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Hermite collocation method for solving Hammerstein integral equations

Y. A. Amer[1], A. M. S. Mahdy[1, 2] and H. A. R. Namoos[3]
[1] Department of Mathematics, Faculty of Science, ZagazigUniversity, Zagazig, Egypt
yaser31270@yahoo.com
[2]Department of Mathematics, Faculty of Science, Taif University, Saudi Arabia
amr_mahdy85@yahoo.com
[3]Department of Mathematics, Faculty of Education Ibn-Al-Haitham, Baghdad University, Baghdad- Iraq
abdulah123s@yahoo.com

ABSTRACT

In this paper, we are presenting Hermite collocation method to solve numerically the Fredholm-Volterra-Hammerstein integral equations. We have clearly presented a theory to find ordinary derivatives. This method is based on replacement of the unknown function by truncated series of well knownHermite expansion of functions. The proposed method converts the equation to matrix equation which corresponding to system of algebraic equations with Hermite coefficients. Thus, by solving the matrix equation, Hermite coefficients are obtained. Some numerical examples are included to demonstrate the validity and applicability of the proposed technique.

Keywords

Fredholm-Hammerstein integral equations, Hermite polynomials, Volterra integral equation.

1. INTRODUCTION

The linear and non linearFredholm and Volterra integral equation have been a growing interest inrecent years ([21], [22]). This are an important branch of modern mathematics and arise frequently in many applied areas which include engineering, mechanics, physics, chemistry, astronomy and biology ([1], [5]). There are several methods for approximating the solution of linear, non-linear integral equations ([6]-[10], [13]). and solving fractional integro-differential equations ([11],[12]).

We consider the Hammerstein integral equations in the forms ([21],[22]):-

$$z(y) = f(y) + \lambda_1 \int_0^1 K_1(y,r)F\big(z(r)\big)dr + \lambda_2 \int_0^y K_2(y,r)G\big(z(r)\big)dr. \quad (1)$$

Where $f(y)$, $K_1(y,r)$ and $K_2(y,r)$ are given functions, $0 \le y, r \le 1$, and λ_1, λ_2 are arbitrary constants.

Hermite polynomials are widely used in numerical computation. One of the advantages of using Hermite polynomials as a tool for expansion functions is the good representation of smooth functions by finite Hermite expansion provided that the function $z(y)$ is infinitely differentiable ([2], [15], [18], [20]). The Hermite collocation method in [16] solving convection diffusion equation, in [17] solving linear differential equations and in [19] solving linear complex differential equation.

The paper is organized as follows: Section 2, we will study some properties of the Hermite polynomials. In Section 3, we take his idea about an approximate formula of the integral derivative. In Section 4, procedure solution using the proposed numerical method. In Section 5, we give numerical implementation. In Section 6, the paper ends with a brief conclusion.

2. Some properties of the Hermite polynomials

Definition:-The Hermite polynomials are given by ([2], [20]):-

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dz^n} e^{-y^2}.$$

Some main properties of these polynomials are :
The Hermite polynomials evaluated at zero argument $H_n(0)$ and are called Hermite number as follows ([2], [20]):-

$$H_n(0) = \begin{cases} 0, & if\ n\ is\ odd, \\ (-1)^{\frac{n}{2}} 2^{\frac{n}{2}} (n-1)! & if\ n\ is\ even, \end{cases} \quad (2)$$

Where $(n-1)!$ is the factorial.

