2) holds. Since $z(t)$ is nonincreasing, we have $z(t) \to L \geq 0.$ If $L > 0,$ then for any $\varepsilon > 0,$ there exists $t_4 \geq t_3$ suchthat $L + \varepsilon > z(t) > L,$ eventually for $t \geq t_4.$ Choose $\varepsilon = \frac{L(1-p)}{2p}.$ Then for $t \neq t_k, t \geq t_4$ we have

$$x(t) = z(t) - p(t)x(t-\tau) > L - pz(t-\tau)$$

$$> L - p(L+\varepsilon)$$

$$= \frac{L(1-p)}{2} = L_1, say.$$

From equation (1.1), we have

$$(a(t)(b(t)(x(t)+p(t)x(t-\tau))')')' = -q(t)x(t-\sigma) \leq -L_1 q(t), t \neq t_k, t \geq t_4. \tag{2.19}$$

For $k = 1,2,\ldots$

$$z'(t_k^+) = x'(t_k^+) + p'(t_k^+)x(t_k^+ - \tau) + p(t_k^+)x'(t_k^+ - \tau)$$

$$= b_k x'(t_k) + p'(t_k)x'(t_k - \tau) + p(t_k)x'(t_k - \tau)$$

$$\leq b_k z'(t_k). \tag{2.20}$$

Also

$$a(t_k^+)(b(t_k^+)z'(t_k^+))' = a(t_k^+)(b'(t_k^+)z'(t_k^+) + b(t_k^+)z''(t_k^+))$$

$$= a(t_k)(b'(t_k)b_k z'(t_k) + b(t_k)z''(t_k))$$

$$\leq b_k a(t_k)(b(t_k)z'(t_k))'. \tag{2.21}$$

Using Lemma 2.1 in (2.19) and (2.21), we obtain

$$(a(t)(b(t)z'(t))') \le (a(t_1)(b(t_1)z'(t_1))') \prod_{t_1<t_k<t} b_k - L_1 \int_{t_1}^{t} \prod_{s<t_k<t} b_k q(s)ds$$

$$(b(t)z'(t))' \le \frac{L_2}{a(t)} \prod_{t_1<t_k<t} b_k - \frac{L_1}{a(t)} \int_{t_1}^{t} \prod_{s<t_k<t} b_k q(s)ds$$

where $L_2 = a(t_1)(b(t_1)z'(t_1))' > 0.$

Again using Lemma 2.1 in the last inequality, we have

$$b(t)z'(t) \le b(t_2)z'(t_2) \prod_{t_2<t_k<t} b_k + \int_{t_2}^{t} \prod_{u<t_k<t} b_k [\prod_{t_1<t_k<u} b_k \frac{L_2}{a(u)} - \frac{L_1}{a(u)} \int_{t_1}^{u} \prod_{s<t_k<u} b_k q(s)ds]du$$

or

$$z'(t) \le \frac{1}{b(t)} \int_{t_2}^{t} \prod_{t_1<t_k<t} b_k [\frac{L_2}{a(u)} - \frac{L_1}{a(u)} \int_{t_1}^{u} \prod_{t_1<t_k<s} \frac{1}{b_k} q(s)ds]du. \tag{2.22}$$

Using Lemma 2.1 in (2.15) and (2.22), we obtain

$$z(t) \le z(t_3) - \int_{t_3}^{t} [\frac{1}{b(v)} \int_{t_2}^{v} \prod_{t_1<t_k<v} b_k \quad \frac{1}{a(u)} (L_1 \int_{t_1}^{u} \prod_{t_1<t_k<s} \frac{1}{b_k} q(s)ds - L_2)du]dv.$$

Taking limit as $t \to \infty$ in the last inequality we get a contradiction with (2.5).Therefore $\lim_{t\to\infty} z(t) = 0.$ Since $x(t) \le z(t),$ we have $\lim_{t\to\infty} x(t) = 0.$ This completes the proof.

