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MERCER–TRAPEZOID RULE FOR THE RIEMANN–STIELTJES INTEGRAL WITH APPLICATIONS

Mohammad W. Alomari[1]
[1]Department of Mathematics, Faculty of Science,
Jerash University, 26150 Jerash, Jordan
mwomath@gmail.com
Sever S. Dragomir[2]
[2]Mathematics, School of Engineering & Science,
Victoria University, PO Box 14428,
Melbourne City, MC 8001, Australia
[2]School of Computational & Applied Mathematics,
University of the Witwatersrand,
Private Bag-3, Wits-2050, Johannesburg, South Africa
sever.dragomir@vu.edu.au

Abstract.

In this paper several new error bounds for the Mercer–Trapezoid quadrature rule for the Riemann-Stieltjes integral under various assumptions are proved. Applications for functions of selfadjoint operators on complex Hilbert spaces are provided as well.

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1. Introduction

1.1. Approximating the Riemann-Stieltjes Integral. The Riemann–Stieltjes Integral $\int_a^b f(t)dg(t)$ is an important concept in Mathematics with multiple applications in several subfields including Probability Theory & Statistics, Complex Analysis, Functional Analysis, Operator Theory and others. In Numerical Analysis, the number of proposed quadrature rules to approximate this type of integrals is very small by comparison with the huge number of methods available to approximate the classical Riemann integral. $\int_a^b f(t)$

In recent years, R the approximation problem of the Riemann–Stieltjes integral $\int_a^b dg$ has been studied with the methods of modern Inequalities Theory and several error approximation bounds for the proposed quadrature rules had been established. Some of the most interesting approximations have been done using Ostrowski and Generalized Trapezoid type rules.

Dragomir [17] has introduced the following Ostrowski type quadrature rule:

$$\int_a^b f(t)\,du(t) \simeq f(x)[u(b)-u(a)], \quad x \in [a,b].$$

For several error bounds of this approximation rule under various assumptions for the functions involved, the reader may refer to [8], [9], [11]–[18], [30], [31], [33]–[35] and the references therein, as well as the recent works [3, 6]. From a different point of view, the authors of [19] considered the problem of approximating the Riemann–Stieltjes integral $\int_a^b f(t)du(t)$ with the Generalized Trapezoid rule $[u(x)+u(a)]f(a) + [u(bx)+u(x)]f(b)$, i.e.,

$$\int_a^b f(t)\,du(t) \simeq [u(x)-u(a)]\,f(a) + [u(b)-u(x)]\,f(b), x \in [a,b].$$

For various bounds of the above Generalized Trapezoid rule the reader may refer to [19]–[23] and the references therein. For new quadrature rules regarding Riemann–Stieltjes integral see [1], [2] and [4,5].

In order to approximate the Riemann-Stieltjes integral $\int_a^b f(x)du(x)$by the Riemann Integral $\int_a^b f(t)dt$, Dragomir and Fedotov [27], have introduced the following functional**:**

$$\mathcal{D}(f;u) := \int_a^b f(x)\,du(x) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt, \tag{1.1}$$

provided that the Riemann-Stieltjes integral $\int_a^b f(x)du(x)$and the Riemann integral $\int_a^b f(t)dt$ exist.

In the same paper [27], the authors have proved the following result:

Theorem 1. *Let $f, u: [a,b] \to \mathbb{R}$ such that u is of bounded variation on $[a,b]$ and$f$ Lipschitzianwith constant $K > 0$. Then we have*

$$|\mathcal{D}(f;u)| \le \frac{1}{2}K\,(b-a)\bigvee_a^b(u). \tag{1.2}$$

The constant is sharp in the sense that it cannot be replaced by a smaller quantity.

In [22], Dragomir has obtained the following inequality:

Theorem 2. *Let $f, u: [a,b] \to \mathbb{R}$ such that u is $L-$Lipschitzian on $[a,b]$, i.e.,*

$$|u(y)-u(x)| \le L\,|x-y|, \forall x,y \in [a,b], \quad (L>0)$$

and f is Riemann integrable on $[a,b]$.