The polynomials $H_n(y)$ are orthogonal with respect to the weight function $\omega(y) = e^{-y^2}$ with the following condition [2]:-

$$\int_{-\infty}^{\infty} H_n(y)H_m(y)\omega(y)dt = \sqrt{\pi}2^n n!\,\delta_{nm}.$$

3.An approximate formula of the fractional derivative

The Hermite polynomials are defined on $\mathbb{R}$ and can be determined with the aid of the following recurrence formula ([2], [14]):-

$$H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}(y), \quad H_0(y) = 1,\ H_1(y) = 2y, \quad n = 1,2,\ldots$$

The analytic form of the Hermite polynomials of degree n is given by [2]

$$H_n(y) = n!\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(-1)^k 2^{n-2k}}{(k)!(n-2k)!} y^{n-2k}. \tag{3}$$

In consequence, for the p-th derivatives of Hermite polynomials the following relation hold:

$$H_n^{(p)}(y) = 2^p \frac{n!}{(n-p)!} H_{n-p}(y) = v_{n,p} H_{n-p}(y), \quad v_{n,p} = 2^p \frac{n!}{(n-p)!}. \tag{4}$$

The function $z(y) \in L^2_{\omega(y)}(\mathbb{R})$, may be expressed in terms ofHermite polynomials as follows

$$z(y) = \sum_{k=0}^{\infty} c_k H_k(y) \tag{5}$$

where the coefficients c_n are given by

$$c_n = \frac{1}{\sqrt{\pi}2^n n!}\int_{-\infty}^{\infty} z(y)H_n(y)\,\omega(y)dy, \quad n = 0,1,\ldots \tag{6}$$

In practice, only the first $(m+1)$ $-$terms of Hermite polynomials are considered. Then we have [2]

$$z_m(y) = \sum_n^m c_n H_n(y). \tag{7}$$

The main approximate formula of the ordinary (integral) derivative is given in the following theorem.

Theorem 1. Let $z(y)$ be approximated by Hermite polynomials as (7) and also suppose $q > 0$, then

$$D^q\big(z_m(y)\big) \cong \sum_{n=q}^{m}\left[n!\sum_{k=q}^{\ell} c_n B_{n,k}^{(q)}\, y^{n-2k-q}\right], \tag{8}$$

where$\ell = \frac{n-q}{2}$ and $B_{n,k}^{(q)}$is given by

$$B_{n,k}^{(q)} = \frac{(-1)^k 2^{n-2k}}{(k)!\,(n-2k-q)!}$$

Proof: since the non linear operation we have

$$D^q\big(z_m(y)\big) = \sum_{n=0}^{m} c_n D^q(H_n(y)). \tag{9}$$

It is clear that $D^q H_n(y) = 0,\ n = 0,1,\ldots,q-1,\ q > 0$ Therefore, for $N = q, q+1, \ldots, m$.

Substituting equation (3) in (9) we have

$$D^q\big(z_m(y)\big) = \sum_{n=0}^{m} n!\,c_n \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(-1)^k 2^{n-2k}}{(k)!\,(n-2k)!} D^q y^{n-2k}.$$

$$D^q\big(z_m(y)\big) = \sum_{n=q}^{m} n!\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} c_n \frac{(-1)^k 2^{n-2k}}{(k)!(n-2k-q)!} y^{n-2k-q}. \tag{10}$$

4. Procedure solution using the proposed numerical method

We consider the Fredholm-Volterra integral equation (1), ([21], [22]).

We substitute the Eq. (7) into Eq. (1) we get:-

$$\sum_{n=0}^{m} c_n H_n(y) = f(y) + \lambda_1 \int_0^1 K_1(y,r) F\left(\sum_{n=0}^{m} c_n H_n(r)\right) dr + \lambda_2 \int_0^y K_2(y,r) G\left(\sum_{n=0}^{m} c_n H_n(r)\right) dr. \quad (11)$$

On the Hermite collocation points depended by:-

$$y_i = -1 + \frac{2}{m} i, \qquad i = 0,1,\dots,m \quad (12)$$

we collocate Eq.(11) with the points (12) to obtain

$$\sum_{n=0}^{m} c_n H_n(y_p) = f(y_j) + \lambda_1 \int_0^1 K_1(y_p,r) F\left(\sum_{n=0}^{m} c_n H_n(r)\right) dr + \lambda_2 \int_0^{y_p} K_2(y_p,r) G\left(\sum_{n=0}^{m} c_n H_n(r)\right) dr. \quad (13)$$

