ISSN 2347-1921

Average number of Real Roots of Random Trigonometric Polynomial follows non-symmetric Semi-Cauchy Distribution

N.K.Sahoo
Faculty of Science and Technology
The ICFAI University Tripura
Kamalghat,Sadar, pin-799210, India.
nirmalkumarsahoo9@gmail.com
Dr.N.N.Nayak,
Ex-Prof of Mathematics Orissa University of Agriculture & Technology,
Bhubaneswar-3, Plot No.1242/2171, Siripur Nuasshi, India.
nnnayak1962@gmail.com

Abstract

Let $a_1(\omega), a_2(\omega), a_3(\omega) \ldots\ldots\ldots \ldots\ldots\ a_n(\omega)$ be a sequence of mutually independent, identically distributed random variables following semi-cauchy distribution with characteristic function $exp(-(C + coslog|t|)|t|), C > 1$. In this work, we obtain the average number of real zeros in the interval $(0,2\pi)$of trigonometric polynomials of the form

$$\binom{n}{1}a_1(\omega)cos\theta + \binom{n}{2}a_2(\omega)cos2\theta + \binom{n}{3}a_3(\omega)cos3\theta + \cdots + \binom{n}{n}a_n(\omega)cosn\theta$$

for large n. Here the required average is $\sim\left(\frac{2n}{\sqrt{2\pi-1}}\right), n \to \infty$.

Key Words and Phrases: Random variables; Joint distribution; Characteristic function; Semi-Cauchy distribution; Random trigonometric equations.

Council for Innovative Research
Peer Review Research Publishing System
Journal: Journal of Advances in Mathematics

Vol 2, No 2
editor@cirworld.com
www.cirworld.com, member.cirworld.com

ISSN 2347-1921

1. Introduction

Let $a_1(\omega), a_2(\omega), a_3(\omega) \dots\dots\dots \dots\dots\; a_n(\omega)$be a sequence of mutually independent, identically distributed random variables. Let $T_n(\theta)$denote the trigonometric polynomial

$$T_n(\theta) = \sum_{k=1}^{n} \binom{n}{k} a_k(\omega) cosk\theta = \sum_{k=1}^{n} b_k a_k(\omega) cosk\theta \qquad (1.1.1)$$

Where $b_k = \binom{n}{k}$. Let $N_n(\beta,\gamma)$ denote the number of real roots of $T_n(\theta) = 0$ in interval $\beta \le \theta \le \gamma$. Das [1] first time studied (1.1.1) polynomial, assuming that $b_k = k^\sigma$, he proved that the average number of zeros is $\left(\frac{2\sigma+1}{2\sigma+3}\right)^{\frac{1}{2}} 2n + 0(n)$ Where $b_k = \binom{n}{k}$. Let $N_n(\beta,\gamma)$ denote the number of real roots of $T_n(\theta) = 0$ in interval $\beta \le \theta \le \gamma$. Das [1] first time studied (1.1.1) polynomial, assuming that $b_k = k^\sigma$, he proved that the average number of zeros is $\left(\frac{2\sigma+1}{2\sigma+3}\right)^{\frac{1}{2}} 2n + 0(n)\; for\; \sigma \ge -\frac{1}{2}$ and of order $n^{(3/2+\sigma)}$ remaining cases. Also Sambandham [6] studied for non-identically distributed case taking $b_k = k^\sigma$ $(\sigma \ge 0)$ and showed that the average number is $\left(\frac{2\sigma+1}{2\sigma+3}\right)^{\frac{1}{2}} 2n + 0\left(n^{11+\frac{13}{n}}\right)$ except for a set of probability at $\frac{1}{n^{2\eta}}$ most where $0 < \eta < \frac{1}{13}$.

A recent work of Faramand[3], shows that the asymptotic formula for the expected number of real zeros of a algebraic polynomial of the form $\sum_{k=0}^{n} \binom{n}{k}^{\frac{1}{2}} a_k(\omega) x^k$ for large n, where the coefficients have identical normal distributions with $\mu \ne 0$ and unit variance, then $EN_n(-\infty,\infty) \sim \frac{\sqrt{n}}{2}$.

