The solvable subgroups of large order of L2(p) , p≥5

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Abstract

By using the following theoretical and computational algorithms , we determined the solvable subgroups of large order of the finite non-abelian simple linear groups G = $L_2(p)$ = PSL(2,p) , for p≥5 and p is a prime number , also their presentations and permutation representations have been found .

Theoratical algorithm

In this section we study theoreticaly the following :

- Determining the solvable subgroups of large order S of $L_2(p)$, p≥5 and finding their structures up to isomorphisms.
- Finding the presentation of S ,also we find its generators from its character table.
- Finding the permutation representations of S .

Determining S

Since any solvable subgroup of large order S of G is either one of the maximal subgroups of G or it is contained in one of them , so we have to deal with the maximal subgroups of G . We begin by stating Dickson's results [8] about the maximal subgroups of PSL(2,p) = $L_2(p)$, p is an odd prime number . The result is divided according to p .

Theorem [8]

Let p = 2^f ≥ 4 . Then the maximal subgroups of PSL(2,p) are :

(1) . $C_2^f \rtimes C_{p-1},$ that is , the stabilizer of a point of the projective line ,
(2) . $D_{2(p-1)},$
(3) . $D_{2(p+1)},$
(4) . PGL(2,p_0), where p= p_0^r for some prime r and $p_0 \neq 2$.

Theorem [8]

Let $q = p^f \geq 5$ with p an odd prime. Then the maximal subgroups of $PSL(2,q)$ are:

(1) . $C_p^f \rtimes C_{(q-1)/2,}$ that is , the stabiliser of a point of a projective line ,
(2) . $D_{q-1,}$ for $q \geq 13$,
(3) . $D_{q+1,}$ for $q \neq 7,9$,
(4) . $PGL(2,q_0)$ for $q = q_0^2$ (2 conjugacy classes),
(5) . $PSL(2,q_0)$, for $q = q_0^r$ where r an odd prime,
(6) . $A_{5,}$ for $q \equiv \pm 1 (mod\ 10)$, where either $q = p$ or $q = p^2$and $p \equiv \pm 3$ $(mod\ 10)$ (2conjugacy classes),
(7) . $A_{4,}$ for $q = p \equiv \pm 3 (mod\ 8)$ and $q \not\equiv \pm 1 (mod\ 10)$,
(8) . $S_{4,}$ for$q = p \equiv \pm 1 (mod\ 8)$ (2 conjugacy classes).

Now , if we put $q = p^1 \geq 5$, we get the following corollary as a result of the above theorem:

Corollary.

Let p ≥ 5 , p is a prime number . Then the maximal subgroups of L_2 (p) are :

(1) $C_p \rtimes C_{(p-1)/2}$
(2) D_{p-1} , for p ≥ 13
(3) D_{p+1}, for p ≠ 7,9.
(4) A_5 , for p ≡ ±1 (mod 10)
(5) A_5 , for p ≡ ±3 (mod 10) (2 conjugacy classes)

(6) A_4, for $p \equiv \pm 3 \pmod 8$ and $p \not\equiv \pm 1 \pmod{10}$
(7) S_4, for $p \equiv \pm 1 \pmod 8$ (2 conjugacy classes)

Proposition.

(1) The dihedral groups D_{2n} of order 2n are solvable .
(2) The symmetric group S_4 is solvable
(3) A_5 is not solvable
(4) S_3 and A_4 are solvable
(5) If $H \lhd G$ and both H and G / H are solvable then G is solvable .
(6) The semi direct product $C_p \rtimes C_q$, where p and q are odd primes , is solvable

Theorem. Let H be the solvable subgroup of large order of $L_2(p)$. Then

(1) $H \cong A_4$, for p=5
(2) $H \cong S_4$, for p=7
(3) $H \cong C_p \rtimes C_{(p-1)/2}$, for p≥11

Proof :

(1) For p=5 , the maximal subgroups of $L_2(5)$ are A_4 of order 12 , D_{10} of order 10 and S_3 of order 6 and all are solvable . But 12 is the largest order , so A_4 is the solvable subgroup of large order in $L_2(5)$.