The integral terms in Eq. (13) can be found using composite trapezoidal integration technique as:

$$\int_0^1 K_1(y_p,r) F(z(r)) dr \cong \frac{h}{2}\left(\Omega_1(r_0) + \Omega_1(r_m) + 2\sum_{k=1}^{m-1} \Omega_1(r_k)\right). \quad (14)$$

Where $\Omega_1(r) = K_1(y_j,r)F(z(r)), h = \frac{1}{m}$, for an arbitrary integer m, $r_i = ih, i = 0,1,\dots,m$

And

$$\int_0^{y_p} K_2(y_p,r) G(z(r)) dr \cong \frac{h_p}{2}\left(\Omega_2(\overline{r_0}) + \Omega_2(\overline{r_m}) + 2\sum_{k=1}^{m-1} \Omega_2(\overline{r_k})\right). \quad (15)$$

Where $\Omega_2(r) = K_2(y_p,r)G(z(r)), h_p = \frac{y_p}{m}$, for an arbitrary integer m, $\overline{r_i} = ih$.

Eq. (13) gives$(m+1)$ system of linear or non-linear algebraic equations, which can be solved for $c_k, k = 0,1,\dots,m$.So the unknown function $z(y)$ can be found.

5. Numerical Implementation

In this section, to a chive the validity, the accuracy and support our theoretical discussion of the proposed method, we give some computational results of numerical examples.

Example1. Consider Eq. (1) with the following functions and coefficients ([21], [22]).

$$f(y) = y^3 - (6-2e)e^y, \ \lambda_1 = 1, \quad \lambda_2 = 1,$$

$$K_1(y,r) = e^{(y+r)}, \ K_2(y,r) = 0, \ F(z(r)) = z(r), \quad G(z(r)) = 0.$$

Eq. (1) takes following the form

$$z(y) = y^3 - (6-2e)e^y + \int_0^1 e^{(y+r)} z(r) dr. \quad (16)$$

We apply the suggested method with $m = 3$, and approximate the solution $z(y)$ as follows

$$z_m(y) = \sum_{n=0}^{3} c_n H_n(y). \quad (17)$$

By the same procedure in the previous section and using Eq. (13) we have

$$\sum_{n=0}^{m} c_n H_n(y) = y^3 - (6-2e)e^y + \frac{h}{2}\left(\Omega_1(r_0) + \Omega_1(r_m) + 2\sum_{k=1}^{m-1} \Omega_1(r_k)\right),$$

$$\sum_{n=0}^{3} c_n H_n(y_p) - \left(y_p^3 - (6-2e)e^{y_p}\right) - \frac{h}{2}\left(\Omega_1(r_0) + \Omega_1(r_3) + 2\sum_{k=1}^{2} \Omega_1(r_k)\right) = 0, \quad p = 0,1,2. \quad (18)$$

Where $\Omega_1(r) = e^{(r+y_p)} \sum_{n=0}^{3} c_n H_n(y_p)$ and the nodes$r_{\ell+1} = r_\ell + h, \ \ell = 0,1,\dots,m. r_0 = 0$ and $h = \frac{1}{m}$.

Eq. (18) represents linear system of $m+1$ algebraic equations in the coefficients c_i, by solving it using the conjugate gradient method.
The exact solution of this example is $z(y) = y^3$ ([21], [22]).