If $m, M \in \mathbb{R}$, are such that $m \le f(x) \le M$ for any $x \in [a,b]$, then we have the inequality

$$|\mathcal{D}(f;u)| \le \frac{1}{2}L\,(M-m)\,(b-a). \tag{1.3}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

Recently, Mercer [32] has obtained some new midpoint and trapezoid type inequalities for the Riemann–Stieltjes integral which provide a natural generalization of Hermite-Hadamard's integral inequality, as follows:

Theorem 3. *Let g be continuous and increasing on $[a,b]$, let $c \in [a,b]$ satisfies*

$$\int_a^b g(t)\,dt = (c-a)\,g(a) + (b-c)\,g(b).$$

If $f'' \geq 0$, then we have

$$f(c)\,[g(b)-g(a)] \leq \int_a^b f dg \leq [G - g(a)]\,f(a) + [g(b) - G]\,f(b) \tag{1.4}$$

where

$$G := \frac{1}{b-a}\int_a^b g(t)\,dt.$$

For a generalization of this result where the positivity of the second derivative $f''$is replaced by the convexity of f, see [28].

However, it seems that Mercer didn't notice that the following significant relation between the right-hand side of (1.4) and the functional $\mathcal{D}(g,f)$ exists:

$$\begin{aligned}
&\int_a^b f(t)\,dg(t) - [G - g(a)]\,f(a) - [g(b) - G]\,f(b) \\
&= \frac{f(b)-f(a)}{b-a}\int_a^b g(t)\,dt - \int_a^b g(t)\,df(t) := -\mathcal{D}(g;f).
\end{aligned} \tag{1.5}$$

This follows by the integration by parts formula

$$\int_a^b f(t)\,dg(t) = f(b)\,g(b) - f(a)\,g(a) - \int_a^b g(t)\,df(t),$$

and some simple calculations.

1.2. Comparing Two Integral Means. In order to study the difference between two Riemann integral means, Barnett et al. [7] have proved the following estimates:

Theorem 4. *Let $f: [a,b] \to \mathbb{R}$ be an absolutely continous function with the property that $f' \in L_\infty[a,b]$, i.e.,*

$$\|f'\|_\infty := \operatorname*{ess\ sup}_{t\in[a,b]} |f'(t)|.$$

Then for $a \leq c < d \leq b$, we have the inequality

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(s)\,ds\right| \\
&\leq \left[\frac{1}{4} + \left(\frac{(a+b)/2 - (c+d)/2}{(b-a)-(d-c)}\right)^2\right][(b-a)-(d-c)]\,\|f'\|_\infty \\
&\leq \frac{1}{2}[(b-a)-(d-c)]\,\|f'\|_\infty .
\end{aligned} \tag{1.6}$$

The constant 1/4 in the first inequality and 1/2 in the second inequality are the best possible.

After that, Cerone and Dragomir [10] have obtained the following three results as well:

Theorem 5. *Let $f: [a,b] \to \mathbb{R}$ be an absolutely continous function mapping. Then for $a \leq c < d \leq b$, we have the inequality*

$$(1.7)\qquad \left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{d-c}\int_c^d f(s)\,ds\right|$$
$$\leq\begin{cases}\frac{(b-a)}{(q+1)^{1/q}}\left[1+\left(\frac{\rho}{1+\rho}\right)^q\right]^{1/q}\left[v^{q+1}+\lambda^{q+1}\right]^{1/q}\|f'\|_p, \\ \qquad f'\in L_p[a,b],\ 1\leq p<\infty,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\left[1-\rho+|v-\lambda|\right]\|f'\|_1, \qquad f'\in L_1[a,b];\end{cases}$$

where $(b-a)v=c-a, (b-a)\rho=d-c$ *and* $(b-a)\lambda=b-d$. *Both inequalities in (1.7) are sharp.*

Theorem 6. *Assume that mapping* $f:[a,b]\to\mathbb{R}$ *is of* $r-H-$ *Holder type on* $[a,b]$. $a\leq c<d\leq b$, *we have the inequality*

$$(1.8)\qquad \left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{d-c}\int_c^d f(s)\,ds\right|\leq H\frac{(c-a)^{r+1}+(b-d)^{r+1}}{(r+1)\left[(b-a)-(d-c)\right]}.$$

Inequality (1.8) is best possible in the sense that we cannot put in the right-hand side a constant less than 1.