(2) For p=7 , the maximal subgroups of $L_2(7)$ are S_4 of order 24 and $C_7 \rtimes C_3$ of order 21 and both are solvable . But $24 > 21$, S_4 is the solvable subgroup of large order in $L_2(7)$.

(3) For p≥11 , by Corollary 4.1.1.3 , the orders of the maximal subgroups of $L_2(p)$ are as follows:

$\left| \ C_p \rtimes C_{(p-1)/2} \right| = \frac{p(p-1)}{2}$ (solvable)

$|D_{p-1}| = p - 1$ (solvable)

$|D_{p+1}| = p + 1$ (solvable)

$|A_5| = 60$($A_5$is not solvable and also it does not contain a subgroup of order large than 12)

$|A_4| = 12$ (solvable)

$|S_4| = 24$ (solvable)

Now , it is clear that $\frac{p(p-1)}{2}$ is greater than both (p-1) and (p+1) . Also for the smallest p=11 , we have $\frac{p(p-1)}{2} = \frac{11(11-1)}{2} = \frac{11\times 10}{2} = 55$ and $55 > 12$ and $55 > 24$

So , the largest order is $\frac{p(p-1)}{2}$ and then $C_p \rtimes C_{\frac{(p-1)}{2}}$ is the solvable subgroup of large order of $L_2(p)$, where p≥11.

presentation of S .

The finite non-abelian simple group $L_2(p)$, p≥5 and p is a prime number , of order $\frac{p(p-1)(p+1)}{2}$ can be presented as, [2] :

$L_2(p) = \langle a, b : a^2 = b^3 = (ab)^p = 1 \rangle$, for p≥5

The presentations of the solvable subgroup of large order S of $L_2(p)$,p≥5 are as follows :(By using theorem 4.1.1.5and [11])

p	G	S	$\lvert S \rvert$	[G : S]	Presentation of S
5	$L_2(5)$	A_4	12	5	$\langle a, b : a^2 = b^5 = (ab)^2 = 1 \rangle$ [ ]
7	$L_2(7)$	S_4	24	7	$\langle a, b : a^2 = b^7 = (ab)^2 = 1 \rangle$ [ ]
p≥11	$L_2(p)$,	$C_p \rtimes C_{\frac{(p-1)}{2}}$	$\frac{p(p-1)}{2}$	P+1	$S = \left\langle a, b \ \middle|\ a^p = b^{\frac{(p-1)}{2}} = e , aba^{-1} = b^k \right\rangle$ with$\left(\frac{(p-1)}{2}, k\right)$=1 **And it is a (p , $\frac{(p-1)}{2}$, $\frac{(p-1)}{2}$) –subgroup in $L_2(p)$**

The permutation representations of S :

Let $G = L_2(5)$ and $S \cong A_4$

From the Character table of G and S

The character table of $A_5 \cong L_2(5)$

```
       2  2  2  .  .  .
       3  1  .  1  .  .
       5  1  .  .  1  1

          1a 2a 3a 5a 5b
       2P 1a 1a 3a 5b 5a
       3P 1a 2a 1a 5b 5a
       5P 1a 2a 3a 1a 1a

X.1       1  1  1  1  1
X.2       3 -1  .  A *A
X.3       3 -1  . *A  A
X.4       4  .  1 -1 -1
X.5       5  1 -1  .  .

       A = (1-Sqrt(5))/2
```

The character table of A_4

```
       2  1  1  .  .
       3  1  .  1  1

          1a 2a 3a 3b
       2P 1a 1a 3b 3a
       3P 1a 2a 1a 1a

X.1       1  1  1  1
X.2       1  1  A /A
X.3       1  1 /A  A
X.4       3 -1  .  .

A = (-1-Sqrt(-3))/2
```