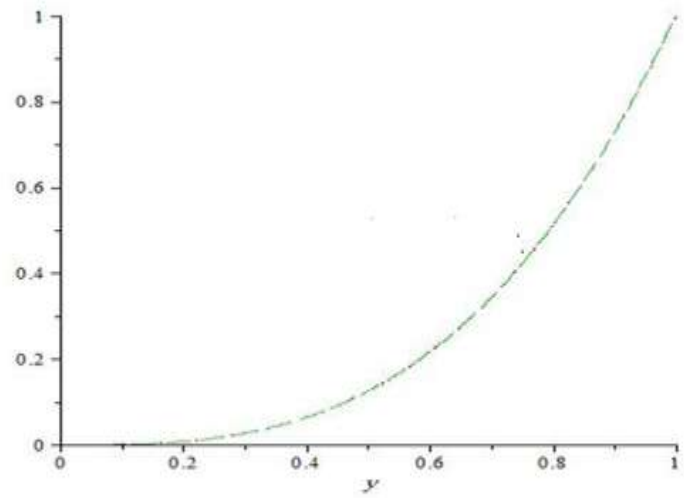

Fig. 1.The behavior of the exact solution and the approximate solution at $m = 3$.

The behavior of the approximate solution using the proposed method with $m = 3$ and the exact solution are presented in Figures, 1-4. From this Figures 1-4, it is clear that the proposed method can be considered as an efficient method to solve the linear integral equations ([21], [22]).

Example 2. Consider Eq. (1) with the following functions and coefficients ([21], [22]).

$$f(y) = 2ye^{y} - e^{y} + 1, \ \lambda_1 = 1, \quad \lambda_2 = -1,$$

$$K_1(y,r) = 0, \ \ K_2(y,r) = (r+y), \ \ F\big(z(r)\big) = 0, \ \ G\big(z(r)\big) = e^{z(r)}.$$

Eq. (1) takes following the form

$$z(y) = 2ye^{y} - e^{y} + 1 - \int_0^y (r+y)e^{z(r)}\, dr. \qquad (19)$$

We apply the suggested method with $m = 3$, and approximate the solution $z(y)$ as follows

$$z_m(y) = \sum_{n=0}^{3} c_n H_n(y). \qquad (20)$$

By the same procedure in the previous section and using Eq. (13) we have

$$\sum_{n=0}^{m} c_n H_n(y) = 2ye^{y} - e^{y} + 1 - \frac{h}{2}\left(\Omega_2(\overline{r_0}) + \Omega_2(\overline{r_m}) + 2\sum_{k=1}^{m-1} \Omega_2(\overline{r_k})\right),$$

$$\sum_{n=0}^{3} c_n H_n(y_p) - \left(2y_p e^{y_p} - e^{y_p} + 1\right) + \frac{h}{2}\left(\Omega_2(\overline{r_0}) + \Omega_2(\overline{r_3}) + 2\sum_{k=1}^{2} \Omega_2(\overline{r_k})\right) = 0, \quad p = 0,1,2. \qquad (21)$$

Where the nodes$r_{\ell+1} = r_\ell + h, \ell = 0,1,\dots,m, r_0 = 0$and $h_p = \frac{y_p}{m}$, $\Omega_2(r)(r + y_p)e^{z(r)}$.

Eq. (21) presents non-linear system of $m + 1$ algebraic equations in the coefficientsc_i. by solving it by using the Newton iteration method with suitable initial solution.
The exact solution of this problem is $z(y) = y$ ([21], [22]).

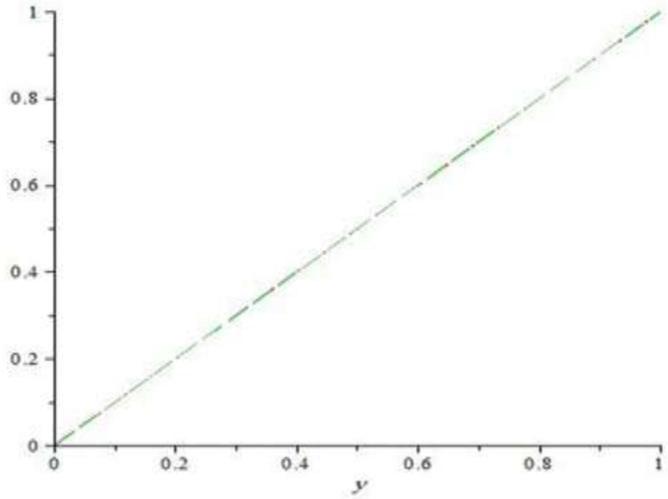

Fig. 2. The behavior of the exact solution and the approximate solution at $m = 3$.