Theorem 7. *Let* $f:[a,b]\to\mathbb{R}$ *be a mapping of bounded variation on* $[a,b]$. *The following bounds holds*

$$(1.9)\qquad \left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{d-c}\int_c^d f(s)\,ds\right|$$
$$\leq\begin{cases}\left[\frac{b-a-(d-c)}{2}+\left|\frac{c+d}{2}-\frac{a+b}{2}\right|\right]\frac{\bigvee_a^b(f)}{b-a}; \\ L\frac{(c-a)^2+(b-d)^2}{2[(b-a)-(d-c)]};\quad \text{if } f \text{ is L-Lipschitzian} \\ \left(\frac{b-d}{b-a}\right)f(b)-\left(\frac{c-a}{b-a}\right)f(a)+\left[\frac{c+d-(a+b)}{b-a}\right]f(s_0); \\ \text{if } f \text{ is monotonic nondecreasing}\end{cases}$$

where, $s_0=\frac{cb-ad}{(b-a)-(d-c)}\in[c,d]$.

In this paper by utilising amongst others the inequalities from Theorems 4–7, several new bounds for the *Mercer–Trapezoid Quadrature rule error*

$$\int_a^b f(t)\,dg(t)-[G-g(a)]f(a)-[g(b)-G]f(b),$$

and, equivalently, for the *Dragomir-Fedotov functional,*

$$\mathcal{D}(g;f)=\int_a^b g(t)\,df(t)-\frac{f(b)-f(a)}{b-a}\int_a^b g(t)\,dt$$

are provided.

The inequalities (1.6)–(1.9) are used in an essential way to obtain new error bounds for the above quadrature rule and hence for the functional $\mathcal{D}(g,f)$, which gives a significant application for these inequalities. Applications for compounding quadrature rules and for functions of self adjoint operators on complex Hilbert Spaces are provided as well.

2. The Case of Bounded Variation f'

We may start with the following result:

Theorem 8. *Let $f, g: [a,b] \to \mathbb{R}$ be a such that f' is of bounded variation on $[a,b]$ and g is absolutely continous on $[a,b]$, then*

$$\left|\int_a^b f(t)\,dg(t) - [G - g(a)]\,f(a) - [g(b) - G]\,f(b)\right| \tag{2.1}$$

$$\leq \bigvee_a^b (f') \cdot \begin{cases} \frac{1}{8}(b-a)^2 \|g'\|_\infty, & g' \in L_\infty[a,b]; \\ \frac{(b-a)}{2}\left(\frac{1+\frac{1}{2^q}}{2(q+1)}\right)^{1/q} \|g'\|_p, & g' \in L_p[a,b]; \\ \frac{1}{8}(b-a)\|g'\|_1, & g' \in L_1[a,b], \end{cases}$$

where $\|\cdot\|_p$ are the usual Lebesgue norms, i.e.,

$$\|h\|_p := \left(\int_a^b |h(t)|^p\,dt\right)^{1/p}, \ \text{for } p \geq 1$$

and

$$\|h\|_\infty := ess \sup_{t\in[a,b]} |h(t)|.$$

Proof. Define $h(t) = g(t) - G$, so that $H(t) = \int_a^t h(u)\,du$ satisfies $H(a) = H(b) = 0$, (see Theorem 1 in [32]). Using integration by parts (twice), we have

$$\begin{aligned}\int_a^b H(t)\,d(f'(t)) &= -\int_a^b f'(t)\,dH(t) \\ &= -\int_a^b (g(t) - G)\,df(t) \\ &= \int_a^b f(t)\,d(g(t) - G) - [G - g(a)]\,f(a) - [g(b) - G]\,f(b) \\ &= \int_a^b f(t)\,dg(t) - [G - g(a)]\,f(a) - [g(b) - G]\,f(b). \end{aligned} \tag{2.2}$$