We have :$L_2(5)$ have 5 conjugacy classes of elements : 1a,2a,3a,5a,5b (of order 1,2,3,5,5 respectively) and A_4 have 4 conjugacy classes of elements 1a,2a,3a,3b (of orders 1,2,3,3 respectively) . So, we have :

$\lvert C_G\,(a)\rvert$	60	4	3	5	5
CL(G)	1a	2a	3a	5a	5b
CL(S) fused up to CL(G)	1a	2a	3a 3b		
$\lvert C_S\,(a)\rvert$	12	4	3 3		
Permutation character $\chi = 1_S \uparrow^G = \frac{\lvert C_G\,(a)\rvert}{\lvert C_S\,(a)\rvert}$ (reducible character)	5	1	1+1=2	0	0
χ splits to 2 irreducible characters	1	1	1	1	1
	4	.	1	-1	-1

And so , the induced Character is : $1_S \uparrow^G = 1a + \ 4a$

Let G = $L_2(7)$ and S=S_4: From the Character table of G and S :

The character table of S_4

```
       2  3  2  3  .  2
       3  1  .  .  1  .

          1a 2a 2b 3a 4a
       2P 1a 1a 1a 3a 2b
       3P 1a 2a 2b 1a 4a

X.1       1 -1  1  1 -1
X.2       3 -1 -1  .  1
X.3       2  .  2 -1  .
X.4       3  1 -1  . -1
X.5       1  1  1  1  1
```

The character table of $L_2(7)$

```
       2  3  .  .  .  3  2
       3  1  1  .  .  .  .
       7  1  .  1  1  .  .

          1a 3a 7a 7b 2a 4a
       2P 1a 3a 7a 7b 1a 2a
       3P 1a 1a 7b 7a 2a 4a
       5P 1a 3a 7b 7a 2a 4a
       7P 1a 3a 1a 1a 2a 4a

X.1       1  1  1  1  1  1
X.2       3  .  A /A -1  1
X.3       3  . /A  A -1  1
X.4       6  . -1 -1  2  .
X.5       7  1  .  . -1 -1
X.6       8 -1  1  1  .  .

A = (-1-Sqrt(-7))/2
```

We have $L_2(7)$ have 6 conjugacy classes of elements : 1a ,2a ,3a ,4a, nd S_4 have 5 conjugacy classes of elements 1a,2a,2b,3a,4a (of orders 1,2,2,3

C_G (a)|	168	8	3	4	7	7
CL(G)	1a	2a	3a	4a	7a	7b
CL(S) fused up to CL(G)	1a	2a 2b	3a	4a		
|C_S (a)|	24	4 8	3	4		
Permutation character $\chi = 1_S \uparrow^G = \frac{\|C_G(a)\|}{\|C_S(a)\|}$ (reducible)	7	2+1=3	1	1	0	0
χ splits to 2 irreducible characters of G	1	1	1	1	1	1
	6	2	0	0	-1	-1

And So , the induced Character is : $1_S \uparrow^G = 1a + 6a$

G=L_2(p),and S= $C_p \rtimes C_{\frac{(p-1)}{2}}$(where p ≥ 11).

The conjugacy classes , representations and the character tables of G have been found by adams [2] , as follows :

Conjugacy Classes of G	Representations of G
1. I 2. $c_2(\epsilon,\gamma) = \begin{pmatrix} \epsilon & \gamma \\ 0 & \epsilon \end{pmatrix}(\epsilon = \pm 1, \gamma \in \{1,\Delta\})$ 3. $c_3(x)(x \neq \pm 1), c_3(x) = c_3(-x) = c_3\left(\frac{1}{x}\right) = c_3\left(-\frac{1}{x}\right)$ 4. $c_4(z)(z \in \mathbb{E}^1, z \neq \pm 1), c_4(z) = c_4(\bar{z}) = c_4(-z) = c_4(-\bar{z})$	1. $\rho(\alpha)(\alpha^2 \neq 1), \rho(\alpha) \simeq \rho(\alpha^{-1})$ 2. $\bar{\rho}(1)$ 3. $\rho'(1)$ 4. $\pi(\chi)(\chi^2 \neq 1, \chi \neq \bar{\chi}), \pi(\chi) \simeq \pi(\bar{\chi})$ 5. $\omega_e^{\pm}$ if ζ (-1) = 1 6. $\omega_o^{\pm}$ if ζ (-1) = -1