In this paper we consider random trigonometric polynomial whose coefficients are independent but not identically distributed. Also we consider $b_k = \binom{n}{k}$. with semi-Cauchy distribution. Das[1] and Sambandham[6] studied the case of normal distribution, which is a smooth curve where variance and moment are finite. But in semi-Cauchy distribution both mean and variance does not exist. We studied for all cases described the above and as such we have most generalized the case of Das and Sambandham.

Theorem: Let $T_n(\theta) = \sum_{k=1}^{n} b_k a_k(\omega) cosk\theta$, be a random trigonometric polynomial, where $a_k(\omega)$ are non-identically distributed random variables following semi-cauchy distribution with characteristic function $exp(-(C + coslog\mid t\mid)|t|)$, $C > 1$. If $N_n(0,2\pi)$ denote average number of real roots of $T_n(\theta) = 0$ in $[0, 2\pi]$, then the required average is $\sim\left(\frac{2n}{\sqrt{2\pi - 1}}\right)$, (as $n \to \infty$).

2. Preliminaries

First we partition the interval $(0,2\pi)$ into two types of intervals namely,

Type -1

$$\left\{\begin{array}{l} \left(\epsilon, \frac{\pi}{2k} - \epsilon\right) \bigcup_{k=2}^{n} \left(\frac{\pi}{2k} + \epsilon, \frac{\pi}{2(k-1)} - \epsilon\right) \bigcup_{k=2}^{n} \left(\pi - \frac{\pi}{2(k-1)} + \epsilon, \pi - \frac{\pi}{2k} - \epsilon\right) \\ \bigcup_{k=2}^{n} \left(\pi - \frac{\pi}{2k} + \epsilon, \pi + \frac{\pi}{2k} - \epsilon\right) \bigcup_{k=2}^{n} \left(\pi + \frac{\pi}{2k} + \epsilon, \pi + \frac{\pi}{2(k-1)} - \epsilon\right) \\ \bigcup_{k=2}^{n} \left(2\pi - \frac{\pi}{2(k-1)} + \epsilon, 2\pi - \frac{\pi}{2k} - \epsilon\right) \bigcup_{k=2}^{n} \left(2\pi - \frac{\pi}{2k} + \epsilon, 2\pi - \epsilon\right) \end{array}\right\} = \lambda(say)$$

Type-2

$$\left\{\begin{array}{c}\bigcup_{k=1}^{n}\left[\frac{\pi}{2k}-\epsilon,\frac{\pi}{2k}+\epsilon\right]\bigcup_{k=1}^{n}\left[\pi+\frac{\pi}{2k}-\epsilon,\pi-\frac{\pi}{2k}+\epsilon\right]\\ \bigcup_{k=1}^{n}\left[\pi+\frac{\pi}{2k}-\epsilon,\pi+\frac{\pi}{2k}+\epsilon\right]\bigcup_{k=1}^{n}\left[\pi-\frac{\pi}{2k}-\epsilon,\pi+\frac{\pi}{2k}+\epsilon\right]\\ \bigcup_{k=1}^{n}\left[2\pi-\frac{\pi}{2k}-\epsilon,\pi-\frac{\pi}{2k}+\epsilon\right]\bigcup[2\pi-\epsilon,2\pi]\end{array}\right\}=\lambda'(say)$$

Since the length of smallest interval like $\left(\frac{\pi}{2k},\frac{\pi}{2(k-1)}\right)$ etc, for $k=2,3,4\ldots\ldots.n$ is $\frac{\pi}{2n(n-1)}$, we take $\epsilon=\frac{\pi}{4n^3}$ which is less than one half of the subintervals. It can be easily verified that with this value of ϵ, all the subintervals type(1) and type(2) are well defined and no two subintervals of any type overlap.

We denote the average number of real zeros of $T_n(\theta)$ in any subinterval of type (1) by $E_n(\beta,\gamma)$and that any subinterval of type (2) by $M_n(\overline{\omega}-\epsilon,\overline{\omega}+\epsilon)$ where,

$$\overline{\omega}\in\left\{\left(\frac{\pi}{2k}\right)_{k=1}^{n},\left(\pi-\frac{\pi}{2k}\right)_{k=2}^{n},\left(\pi+\frac{\pi}{2k}\right)_{k=1}^{n},\left(\pi+\left(\pi-\frac{\pi}{2k}\right)\right)_{k=2}^{n},2\pi\right\} \qquad (2.2.1)$$