It is known that for a continuous function $p: [a,b] \to \mathbb{R}$ and a function $\nu: [a,b] \to \mathbb{R}$ of bounded variation, the Riemann–Stieltjes integral $\int_a^b p(t)d\nu(t)$ exists and one has the inequality

$$\left|\int_a^b p(t)\,d\nu(t)\right| \leq \sup_{t\in[a,b]} |p(t)| \bigvee_a^b (\nu). \tag{2.3}$$

As f' is of bounded variation on [a, b], by (2.3) we have

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$$\text{(2.4)} \qquad \begin{aligned} &\left| \int_a^b H(t)\, d(f'(t)) \right| \\ &\leq \sup_{t\in[a,b]} |H(t)| \bigvee_a^b (f') \\ &= \sup_{t\in[a,b]} \left| \int_a^t h(u)\, du \right| \bigvee_a^b (f') \\ &= \sup_{t\in[a,b]} \left| \int_a^t [g(u) - G]\, du \right| \bigvee_a^b (f') \\ &= \sup_{t\in[a,b]} \left| \int_a^t g(u)\, du - (t-a) G \right| \bigvee_a^b (f') \\ &= \sup_{t\in[a,b]} \left| (t-a) \left[\frac{1}{t-a} \int_a^t g(u)\, du - G \right] \right| \bigvee_a^b (f') \\ &= \sup_{t\in[a,b]} \left| (t-a) \left[\frac{1}{t-a} \int_a^t g(u)\, du - \frac{1}{b-a} \int_a^b g(u)\, du \right] \right| \bigvee_a^b (f'). \end{aligned}$$

In the inequality (1.6), setting $d = t$ and $c = a$, we get

$$\text{(2.5)} \qquad \left| \frac{1}{t-a} \int_a^t g(s)\, ds - \frac{1}{b-a} \int_a^b g(s)\, ds \right| \leq \frac{1}{2} (b-t) \|g'\|_\infty .$$

Substituting (2.5) in (2.4), we get

$$\begin{aligned} \left| \int_a^b H(t)\, d(f'(t)) \right| &\leq \frac{1}{2} \|g'\|_\infty \sup_{t\in[a,b]} \{(t-a)(b-t)\} \bigvee_a^b (f') \\ &= \frac{1}{8} (b-a)^2 \|g'\|_\infty \bigvee_a^b (f'), \end{aligned}$$

since $\sup\limits_{t\in[a,b]} \{(t-a)(b-t)\}$, occurs at $t = \frac{a+b}{2}$, therefore, $\sup\limits_{t\in[a,b]} \{(t-a)(b-t)\} = \frac{1}{4}(b-a)^2$, which proves the first inequality in (2.1).

In the inequality (1.7), setting $d = t$ and $c = a$, we get

$$\text{(2.6)} \qquad \begin{aligned} &\left| \frac{1}{t-a} \int_a^t g(s)\, ds - \frac{1}{b-a} \int_a^b g(s)\, ds \right| \\ &\leq \begin{cases} \frac{(b-t)^{1+\frac{1}{q}}}{(q+1)^{1/q} (b-a)^{\frac{1}{q}}} \left[1 + \left(\frac{t-a}{b+t-2a} \right)^q \right]^{1/q} \|g'\|_p, & g' \in L_p[a,b], \\ \frac{1}{2} \frac{b-t}{b-a} \|g'\|_1, & g' \in L_1[a,b]. \end{cases} \end{aligned}$$