Character Table of $PSL(2,q), q \equiv 1$ mod (4)							
		Number :	1	2	$\frac{q-5}{4}$	1	$\frac{q-1}{4}$
		Size :	1	$(q^2-1)/2$	$q(q+1)$	$\frac{q(q+1)}{2}$	$q(q-1)$
Rep	Dimension	Number	1	$c_2(\gamma)$	$c_3(x)$	$c_3(\sqrt{-1})$	$c_4(z)$
$\rho(\alpha)$	$q+1$	$\frac{q-5}{4}$	$(q+1)$	1	$\alpha(x)+\alpha(x^{-1})$	$2\alpha(\sqrt{-1})$	0
$\bar{\rho}(1)$	q	1	q	0	1	1	-1
$\rho'(1)$	1	1	1	1	1	1	1
$\pi(\chi)$	$q-1$	$\frac{q-1}{4}$	$(q-1)$	-1	0	0	$-\chi(z) - \chi(z^{-1})$
$\omega_e^{\pm}$	$\frac{q+1}{2}$	2	$\frac{q+1}{2}$	$\omega_e^{\pm}(1,\gamma)$	$\zeta(x)$	$\zeta(\sqrt{-1})$	0

Character Table of $PSL(2,q), q \equiv 3$ mod (4)							
		Number :	1	2	$\frac{q-3}{4}$	$\frac{q-7}{4}$	1
		Size :	1	$(q^2-1)/2$	$q(q+1)$	$q(q-1)$	$\frac{q(q-1)}{2}$
Rep	Dimension	Number	1	$c_2(\gamma)$	$c_3(x)$	$c_4(z)$	$c_4(\delta)$

$\rho(\alpha)$	$q+1$	$\frac{q-3}{4}$	$(q+1)$	1	$\alpha(x)+\alpha(x^{-1})$	0	0
$\bar{\rho}(1)$	q	1	q	0	1	-1	1
$\rho^{'}(1)$	1	1	1	1	1	1	1
$\pi(\chi)$	$q-1$	$\frac{q-3}{4}$	$(q-1)$	-1	0	$-\chi(z)$ $-\chi(z^{-1})$	$-2\chi(\delta)$
$\omega_o^{\pm}$	$\frac{q-1}{2}$	2	$\frac{q-1}{2}$	$\omega_o^{\pm}(1,\gamma)$	0	$-\chi_0(z)$	$-\chi_0(\delta)$

Property [10]

Let A be a normal subgroup of G such that A is the centralizer of every non-trivial element in A . If further G/A is abelian, than G has $|G{:}A|$linear characters, and $(|A|-1)/|G:A|$non-linear irreducible characters of degree =$|G{:}A|$.

Theorem

Let G = $L_2(p)$, p ≥ 11 , *and let* S= $C_p \rtimes C_{\frac{(p-1)}{2}}$. Then S has $\frac{p+3}{2}$ conjugacy classes of elements .

Proof:

Since S= $C_p \rtimes C_{\frac{(p-1)}{2}} \Rightarrow$ (*from the definition*) , $C_p \trianglelefteq S \Rightarrow$ every non-trivial element of C_p has centralizer of order p and isomorphic to C_p . Now, S/$C_p \cong C_{\frac{(p-1)}{2}}$ is cyclic , and so it is abelian. So , by applying theorem 4.1.3.4 ⇒ S has [S : C_p] = $\frac{(p-1)}{2}$ linear characters and $(|C_p|-1)/$[S : C_p] = $\frac{p-1}{\frac{p-1}{2}}$ =2 non-linear irreducible characters of degree $\frac{p-1}{2}$. Then , totally , S has $\frac{p-1}{2}+2=\frac{p+3}{2}$ irreducible characters and so by corollary1.9.7.(The number of conjugacy classes is equal to the number of irreducible characters) , ⇒The number of conjugacy classes of S = $\frac{p+3}{2}$, p≥11.

