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# On the Extended Hardy Transformation of Generalized Functions

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**Abstract**

The classical Hardy transformation is extended to a certain class of generalized functions namely ultradistributions. The derivative of the extended Hardy transformation is obtained.

**Key words:** Hardy Transform; Function Space; Generalized Function.

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## 1. Introduction

The classical Hardy transformation for a function $f$ is defined by

$$f(x)=\int_0^\infty F(tx)tdt\int_0^\infty C_v(ty)yf(y)dy \qquad (1.1)$$

where

$$C_v(z)=\cos(p\pi)J_v(z)+\sin(p\pi)Y_v(z) \qquad (1.2)$$

and

$$F_v(z)=\sum_0^\infty \frac{(-1)^m\left(\frac{1}{2}z\right)^{v+2p+2m}}{\prod(p+m+1)\prod(p+m+v+1)}=\frac{2^{2-v-2p}S_{v+2p^{-1},v}}{\prod(p)\prod(v+p)},$$

$S_{\mu,v}(z)$ is the Lommel's function [5].

If $p=0,$ then $C_v(z)=J_vC(z)$ and $F_v(z)=J_v\le(z)$. Hence, the transform reduces to the Hankels' formula

$f(x)=\int_0^\infty J_vC(xt)tdt\int_0^\infty J_vC(yt)yf(y)dy$. If $p=\frac{1}{2}$ then $C_v(z)=Y_v$ and $F_v(z)=H_v(z)$. This leads to the $Y$ - transformation

$$f(x)=\int_0^\infty H_v(xt)tdt\int_0^\infty Y_v(ty)yf(y)dy,$$

where $H_v(z)$ is the Struve function [5, p.328].

In what follows, we extend the classical Hardy transformation to a space of generalized functions by constructing a space of test functions where the classical transformation turns to be a closed mapping from the constructed space into itself. The extended transformation is then defined as an adjoint operator.

## 2. The Test Function Space $E(n_i,\alpha)$

Let $I$ be the open interval $(0,\infty)$ and $\alpha$ be a positive number such that $|v|\le\alpha\le\frac{1}{2}$. Let $n_i, i=1,2,3,\ldots$ be a sequence of positive real numbers. An infinitely smooth function $\phi(x)$ defined on $I$ is said to belong to the space $E(n_i,\alpha)$ if

$$y_x(\phi)\overset{\triangle}{=}\sup_{x\in I}\left|\xi(x)\triangle_x^k\phi(x)\right|<N_{n_i,n^i} \qquad (2.1)$$

for some positive constant $N$ and arbitrary $n,$ where

$$\xi_k(x)\triangleq\begin{cases}x^\alpha, 0<x\le 1\\ x^{\alpha-2}, x>1\end{cases},\ k=0,1,2,\ldots.$$

and

$\triangle_x=D_x^2+\frac{1}{x}D_x-\frac{v^2}{x^2}, D_x$ stands for the usual derivative with respect to $x$.

Let $E'(n_i,\alpha)$ be the dual of $E(n_i,\alpha)$, then the space $E'(n_i,\alpha)$ is a dual space of ultradifferentiable functions and wider than the space of Schwartz' distributions ; See [1,2,3] for further discussions.

**Lemma 2.1** Let $|v| \le \alpha \le \frac{1}{2}$. and $t,x>0$ then, for $x>0$, being fixed, we have

$$C_v(xt) \in E(n_i,\alpha)$$

**Proof** It can be easily observed that for some positive constant $N$ and all $y \in I$ we have

$$\left|y^{\alpha} C_v(y)\right| < N n^i n_i \,.$$

Hence

$$\sup_{y\in I}\left|y^{\alpha} C_v(y)\right| < N n^i n_i \,.$$

We have

$$\left|\xi(t)\Delta_t^x C_v(xt)\right| = \left|(-1)^x x^{2x} \xi(t) C_v(xt)\right| = \left|x^{2k-\alpha}(xt)^{\alpha} C_v(xt)\xi(t)t^{-\alpha}\right| \quad . \qquad (2.2)$$

Hence, multiplying (2.2) by $\frac{1}{Nn_i n^i}$ and considering the supremum over all $t\in(0,\infty)$ and for $n>0$ (being fixed), $|v| \le \alpha \le \frac{1}{2}$ we have ,

$$\sup_{0<t<\infty} \overset{\Delta}{=} \left|\xi(t)\Delta_t^k C_v(xt)\right| < N n_i n^i,$$

for some positive number $N$ and all $n$.