Substituting (2.6) in (2.4), we get

$$\left|\int_a^b H(t)\,d(f'(t))\right|$$
$$\leq \bigvee_a^b (f')$$
$$\times \sup_{t\in[a,b]} (t-a)\cdot\begin{cases} \frac{(b-t)^{1+\frac{1}{q}}}{(q+1)^{1/q}(b-a)^{\frac{1}{q}}}\left[1+\left(\frac{t-a}{b+t-2a}\right)^q\right]^{1/q}\|g'\|_p, & g'\in L_p[a,b], \\ \frac{1}{2}\frac{b-t}{b-a}\|g'\|_1, & g'\in L_1[a,b] \end{cases}$$
$$\leq \bigvee_a^b (f')\times\begin{cases} \frac{(b-a)}{2^{1+1/q}(1+q)^{1/q}}\left[1+\left(\frac{1}{2}\right)^q\right]^{1/q}\|g'\|_p, & g'\in L_p[a,b]; \\ \frac{1}{8}(b-a)\|g'\|_1, & g'\in L_1[a,b]; \end{cases}$$

where, $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, which proves the second and the third inequalities in (2.1). ∎

We have the following bound for the trapezoidal rule:

Corollary 1. *In Theorem 8, by setting $g(t)=t$, we obtain*

$$\left|\int_a^b f(t)\,dt-(b-a)\frac{f(a)+f(b)}{2}\right| \leq \frac{1}{8}(b-a)^2\bigvee_a^b (f'). \tag{2.7}$$

Another result when g is of r-H–Hölder type is as follows:

Theorem 9. *Let $f,g:[a,b]\to\mathbb{R}$ be such that f' is of bounded variation on $[a,b]$ and g is of r-H–Hölder type on $[a,b]$, then*

$$\left|\int_a^b f(t)\,dg(t)-[G-g(a)]f(a)-[g(b)-G]f(b)\right| \leq H\frac{r^r}{(r+1)^{r+1}}(b-a)^{r+1}\bigvee_a^b (f'). \tag{2.8}$$

Proof. As f' is of bounded variation and g is of r-H–Hölder type on $[a,b]$, by (2.4) and using (1.8) we have

$$\left|\int_a^b H(t)\,d(f'(t))\right|$$

$$\leq \sup_{t\in[a,b]} \left|(t-a)\left[\frac{1}{t-a}\int_a^t g(u)\,du - \frac{1}{b-a}\int_a^b g(u)\,du\right]\right| \bigvee_a^b (f')$$

$$\leq \frac{H}{r+1}\sup_{t\in[a,b]} (t-a)(b-t)^r \bigvee_a^b (f')$$

$$= H\frac{r^r}{(r+1)^{r+1}}(b-a)^{r+1}\bigvee_a^b (f'),$$

since $\sup_{t\in[a,b]} (t-a)(b-t)^r = \frac{r^r}{(r+1)^{r+1}}(b-a)^{r+1}$, which completes the proof. ∎

Theorem 10. *Let $f, g : [a,b] \to \mathbb{R}$ be such that f' is of bounded variation on $[a,b]$ and g is monotonic nondecreasing on $[a,b]$, then*

$$\left|\int_a^b f(t)\,dg(t) - [G - g(a)]\,f(a) - [g(b) - G]\,f(b)\right| \leq \frac{1}{6}(b-a)^2[g(b)-g(a)]\bigvee_a^b (f'). \tag{2.9}$$

Proof. As f' and g is of bounded variation on $[a,b]$ and g is monotonic nondecreasing on $[a,b]$, by (2.3) and using (2.4) we have

$$\left|\int_a^b H(t)\,d(f'(t))\right| \leq \int_a^b \left|(t-a)\left[\frac{1}{t-a}\int_a^t g(u)\,du - \frac{1}{b-a}\int_a^b g(u)\,du\right]\right| dt \bigvee_a^b (f') \tag{2.10}$$