We denote

$M_n(\lambda)$=Sum of expectations of all subintervals of type (1) and

$M_n(\lambda')$=Sum of expectations of all subintervals of type (2)

Let

$$T_n(\theta)=\sum_{k=1}^{n}b_k a_k(\omega)\cos k\theta=X \qquad (2.2.2)$$

$$T_n'(\theta)=-\sum_{k=1}^{n}kb_k a_k(\omega)\sin k\theta=Y \qquad (2.2.3)$$

Then the joint characteristics function of X and Y is given by

$$G(z,\omega)=e^{-\left(c+cos\log\left(\sum_{k=1}^{n}|zb_k\cos k\theta-k\omega b_k\sin k\theta|\right)\right)\left(\sum_{k=1}^{n}|zb_k\cos k\theta-k\omega b_k\sin k\theta|\right)} \qquad (2.2.4)$$

as a_k's are independent random variables with common characteristic function $exp\{-(c+cos\log|t|)|t|\}$.

3. Main Results

Proof of the theorem: By Kac-Rice formula, and the procedure of Das,

$$M_n(\beta,\gamma)\le\frac{1}{\pi}\int_{\beta}^{\gamma}\sqrt{\frac{(4XZ-Y^2)}{X^2}}\,d\theta \qquad (3.3.1)$$

where

$$X(\omega) = \sum_{k=1}^{n} \binom{n}{k} |cosk\theta| \qquad (3.3.2)$$

$$Y(\omega) = 2\sum_{k=1}^{n} \binom{n}{k} |ksink\theta - cosk\theta| \qquad (3.3.3)$$

and

$$Z(\omega) = \sum_{k=1}^{n} \binom{n}{k} |seck\theta| (ksink\theta - cosk\theta)^2 \qquad (3.3.4)$$

In order to estimate Z, we proceed to estimate the maximum value of $|Seck\theta|$ in any sub interval of λ. Here we consider different cases corresponding to different kind of sub interval of type-1.

Case-1

If $\epsilon < \theta < \frac{\pi}{2n} - \epsilon$, then

$$|cos\theta| > \left|cos\left(\frac{\pi}{2} - \epsilon\right)\right| = |sin\epsilon|$$

As $|cos\theta|$ is decreasing in $\left(0, \frac{\pi}{2}\right)$

$$\Rightarrow |seck\theta| < \frac{1}{|sin\epsilon|} = \frac{1}{\left|sin\frac{\pi}{4n^3}\right|} \qquad taking\ \epsilon = \frac{\pi}{4n^3}$$

As $\lim_{n\to\infty} \left|\frac{sin\frac{\pi}{4n^3}}{\frac{\pi}{4n^3}}\right| = 1, \quad \left(\frac{\pi}{4n^3} \to 0\ as\ n \to \infty\right)$

We have, for $n > n_0, \left|\frac{sin\frac{\pi}{4n^3}}{\frac{\pi}{4n^3}} - 1\right| < \eta\ (very\ small)$

$$\frac{1}{1-\eta} > \frac{\frac{\pi}{4n^3}}{\left|sin\frac{\pi}{4n^3}\right|} > \frac{1}{1+\eta} \quad for\ \ n > n_0\text{, taking } \eta = 1 - \frac{1}{\pi},$$

We have $\pi > \frac{\frac{\pi}{4n^3}}{\left|sin\frac{\pi}{4n^3}\right|} \Rightarrow \frac{1}{\left|sin\frac{\pi}{4n^3}\right|} < 4n^3, for\ \ n > n_0$

$|seck\theta| < \frac{1}{\left|sin\frac{\pi}{4n^3}\right|}$ *and* $\frac{1}{\left|sin\frac{\pi}{4n^3}\right|} < 4n^3$

Then $|seck\theta| < 4n^3$, *for large n.*

$\frac{\frac{\pi}{4n^3}}{\left|sin\frac{\pi}{4n^3}\right|} > \frac{1}{1+\eta}$ *and* $\eta = 1 - \frac{1}{\pi}$.