Theorem. Let S = $C_p \rtimes C_{\frac{(p-1)}{2}}$, p≥11 . Then S has the following conjugacy classes of elements:

1- The identity .
2- 2 classes of order p .
3- If $\frac{p-1}{2}$ is a prime number , then S has $\frac{p-3}{2}$ classes of elements order $\frac{p-1}{2}$
4- If $\frac{p-1}{2}$ is not a prime number , then S has $\frac{p-3}{2}$ classes of elements of order m | $\frac{p-1}{2}$

Proof :

1- Since S is a group then it hase an identity element which is unique .
2- From the character tables of G mentioned above with respect to both cases p≡1 (mod 4) and p≡3 (mod 4) , we find that G has only 2 conjugate classes of types C_2 (γ) and $\overline{C_2(\gamma\Upsilon)}$ and each class is of size $\frac{p^2-1}{2}$, and the centralizer of an element in each class is of order p. Now the sylow p-subgroup of S= $C_p \rtimes C_{\frac{(p-1)}{2}}$ is isomorphic to C_p and so S has conjugacy classes of elements of order p and they are must be only tow conjugate classes, for , if they are $>2 \Rightarrow$ they must be at least 4 conjugacy classes and 2 of them are fused to $C_2(\gamma) \in$ G and the remaining are fused to $\overline{C_2(\gamma\Upsilon)}$

G: 1a $C_2(\gamma)$ $\overline{C_2(\gamma\Upsilon)}$

1 1 From character tables of G →

if S has at least 4 conjugacy classes $a, \bar{a}, b, \bar{b}$, they will be fused to :

a	$\bar{a}$
$\bar{b}$	$\bar{b}$
2	2

which means the perm. Character is :

which is imposible because the value must be equal 1

⇒ S has only 2 conjugacy classes of e... ...der p , and therefore each class is of size = $\frac{|S|}{p} = \frac{p-1}{2}$

3- If p≥11 and q= $\frac{p-1}{2}$ is also a prime number ⇒ S = $C_p \rtimes C_q$ ⇒ S has elements of only orders 1,p,q and it has no elements of order pq because S is not cyclic . Now, we have the numbers of conjugacy classes of type $\frac{p-1}{2} = \frac{p+3}{2} - 1 - 2 = \frac{p+3-2-4}{2} = \frac{p-3}{2}$ and since $\frac{p-1}{2}$ is prime ⇒ the centralizers of elements of order $\frac{p-1}{2}$ have the same order and so each of these classes contains p elements , and all are lieing in $C_3(x)$ ∈ G and then we have :

	$\lvert C_S(a)\rvert$ | Number |Cs(a) | of classes	Order of elements	Classes Fusions
Any prime p≥11	1 1 $\lvert S\rvert$ 2 p p	1 $\frac{p-1}{2}$	1 $C_2(\gamma)$ For $\lvert C_S(a)\rvert = p$ Which divides only $\lvert C_G(C2(\gamma))\rvert$
When $\frac{p-1}{2}$ is a prime number	$\frac{p-3}{2}$ $\frac{p-1}{2}$ p	$\frac{p-1}{2}$	C_3 (x) For $\lvert C_S(a)\rvert = \frac{p-1}{2}$ Which divides only $\lvert C_G(C3(x))\rvert$
S	$\frac{p+3}{2}$ and $\lvert S\rvert = 1 + \frac{p-1}{2} \times 2 + \left(\frac{p-3}{2}\right) \times p = \frac{p(p-1)}{2}$		