This completes the proof of the lemma.

## 3. The Testing Function Space $\Omega(n_i,\alpha)$

**Definition 3.1** An infinitely smooth function $\phi(x)$ defined on $I=(0,\infty)$ is said to belong to the space $\Omega(n_i,\alpha)$ if

$$\frac{\phi(x)}{m'(x)} \in E(n_i,\alpha), \text{ where } m'(x) = x.$$

Hence, the topology equipped with $\Omega(n_i,\alpha)$ is defined by the collection of seminorms

$$\delta_k(\phi) \overset{\Delta}{=} \sup_{x\in I}\left|\frac{\xi(x)\Delta_x^k \phi(x)}{m'(x)}\right| < N n^i n_i \,, \qquad (3.1)$$

for some positive $N$ and arbitary $n$.

The Dual of $\Omega(n_i,\alpha)$ is denoted by $\Omega'(n_i,\alpha)$ which consists of the set of all continuous linear forms on $\Omega(n_i,\alpha)$.

Let $f \in \Omega'(n_i,\alpha)$.Then, define $m'f \in E'(n_i,\alpha)$ by

$$(m'f,\phi) = \langle f, m'\phi\rangle, \phi \in E(n_i,\alpha).$$

Hence, if $f \in \Omega'(n_i,\alpha)$ then $m'f \in E'(n_i,\alpha)$

**Definition 3.2** Let $f \in \Omega'(n_i, \alpha)$. Define the extended Hardy transformation of $f$ by

$$H(y) = (\hbar f)(y) \overset{\Delta}{=} \langle m'(n) f(x), C_v(xy) \rangle, \quad (3.2)$$

where $y\ (y \neq 0)$ is a real number, $x > 0$.

Let $y\ (y \neq 0)$ be defined, then Lemma 2.1 implies

$$C_v(xy) \in E(n_i, \alpha),\ |v| \leq \alpha \leq \frac{1}{2}.$$

Hence (3.2) is meaningful.

**Theorem 3.3** Let $y\ (y \neq 0)$ be a real number and $H(y)$ be the extended Hardy transformation of $f \in \Omega'(n_i, \alpha)$ then

$$H^{(k)}(y) = \left\langle m'(x) f(x). \frac{\partial^k}{\partial y^k} C_v(xy) \right\rangle,$$

$k = 0, 1, 2, \ldots.$

**Proof** We intend to prove the theorem by the induction method on $k$. For $k = 1,$ we have

$$\frac{H(y+\Delta y) - H(y)}{\Delta y} - \left\langle m'(x) f(x), \frac{\partial^k}{\partial y} C_v(xy) \right\rangle$$

$$= \left\langle m'(x) f(x), \frac{C_v(x[y-\Delta y]) - C_v(xy)}{\Delta y} \frac{\partial}{\partial y} C_v(xy) \right\rangle.$$

It is sufficient to show that

$$B_{\Delta y} \overset{\Delta}{=} \frac{C_v(x[y+\Delta y]) - C_v(xy)}{\Delta y} - \frac{\partial^k}{\partial y} C_v(xy) \to 0$$