Example 3. Consider Eq. (1) with the following functions and coefficients ([21], [22]).

$$f(y) = y^e + 1, \ \ \lambda_1 = -1, \quad \lambda_2 = 1,$$

$$K_1(y,r) = r + y, \ \ K_2(y,r) = 0, \ \ F\big(z(r)\big) = e^{z(r)}, \quad G\big(z(r)\big) = 0.$$

Eq. (1) takes following the form

$$z(y) = y^e + 1 - \int_0^1 (r + y)e^{z(r)}\, dr. \tag{22}$$

We apply the suggested method with $m = 3$, and approximate the solution $z(y)$ as follows

$$z_m(y) = \sum_{n=0}^{3} c_n H_n(y). \tag{23}$$

By the same procedure in the previous section and using Eq. (13) we have

$$\sum_{n=0}^{m} c_n H_n(y) = y^e + 1 - \frac{h}{2}\left(\Omega_1(r_0) + \Omega_1(r_m) + 2\sum_{k=1}^{m-1} \Omega_1(r_k)\right),$$

$$\sum_{n=0}^{3} c_n H_n\big(y_p\big) - (y^e + 1) + \frac{h}{2}(\Omega_1(r_0) + \Omega_1(r_3) + 2\sum_{k=1}^{2} \Omega_1(r_k)) = 0, \quad p = 0,1,2. \tag{24}$$

Where the nodes$r_{\ell+1} = r_\ell + h, \ell = 0,1, \dots, m, r_0 = 0$and $h = \frac{1}{m}$, $\Omega_1(r) = (r + y)e^{z(r)}$.

Eq. (24) presents non-linear system of algebraic equations by solving it using the well known Newton iteration method with suitable initial solution.
The exact solution of this problem is $z(y) = y$ ([21], [22]).

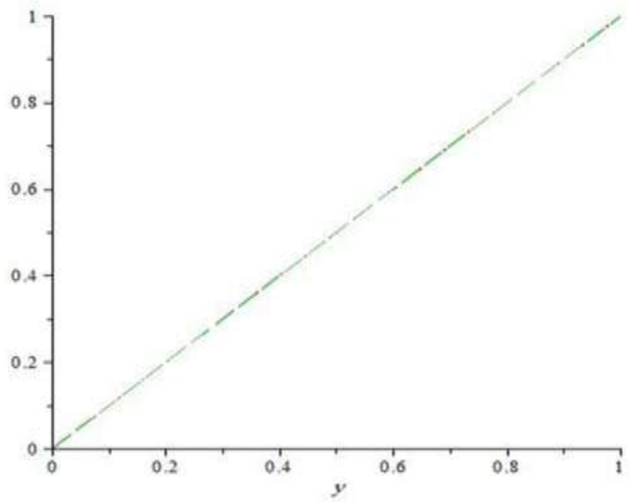

Fig. 3.The behavior of the exact solution and the approximate solution at $m = 3$.

Example 4. Consider Eq. (1) with the following functions and coefficients ([21], [22]).

$$f(y) = \frac{y}{2} - \frac{y^4}{12} - \frac{1}{3}, \quad \lambda_1 = 1, \quad \lambda_2 = -1,$$

$$K_1(y,r) = r + y, \quad K_2(y,r) = r - y, \quad F\big(z(r)\big) = z(r), \quad G\big(z(r)\big) = z^2(r).$$

Eq. (1) takes the following form

$$z(y) = \frac{y}{2} - \frac{y^4}{12} - \frac{1}{3} + \int_0^1 (r + y) z(r)\, dr + \int_0^y (r - y)\, z^2(r) dr. \tag{25}$$