4- If p ≥ 11 and $\frac{p-1}{2}$ is not a prime number,then S has elements of order 1,p and m$\left|\frac{(p-1)}{2}\right.$. We can easily show that S has $\frac{p-3}{2}$ conjugacy classes of order m and each class consist of p elements and has centralizers of order $\frac{p-1}{2}$ and all subgroups of order m in G have been determined in [7] , and we have :

	Number of Classes	Order of element a	Size of $\lvert C_S(a)\rvert$	$C_s(a)$	Classes Fusions up to G
Any prime p≥11	1 2	1 p	1 $\frac{p-1}{2}$	$\lvert S\rvert$ p	1 $C_2(\gamma)$ For $\lvert C_S(a)\rvert = p$ Which divides only $\lvert C_G(C2(\gamma))\rvert$
When $\frac{p-1}{2}$ is not a prime number	$\frac{p-3}{2} d\left\|\frac{p-1}{2}\right.$	total=p	$\frac{p-1}{2}$		C_3 (x) For $\lvert C_S(a)\rvert = \frac{p-1}{2}$ Which divides only $\lvert C_G(C3(x))\rvert$
S	$\frac{p+3}{2}$ and $\lvert S\rvert = 1 + \frac{p-1}{2} \times 2 + \left(\frac{p-3}{2}\right) \times p = \frac{p(p-1)}{2}$				

The permutation representations of S into G, $1_S \uparrow^G$ can be obtained from the following tow tables as follows :

1- **When $G = L_2(p), p \geq 11$ and $p \equiv 1 \pmod 4$**

Number of conjugacy classes:	1	2	$\frac{p-5}{4}$	1	$\frac{p-1}{4}$
Size of each class :	1	$(p^2-1)/2$	p(p+1)	$\frac{p(p+1)}{2}$	p(p-1)
Order of centralizers $C_G(a)$	$\|G\|$	P	$\frac{(p-1)}{2}$	(p-1)	$\frac{(p+1)}{2}$
Order of centralizers $C_S(a)$	$\|S\|$	P	$\frac{(p-1)}{2}$	No element in S has centralizer of order divides (p-1)	No element in S has centralizer of order divides $\frac{(p+1)}{2}$
Type of classes [a]	1	$c_2(\gamma)$	$c_3(x)$	$c_3(\sqrt{-1})$	$c_4(z)$
Irreducible characters	(1+q)	1	$\alpha(x) + \alpha(x^{-1})$	$2\alpha(\sqrt{-1})$	0
	q	0	1	1	-1
Reducible character (induced character $1_S\uparrow^G = \frac{\|C_G(a)\|}{\|C_S(a)\|}$) ←	1	1	1	-1	1
	(1+q)	1	2	0	0

2-When $G = L_2(p), p \geq 11$ and $p \equiv 3 \pmod 4$

Number of conjugacy classes:	1	2	$\frac{p-3}{4}$	1	$\frac{p-7}{4}$
Size of each class :	1	$(p^2-1)/2$	p(p+1)	$\frac{p(p-1)}{2}$	p(p-1)
Order of centralizers $C_G(a)$	$\|G\|$	p	$\frac{(p-1)}{2}$	(p+1)	$\frac{(p+1)}{2}$
Order of centralizers $C_S(a)$	$\|S\|$	P	$\frac{(p-1)}{2}$	No element in S has centralizer of order divides (p-1)	No element in S has centralizer of order divides $\frac{(p+1)}{2}$
Type of classes [a]	1	$c_2(\gamma)$	$c_3(x)$	$c_4(\delta)$	$c_4(z)$
Irreducible characters {	(q+1)	1	$\alpha(x) + \alpha(x^{-1})$	0	0
	q	0	1	1	-1
Reducible character (induced	1	1	1	-1	1
character $1_S\uparrow^G = \frac{\|C_G(a)\|}{\|C_S(a)\|}$)	(1+q)	1	2	0	0

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