Theorem 2.2 *Assume that (1.3) holds and there exists a function* $\rho \in C^1(([t_0,\infty),(0,\infty))$ *such that for all sufficiently large* $t \ge t_3 \ge t_2 \ge t_1 \ge t_0,$ *we have (2.4) and (2.5). If*

$$\lim_{t\to\infty} \int_{t_2}^{t} \prod_{t_2<t_k<s} b_k (\delta(s)q(s)(1-p(s-\sigma)) \int_{t_1}^{s-\sigma} \frac{du}{b(u)} - \frac{1}{4\delta(s)a(s)})ds = \infty, \tag{2.23}$$

where

$$\delta(t) := \int_{t}^{\infty} \frac{1}{a(s)} ds, \tag{2.24}$$

then every solution $x(t)$ of equation (1.1) is either oscillatory or $\lim_{t\to\infty} x(t) = 0.$

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may suppose that $x(t) > 0, x(t-\tau) > 0,$ and $x(t-\sigma) > 0$ for $t \ge t_1 \ge t_0.$ For $t \ne t_k, t \ge t_1$ from (1.3), there exist three possible cases (1), (2)(as in Theorem 2.1) and

(3) $z(t) > 0, z'(t) > 0, (b(t)z'(t))' < 0, [a(t)(b(t)z'(t))']' \le 0.$

For the cases (1) and (2), we obtain the conclusion from Theorem 2.1. Now assume that case (3) holds. Since $a(t)(b(t)z'(t))'$ is nonincreasing, we have

$$a(s)(b(s)z'(s))' \le a(t)(b(t)z'(t))', s \ge t \ge t_5 \ge t_4.$$

Dividing the above inequality by $a(s)$ and integrating from t to $l,$ we obtain

$$b(l)z'(l) \le b(t)z'(t) + a(t)(b(t)z'(t))' \int_t^l \frac{ds}{a(s)}.$$

Letting $l \to \infty,$ we have

$$0 \le b(t)z'(t) + a(t)(b(t)z'(t))' \int_t^\infty \frac{ds}{a(s)}.$$

That is,

$$-\frac{a(t)(b(t)z'(t))'}{b(t)z'(t)} \int_t^\infty \frac{ds}{a(s)} \le 1. \tag{2.25}$$

Define a function ϕ by

$$\phi(t) = -\frac{a(t)(b(t)z'(t))'}{b(t)z'(t)}, t \ne t_k, t \ge t_5. \tag{2.26}$$

Then $\phi(t_k^+) > 0, k = 1,2,\ldots and \ \phi(t) > 0,$ for $t \ge t_5.$ Hence from (2.25) and (2.26), we obtain

$$\delta(t)\phi(t) \le 1. \tag{2.27}$$

Differentiating (2.26) gives

$$\phi'(t) = -\frac{(a(t)(b(t)z'(t))')'}{b(t)z'(t)} + \frac{a(t)(b(t)z'(t))'(b(t)z'(t))'}{(b(t)z'(t))^2}, t \ne t_k, t \ge t_5.$$

From equation (1.1), (2.8) and (2.26), we obtain

$$\phi'(t) = q(t)(1 - p(t-\sigma))\frac{z(t-\sigma)}{b(t)z'(t)} + \frac{\phi^2(t)}{a(t)}, t \ne t_k, t \ge t_5. \tag{2.28}$$

From the third inequality in case (3), we see that

$$z(t) \ge b(t)\int_{t_5}^t \frac{ds}{b(s)} z'(t). \tag{2.29}$$

Hence,

$$\left(\frac{z(t)}{\int_{t_5}^t \frac{ds}{b(s)}}\right)' \le 0, t \ne t_k, t \ge t_5$$

which implies that

$$\frac{z(t-\sigma)}{z(t)} \ge \frac{\int_{t_5}^{t-\sigma} \frac{ds}{b(s)}}{\int_{t_5}^{t} \frac{ds}{b(s)}}. \tag{2.30}$$

Using (2.28) and (2.29) in (2.30), we have

$$\phi'(t) \ge (t)(1 - p(t-\sigma))\int_{t_5}^{t-\sigma} \frac{ds}{b(s)} + \frac{\phi^2(t)}{a(t)}, t \ne t_k.$$

Multiplying the last inequality by $\delta(t),$ we have

$$\delta(t)\phi'(t) \geq \delta(t)q(t)(1-p(t-\sigma))\int_{t_5}^{t-\sigma}\frac{ds}{b(s)}+\frac{\phi^2(t)}{a(t)}\delta(t), t \neq t_k. \tag{2.31}$$

Now

$$\begin{aligned}(\delta(t)\phi(t))' &= \delta(t)\phi'(t)+\delta'(t)\phi(t)\\ &= \delta(t)\phi'(t)-\frac{1}{a(t)}\phi(t)\\ &\geq \delta(t)q(t)(1-p(t-\sigma))\int_{t_5}^{t-\sigma}\frac{ds}{b(s)}+\frac{\phi^2(t)\delta(t)}{a(t)}-\frac{\phi(t)}{a(t)}.\end{aligned} \tag{2.32}$$