We apply the suggested method with $m = 3$, and approximate the solution $z(y)$ as follows

$$z_m(y) = \sum_{n=0}^{3} c_n H_n(y). \tag{26}$$

By the same procedure in the previous section and using Eq. (13) we have

$$\sum_{n=0}^{m} c_n H_n(y) = \frac{y}{2} - \frac{y^4}{12} - \frac{1}{3} + \frac{h}{2}\left(\Omega_1(r_0) + \Omega_1(r_m) + 2\sum_{k=1}^{m-1} \Omega_1(r_k)\right)$$
$$- \frac{h}{2}\left(\Omega_2(\overline{r_0}) + \Omega_2(\overline{r_m}) + 2\sum_{k=1}^{m-1} \Omega_2(\overline{r_k})\right),$$

$$\sum_{n=0}^{3} c_n H_n(y_p) - \left(\frac{y}{2} - \frac{y^4}{12} - \frac{1}{3}\right) - \frac{h}{2}\left(\Omega_1(r_0) + \Omega_1(r_3) + 2\sum_{k=1}^{2} \Omega_1(r_k)\right)$$
$$+ \frac{h}{2}\left(\Omega_2(\overline{r_0}) + \Omega_2(\overline{r_3}) + 2\sum_{k=1}^{2} \Omega_2(\overline{r_k})\right) = 0, p = 0,1,2. \tag{27}$$

Where the nodes$r_{\ell+1} = r_\ell + h,\ \bar{r}_{\ell+1} = \bar{r}_\ell + h,\ \ell = 0,1, \dots, m,\ r_0 = \bar{r}_0 = 0$ and $h = \frac{1}{m}, h_p = \frac{y_p}{m}$ and $\Omega_1(r) = (r + y)z(r),\ \Omega_2(r) = \big(y_p - r\big)z^2(r).$

Eq. (27) presents non-linear system of $m + 1$ algebraic equations by solving it using Newton iteration method.

The exact solution of this problem is $z(y) = y$ ([21], [22]).

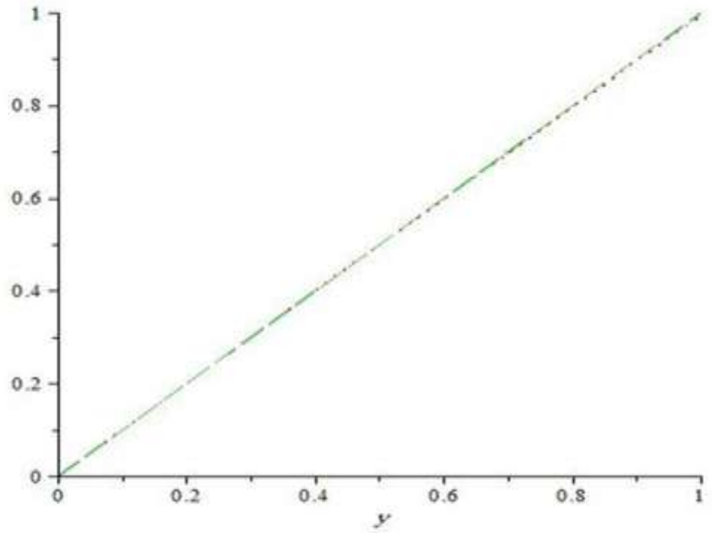

Fig. 4. The behavior of the exact solution and the approximate solution at $m = 3$.

6. Conclusion

In this articals, we approximate method for the solution of linear and non-linear Fredholm-Volterra integral equations using Hermite collocation method. A comparison of the exact solution reveals that the presented method is very effective and convenient. The numerical results show that the accuracy improves with increasing m, hence for better results, using number m is recommended. Also, from the obtained approximate solution, we can conclude that the proposed method gives the solution in an excellent agreement with the exact solution. All computations are done using Maple programming.

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