So we get, $|seck\theta| > \frac{2\pi - 1}{4n^3}$, *for large n.*

Case -2

If $\frac{\pi}{2k} + \epsilon < \theta < \frac{\pi}{2(k-1)} - \epsilon$, proceed as above

As $|cos\theta|$ is increasing in $\left(\frac{\pi}{2}, \pi\right)$

$\Rightarrow |seck\theta| < 4n^3$, *for* $n > n_0$

Similarly we get, $|seck\theta| > \frac{2\pi - 1}{4n^3}$, *for large n.*

Case-3

If $\pi - \frac{\pi}{2(k-1)} + \epsilon < \theta < \pi - \frac{\pi}{2k} - \epsilon$, also as before

$\Rightarrow |seck\theta| < 4n^3$, *for* $n > n_0$

Similarly we get, $|seck\theta| > \frac{2\pi - 1}{4n^3}$, *for large n.*

Case-4

If $\pi - \frac{\pi}{2n} + \epsilon < \theta < \pi + \frac{\pi}{2n} - \epsilon$, *then*, also as before, we get

$|seck\theta| < 4n^3$, *for* $n > n_0$

Similarly we get, $|seck\theta| > \frac{2\pi - 1}{4n^3}$, *for large n.*

We shall leave other two cases by the period π in which we shall we get same upper estimate for$|seck\theta|$.

Case-5

If $2\pi k - \frac{\pi k}{2n} + \epsilon k < k\theta < 2\pi k + \epsilon k$ then

Since $|cos\theta|$ is increasing in $\left(2\pi k - \frac{\pi k}{2n} + \epsilon k,\ 2\pi k\right)$ and decreasing in $(2\pi k,\ 2\pi k + \epsilon k)$

After calculation, we get

$|seck\theta| \; < 4n^3$ and $|seck\theta| > \frac{2\pi-1}{4n^3}$ $, for \; large \; n.$

Now from (3.3.2)

$$X(\omega) = \sum_{k=1}^{n} \binom{n}{k} |cosk\theta|$$

$$= \frac{4n^3}{2\pi - 1} \sum_{k=1}^{n} \binom{n}{k}$$

$$= \frac{4n^3 2^n}{2\pi - 1} \left(1 - \frac{1}{2^n}\right)$$

$$X(\omega) \leq \frac{4n^3 2^n}{2\pi - 1} O(1) \qquad (3.3.5)$$

Now from (3.3.3)

$$Y(\omega) = 2 \sum_{k=1}^{n} \binom{n}{k} |ksink\theta - cosk\theta|$$

$$\leq 2 \sum_{k=1}^{n} \binom{n}{k} (|ksink\theta| + |cosk\theta|)$$

$$\leq 2 \sum_{k=1}^{n} \binom{n}{k} (k+1)$$

$$= 2^n n \left(1 + O\left(\frac{1}{n}\right)\right)$$

$$Y(\omega) \leq \; 2^n n \left(1 + O\left(\frac{1}{n}\right)\right) \qquad (3.3.6)$$

Now from (3.3.4)

$$Z(\omega) = \sum_{k=1}^{n} \binom{n}{k} |seck\theta| \,(ksink\theta - cosk\theta)^2$$

$$\leq \sum_{k=1}^{n} \binom{n}{k} |seck\theta| \,(ksink\theta + cosk\theta)^2$$

$$\leq \sum_{k=1}^{n} 4n^3 \binom{n}{k} (k+1)^2$$

$$\leq n^5 2^n \left(1 + O\left(\frac{1}{n}\right)\right)$$

$$Z(\omega) \leq n^5 2^n \left(1 + O\left(\frac{1}{n}\right)\right) \qquad (3.3.7)$$

Using the order of X, Y and Z, we get

$$\therefore \sqrt{\left|\frac{4XZ - Y^2}{X^2}\right|} \leq \frac{\frac{4n^4 2^n}{\sqrt{(2\pi-1)}}\sqrt{O(1)}\left(1 + O\left(\frac{1}{n}\right)\right)}{\frac{4n^3 2^n}{(2\pi-1)}O(1)} \left\{\frac{1}{\left(1 + O\left(\frac{1}{n}\right)\right)} - \frac{(2\pi-1)}{16n^6 O(1)}\right\}^{1/2}$$

$$= \frac{n}{\sqrt{(2\pi-1)}} \frac{\left(1 + O\left(\frac{1}{n}\right)\right)}{\sqrt{O(1)}} |\varphi(\Psi_n)|$$

where $\varphi(\Psi_n) = \left\{\frac{1}{\left(1+O\left(\frac{1}{n}\right)\right)} - \frac{(2\pi-1)}{16n^6 O(1)}\right\}^{1/2}$ and $\varphi(\Psi_n) \to 1 \;\; as\; n \to \infty$