as $\Delta y \to 0$ in the topology of $\Omega(n_i, \alpha)$. We have

$$\frac{\xi(x)\Delta_x^k\beta_{\Delta y}}{Nn^i n_i} = \frac{\xi(x)(-1)^k}{Nn^i n_i}\times$$
$$\left[\frac{(y+\Delta y)^{2k}C_v(x(y+\Delta y)) - C_v(xy)y^{2k}}{\Delta y} - \frac{\partial}{\partial y}y^{2k}C_v(xy)\right]$$
$$= \frac{\xi(x)(-1)^k}{Nn^i n_i}$$
$$\times\left[\frac{1}{\Delta y}\int_y^{y+\Delta y}\frac{\partial}{\partial t}\left[t^{2k}C_v(tx)\right]dt - \frac{\partial}{\partial y}\left(y^{2k}C_v(tx)\right)\right]$$
$$= \frac{\xi(x)(-1)^k}{Nn^i n_i}$$
$$\times\left[\frac{1}{\Delta y}\int_y^{y+\Delta y}\frac{\partial}{\partial t}\left[t^{2k}C_v(tx)\right]dt - \left\{\frac{d}{dt}\left[\left(t^{2k}C_v(tx)\right)\right]\right\}_{t=y}\right]$$
$$= \frac{\xi(x)(-1)^k}{Nn^i n_i}\frac{1}{\Delta y}\int_y^{y+\Delta y}dt\int_y^t\frac{\partial^2}{\partial\eta^2}\left\{\eta^{2k}C_v(\eta x)\right\}d\eta.$$

Therefore

$$\left|\frac{\xi(x)\Delta_x^k\beta_{\Delta y}}{Nn^i n_i}\right| \le |\Delta y| \sup_{\substack{x\in I\\ y-|\Delta y|\le\eta\le y+|\Delta y|}} \frac{\left|\xi(x)\frac{d^2}{d\eta^2}\left\{\eta^{2k}C_v(\eta x)\right\}\right|}{Nn^i n_i}.$$

That is,

$$\left|\frac{\xi(x)\Delta_x^k\beta_{\Delta y}}{Nn^i n_i}\right| \le |\Delta y|\times$$
$$\sup_{\substack{x\in I\\ y-|\Delta y|\le\eta\le y+|\Delta y|}} \frac{\left|\xi(x)\left[4k\eta^{2k}xC_v'(\eta x) + 2k(2k-1)\eta^{2k-2}C_v'(\eta x) + \eta^{2k}C_v''(\eta x)x^2\right]\right|}{Nn^i n_i}$$

Due to the fact that

$\xi(x)C_v'(x) < \infty$ and $x\xi(x)C_v'(x) < \infty$ and $x^2\xi(x)C_v''(x) < \infty$ for all $x > 0$, we can find a constant $K$ such that

$$\left|\xi(x)C_v'(x)\right| < K, \left|x\xi(x)C_v'(x)\right| < K \text{ and } \left|x^2\xi(x)C_v''(x)\right| < K.$$

Once again, for $0 < |\Delta y| < \frac{1}{2}y$ and $y - |\Delta y| < \eta < y + |\Delta y|$ we can find a constant $A$ such that

$$\sup_{\substack{x\in I\\ y-|\Delta y|\le\eta\le y+|\Delta y|}} \frac{|\xi(x)|}{|\xi(\eta x)|} < A.$$

Therefore,

$$\left|\frac{\xi(x)\Delta_x^k\beta_{\Delta y}}{Nn^i n_i}\right| \le \frac{|\Delta y|}{Nn^i n_i}\times \sup_{\substack{x\in I\\ y-|\Delta y|\le\eta\le y+|\Delta y|}}\left|\frac{\xi(x)}{\xi(\eta x)}4k\eta^{2k-2}(\eta x)\xi(\eta x)C_v'(\eta x)\right|$$

$$+\frac{|\Delta y|}{Nn^i n_i}\sup_{\substack{x\in I\\ y-|\Delta y|\le\eta\le y+|\Delta y|}}\left|\frac{\xi(x)}{\xi(\eta x)}2k(2k-1)y^{2k-2}\xi(\eta x)C_v(\eta x)\right|$$

$$+\frac{|\Delta y|}{Nn^i n_i}\sup_{\substack{x\in I\\ y-|\Delta y|\le\eta\le y+|\Delta y|}}\left|\eta^{2k-2}(\eta x)^2\xi(\eta x)C_v''(\eta x)\right|$$

$$\le\frac{|\Delta y|}{Nn^i n_i}LQ\left[4k+2k(2k-1)+1\right]\sup_{\frac{y}{2}<\eta<\frac{3y}{2}}\eta^{2k-2}$$

$$\le\frac{|\Delta y|}{Nn^i n_i}LQ\left[4k^2+2k+1\right]\frac{\left(\frac{3}{2}y\right)}{\left(\frac{1}{2}y\right)^2}\to 0$$

as $\Delta y\to 0.$

This completes the proof of the theorem.

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