For $k=1,2,\ldots$ from the definition of $\phi(t)$, we have

$$\begin{aligned}\phi(t_k^+) &= -a(t_k^+)\left(\frac{b(t_k^+)z''(t_k^+)+b'(t_k^+)z'(t_k^+)}{b(t_k^+)z'(t_k^+)}\right)\\ &= -a(t_k^+)\left(\frac{z''(t_k^+)}{z'(t_k^+)}+\frac{b'(t_k^+)}{b(t_k^+)}\right)\\ &\geq -\rho(t_k)a(t_k)\left(\frac{z''(t_k)}{b_k z'(t_k)}+\frac{b'(t_k)}{b(t_k)}\right)\\ &\geq \frac{1}{b_k}\phi(t_k), k=1,2,\ldots.\end{aligned} \tag{2.33}$$

Using Lemma 2.1 in (2.32) and (2.33) for all $t_6 \geq t_5$, we obtain

$$\delta(t)\phi(t) \geq \delta(t_6)\phi(t_6)\prod_{t_6<t_k<t}\frac{1}{b_k} + \int_{t_6}^{t}\prod_{s<t_k<t}\frac{1}{b_k}(\delta(s)q(s)(1-p(s-\sigma))$$
$$\int_{t_5}^{s-\sigma}\frac{du}{b(u)}+\frac{\phi^2(s)\delta(s)}{a(s)}-\frac{\phi(s)}{a(s)})ds,$$

or

$$\delta(t)\phi(t) \geq \prod_{t_6<t_k<t}\frac{1}{b_k}[\delta(t_6)\phi(t_6)+\int_{t_6}^{t}\prod_{t_6<t_k<s}b_k(\delta(s)q(s)(1-p(s-\sigma))$$
$$\int_{t_5}^{s-\sigma}\frac{du}{b(u)}-\frac{1}{4\delta(s)a(s)})ds].$$

Taking limit as $t\to\infty$ in the last inequality, we obtain a contradiction with (2.23) due to (2.27). Now the proof is complete.

Theorem 2.3 *Assume that (1.4) holds and and there exists a function* $\rho \in C^1(([t_0,\infty), (0,\infty))$ *such that for all sufficiently large* $t \geq t_3 \geq t_2 \geq t_1 \geq t_0$, *we have (2.4) ,(2.5) and (2.23). If*

$$\lim_{t\to\infty}\int_{t_1}^{t}\frac{1}{b(v)}\int_{t_1}^{v}\prod_{t_1<t_k<v}b_k\frac{1}{a(u)}\int_{t_1}^{u}\prod_{s<t_k<u}b_k\eta(s)q(s)\xi(t-\sigma)dsdudv=\infty, \tag{2.34}$$

where

$$\eta(t)=(1-p(t-\sigma))\frac{\xi(t-\tau-\sigma)}{\xi(t-\sigma)}>0, \quad \xi(t)=\int_t^{\infty}\frac{1}{b(s)}ds, \tag{2.35}$$

then every solution $x(t)$ of equation (1.1) is either oscillatory or $\lim_{t\to\infty} x(t) = 0.$

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may suppose that $x(t) > 0, x(t-\tau) > 0,$ and $x(t-\sigma) > 0$ for $t \geq t_1 \geq t_0$. For $t \neq t_k, t \geq t_1$ from (1.4), there exist four possible cases (1), (2), (3)(as in Theorem 2.2) and

(4) $z(t) > 0, z'(t) < 0, (b(t)z'(t))' < 0, [a(t)(b(t)z'(t))']' \leq 0,$

For the cases (1),(2) and (3), we obtain the conclusion from Theorem 2.2. Now assume that case (4) holds. Since $b(t)z'(t)$ is non increasing, we have

$$b(s)z'(s) \leq b(t)z'(t), s \geq t \geq t_6 \geq t_5. \tag{2.36}$$

Dividing (2.36) by $b(s)$ and then integrating from t to $\ell,$ and letting $\ell \to \infty,$ we have

$$z(t) \geq -b(t)z'(t)\int_t^{\infty} \frac{1}{b(s)} ds = -b(t)z'(t)\xi(t) := M\xi(t), t \neq t_k. \tag{2.37}$$

where $M = -b(t)z'(t) > 0.$ Hence

$$\left(\frac{z(t)}{\xi(t)}\right) \geq 0. \tag{2.38}$$

From (2.38), we see that

$$\begin{aligned} x(t) &= z(t) - p(t)x(t-\tau) \geq z(t) - p(t)z(t-\tau) \\ &\geq \left(1 - p(t)\frac{\xi(t-\tau)}{\xi(t)}\right) z(t), t \neq t_k. \end{aligned} \tag{2.39}$$