Now from (3.3.1)

$$M_n(\lambda) \leq \frac{1}{\pi} \int_0^{2\pi} \sqrt{\frac{4XZ - Y^2}{X^2}}\, d\theta$$

$$= \frac{2n}{\sqrt{(2\pi-1)}}$$

$$\therefore M_n(\lambda) \leq \frac{2n}{\sqrt{(2\pi-1)}} \qquad (3.3.8)$$

we show that the probability of $T_n(\theta)$ having an appreciable number of zeros in the small interval $\bar{\omega}-\epsilon<\theta<\bar{\omega}+\epsilon$, i.e in any subinterval of type 2 is small.

Following lemma is necessary for estimation of $M_n(\bar{\omega}-\epsilon,\bar{\omega}+\epsilon)$.

Lemma :

$$P\left(n(\epsilon)>1+\frac{3n(3n^3+1)\epsilon}{log2}\right)<\frac{\mu_2}{e^{n^4\epsilon}}$$

where $n(r)$ denote the number of zeros of $T_n(\theta)$ in$|z|\le r$.

Proof: Applying Jensen's theorem to the entire function $T_n(\theta)$ we have

$$n(\epsilon)\le\frac{1}{2\pi log2}\int_0^{2\pi}log\left|\frac{T_n(2\epsilon e^{i\theta})}{T_n(0)}\right|d\theta \qquad (3.3.9)$$

Provided $T_n(0)\neq 0$. By Gnedenko and kolomogorv[4]

$$P(\{\omega:a_k(\omega)>n\})=1-\{F(n)-F(-n)\}<\frac{\mu}{n} \qquad (3.3.10)$$

If $\max_{1\le k\le h}|a_k(\omega)|=h_n$, then from(3.3.10), we have

$T_n(2\epsilon e^{i\theta})<2e^{2n^4\epsilon}e^{2n\epsilon}D_n$, where

$$D_n=\sum_{k=1}^{n}b_k=\sum_{k=1}^{n}\binom{n}{k}$$

except a set of measure at most least $(1-\frac{\mu_1}{e^{n^4\epsilon}})$. From Gredenko ([5] p.229), as $f(t)$ is integrable over the entire real line, we get,

$$F(x+h)-F(x-h)\le\frac{h}{\pi}\int_{-\infty}^{\infty}|f(t)|dt$$

$$=\frac{2h}{\pi}\frac{1}{D_n(c-1)} \qquad (where \quad c>1)$$

Now taking $x=0$ and $h=D_ne^{-n^4\epsilon}$ $,we\ get,$

$$P\left(|T_n(0)|<D_ne^{-n^4\epsilon}\right)=F\left(D_ne^{-n^4\epsilon}\right)-F\left(-D_ne^{-n^4\epsilon}\right)$$

$$<\mu_1e^{-n^4\epsilon} \quad \text{where} \quad \mu_1=\frac{2}{\pi(c-1)}$$

Hence we have

$$|T_n(2\epsilon e^{i\theta})| \leq 2.D_n e^{2n^4\epsilon} e^{2n\epsilon} \quad and \ |T_n(0)| < D_n e^{-n^4\epsilon}$$

Hence

$$P\left(\left|\frac{T_n(2\epsilon e^{i\theta})}{T_n(0)}\right| \leq \frac{2.D_n e^{2n^4\epsilon} e^{2n\epsilon}}{D_n e^{-n^4\epsilon}}\right) > 1 - \frac{\mu_2}{e^{n^4\epsilon}}$$

$$or \quad P\left(\left|\frac{T_n(2\epsilon e^{i\theta})}{T_n(0)}\right| \leq 2e^{3n^4\epsilon+2n\epsilon}\right) > 1 - \frac{\mu_2}{e^{n^4\epsilon}} \qquad (3.3.11)$$

It follows from (3.3.9), we get

$$n(\epsilon) \leq 1 + \frac{3n(n^3+1)\epsilon}{log2} \qquad (3.3.12)$$

Combining (3.3.11) and (3.3.12), we get,

$$P\left\{n(\epsilon) \leq 1 + \frac{3n(n^3+1)\epsilon}{log2}\right\} > 1 - \frac{\mu_2}{e^{n^4\epsilon}}$$

$$or \quad P\left\{n(\epsilon) > 1 + \frac{3n(n^3+1)\epsilon}{log2}\right\} \leq \frac{\mu_2}{e^{n^4\epsilon}}$$

we have observe that in any interval like $[\varpi - \epsilon, \varpi + \epsilon]$ as well as $[0, \ \epsilon]$ and $[2\pi - \epsilon, 2\pi]$, the probability that $T_n(\theta)$ has more than $\left(1 + \frac{3n(n^3+1)\epsilon}{log2}\right)$ zeros does not exceed $\frac{\mu_2}{e^{n^4\epsilon}}$. As there are altogether $4n$ such disjoint intervals in λ', there fore the probability that $T_n(\theta)$ has more than $4n\left(1 + \frac{3n(n^3+1)\epsilon}{log2}\right)$ zeros in λ', does not exceed $4n\frac{\mu_2}{e^{n^4\epsilon}} = \frac{n\mu'}{e^{n^4\epsilon}}$.

$$P\left\{M_n(\lambda') > \left(4n + \frac{12n^2(n^3+1)\epsilon}{log2}\right)\right\} < \frac{n\mu'}{e^{n^4\epsilon}} < \frac{n\mu'}{e^{\frac{n\pi}{4}}}, \quad putting \ \epsilon = \frac{\pi}{4n^3}$$

Thus

$$M_n(\lambda') = O\left\{\frac{n\mu'}{e^{\frac{n\pi}{4}}}\left(4n + \frac{3(n^3+1)\frac{\pi}{n}}{log2}\right)\right\}$$

$$= O\left\{\frac{3n^3\mu'}{e^{\frac{n\pi}{4}}}\left(\frac{4}{3n} + \frac{\left(1+\frac{1}{n^3}\right)\pi}{log2}\right)\right\}$$

$\to 0 \ as\, n \ \to \infty \ as\ e^{\frac{n\pi}{4}}$ is exponentially large .

So $M_n(\lambda') = 0(1) \quad as\, n \ \to \infty \qquad (3.3.13)$

Combining (3.3.8) and (3.3.13), we get,

$$EN_n[0,2\pi] = M_n[0,2\pi]$$

$$= M_n(\lambda) + M_n(\lambda')$$

$$\le \frac{2n}{\sqrt{(2\pi-1)}} + 0(1)$$

Hence we get,

$$EN_n[0,2\pi] = M_n[0,2\pi] \sim \frac{2n}{\sqrt{(2\pi-1)}} \quad as\, n \to \infty.$$

Thus the proof ends.

REFERENCES

[1] Das, M.K., The average number of real zeros of a random trigonometric polynomial, proc. Cambridge. Phil. Soc. 64(1968), 721-729.

[2] Dunnage, J .E. A., The number of real zeros of a random trigonometric Polynomials , Proc . London. Math. Soc. (3) 16 (1966), 53-84. 11

[3] Farahmand, K. and Stretch.T.C., Algebraic polynomial with random non-symmetric coefficients, Statistics and Probability Letters (2011).

[4] Gnedenko, B .V. and Kolmogorov, A .N., Limit Distributions for Sums of Independent random Variables, (Addison-Wesley, Inc,1954) .

[5] Gnedenko, B .V., The theory of probability, Mir Publisher, Moscow,1969.

[6] Sambandham, M., On a random trigonometric polynomial, Indian. Jour. Pure. and Appl. Math. 7 (1976d), 993-998.

[7] Shimizu.R., Characterization of the normal distribution II, Ann. Inst Stat. Math Vol-14(1962) 173-178.

[8] Shimizu.R., Certain class of infinitely divisible characteristic functions, Ann. Inst Stat. Math Vol-17(1965) 115-132.

[9] Shimizu.R., Characteristic functions satisfying a functional equation (I), Ann. Inst Stat. Math Vol-20(1968) 187-209.

[10] Shimizu.R., Characteristic functions satisfying a functional equation (II), Ann. Inst Stat. Math Vol-21(1969) 391-405.

[11] Shimizu.R., On the domain of attraction of semi-stable distributions, Ann. Inst Stat. Math 22(1970)245-255.

[12] Titchmarsh, E.C., The theory of Functions, 2nd end, The English Language Book Society, (Oxford University Press, 1939).

[13] Williamson, B., The Integral Calculus, (Scientific Book Company, India, 1955).