From equation (1.1), (2.37) and (2.39), we have

$$(a(t)(b(t)z'(t))')' \leq -Mq(t)\eta(t)\xi(t-\sigma). \tag{2.40}$$

From equation (2.16), we have

$$b(t_k^+)z'(t_k^+) \leq b_k b(t_k)z'(t_k). \tag{2.41}$$

Using (2.41), we have

$$\begin{aligned} (b(t_k^+)z'(t_k^+))' &= b'(t_k^+)z'(t_k^+) + b(t_k^+)z''(t_k^+) \\ &\leq b'(t_k)b_k z'(t_k) + b(t_k)z''(t_k) \\ &\leq b_k(b(t_k)z'(t_k))'. \end{aligned} \tag{2.42}$$

Using Lemma 2.1 in (2.40) and (2.42) for $t_6 > t_5,$ we obtain

$$a(t)(b(t)z'(t))' \leq a(t_6)(b(t_6)z'(t_6))' \prod_{t_6<t_k<t} b_k - M\int_{t_6}^{t} \prod_{s<t_k<t} b_k q(s) \eta(s)\xi(s-\sigma)ds$$

or

$$(b(t)z'(t))' \leq -\frac{M}{a(t)}\int_{t_6}^{t} \prod_{s<t_k<t} b_k q(s)\eta(s)\xi(s-\sigma)ds. \tag{2.43}$$

Again using Lemma 2.1 in (2.41) and (2.43), we obtain

$$b(t)z'(t) \le b(t_6)z'(t_6)\prod_{t_6<t_k<t} b_k - M\int_{t_6}^{t} \prod_{u<t_k<t} b_k \frac{1}{a(u)}\int_{t_6}^{u}\prod_{s<t_k<u} b_k \, q(s)\eta(s)\xi(s-\sigma)dsdu.$$

Dividing the last inequality by b(t) and using (2.15) in Lemma 2.1, we have

$$z(t) \le z(t_6) - M\int_{t_6}^{t}\frac{1}{b(v)}\int_{t_6}^{v}\prod_{t_6<t_k<v} b_k\left(\frac{1}{a(u)}\int_{t_6}^{u}\prod_{s<t_k<u} b_k q(s)\eta(s)\xi(s-\sigma)ds\right)dudv.$$

Taking limit as $t \to \infty$ in the last inequality we get a contradiction with (2.34). This completes the proof.

3 EXAMPLES

In this section we provide two examples to illustrate the main results.

Example 3.1 *Consider the following third order impulsive differential equation*

$$\begin{cases} [e^{\frac{t}{2}}(x(t)+\frac{1}{2e}x(t-1))'']' + \frac{3}{4}e^{\frac{t}{2}-2}x(t-2) = 0, t \ge 3, t \ne t_k; \\ x(t_k^+) = \left(\frac{1}{k}\right)x(t_k), \quad x'(t_k^+) = \left(\frac{k+1}{k}\right)x'(t_k) \\ x''(t_k^+) = \left(\frac{1}{k}\right)x''(t_k), k = 1,2,\dots \end{cases} \tag{3.1}$$

Here $a(t) = e^{\frac{t}{2}}, b(t) = 1, p(t) = \frac{1}{2e}, q(t) = \frac{3}{4}e^{\frac{t}{2}-2}, \tau = 1, \sigma = 2, a_k = c_k = \frac{1}{k}, b_k = \frac{k+1}{k}$. It is easy to see that all the conditions of Theorem 2.2 are satisfied with $\rho(t) = 1$. Hence any solution of equation (2.34) is either oscillatory or converging to zero.

Example 3.2 *Consider the following third order impulsive differential equation*

$$\begin{cases} [e^{t}(e^{t}(x(t)+\frac{1}{e}x(t-1))')']' + \frac{4}{e^2}x(t-2) = 0, t \ge 3, t \ne t_k; \\ x(t_k^+) = \left(\frac{1}{k}\right)x(t_k), \quad x'(t_k^+) = \left(1+\frac{1}{k}\right)x'(t_k) \\ x''(t_k^+) = x''(t_k), k = 1,2,\dots. \end{cases} \tag{3.2}$$

Here $a(t) = b(t) = e^t, p(t) = \frac{1}{e}, q(t) = \frac{4}{e^2}, \tau = 1, \sigma = 2, a_k = \frac{1}{k}, b_k = 1+\frac{1}{k}, c_k = 1$. It is easy to see that all the conditions of Theorem 2.3 are satisfied with $\rho(t) = 1$. Hence any solution of equation (3.1) is either oscillatory or converging to zero.

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