In the third part of inequality (1.9), setting $d = t$ and $c = a$, we get

$$\left|\frac{1}{t-a}\int_a^t g(s)\,ds - \frac{1}{b-a}\int_a^b g(s)\,ds\right| \leq \frac{b-t}{b-a}[g(b)-g(a)]. \tag{2.11}$$

Substituting (2.11) in (2.10), we get

$$\left|\int_a^b H(t)\,d(f'(t))\right| \leq \frac{g(b)-g(a)}{b-a}\int_a^b (t-a)(b-t)\,dt \bigvee_a^b (f') = \frac{1}{6}(b-a)^2[g(b)-g(a)]\bigvee_a^b (f'),$$

and thus the proof is finished.

3.The Case of *Lipschitzian* f'

In this section, we give some new bounds when f' is $L - Lipschitzian$.

Theorem 11. *Let $f, g : [a,b] \to \mathbb{R}$ be such that f' is L–Lipschitzian on $[a,b]$ and g is an absolutely continuous on $[a,b]$, then*

$$\left|\int_a^b f(t)\,dg(t) - [G - g(a)]\,f(a) - [g(b) - G]\,f(b)\right| \tag{3.1}$$

$$\leq L \begin{cases} \frac{1}{12}(b-a)^3 \|g'\|_\infty, & g' \in L_\infty[a,b]; \\ \frac{q^2\left(1+\frac{1}{2q}\right)^{1/q}}{(2q+1)(3q+1)(q+1)^{1/q}}(b-a)^3 \|g'\|_p, & g' \in L_p[a,b]; \\ \frac{1}{12}(b-a)^2 \|g'\|_1, & g' \in L_1[a,b]; \end{cases}$$

where, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using the fact that for a Riemann integrable function $p : [c,d] \to \mathbb{R}$ and L-Lipschitzian function $\nu : [c,d] \to \mathbb{R}$, one has the inequality

$$\left|\int_c^d p(t)\,d\nu(t)\right| \leq L \int_c^d |p(t)|\,dt. \tag{3.2}$$

As f' is L–Lipschitzian on $[a,b]$, by (3.2) we have

$$\begin{aligned}
\left|\int_a^b H(t)\,d(f'(t))\right| &\leq L \int_a^b |H(t)|\,dt \\
&= L \int_a^b \left|\int_a^t h(u)\,du\right| dt \\
&= L \int_a^b \left|\int_a^t [g(u) - G]\,du\right| dt \\
&= L \int_a^b \left|\int_a^t g(u)\,du - (t-a)\,G\right| dt \\
&= L \int_a^b \left|(t-a)\left[\frac{1}{t-a}\int_a^t g(u)\,du - G\right]\right| dt \\
&= L \int_a^b \left|(t-a)\left[\frac{1}{t-a}\int_a^t g(u)\,du - \frac{1}{b-a}\int_a^b g(u)\,du\right]\right| dt \qquad (3.3) \\
&\leq \frac{1}{2} L \|g'\|_\infty \int_a^b (t-a)(b-t)\,dt = \frac{1}{12} L (b-a)^3 \|g'\|_\infty
\end{aligned}$$

where, for the last inequality we used the inequality (1.6), with $d = t$ and $a = c$, (see (2.5)).

In the inequality (1.7), setting $d = t$ and $a = c$, we get

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$$\text{(3.4)}\qquad \left|\frac{1}{t-a}\int_a^t g(s)\,ds-\frac{1}{b-a}\int_a^b g(s)\,ds\right|$$
$$\leq\begin{cases}\frac{(b-t)^{1+\frac{1}{q}}}{(q+1)^{1/q}(b-a)^{\frac{1}{q}}}\left[1+\left(\frac{t-a}{b+t-2a}\right)^q\right]^{1/q}\|g'\|_p\\ \qquad g'\in L_p[a,b],\ 1\leq p<\infty,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2}\frac{b-t}{b-a}\|g'\|_1, \qquad g'\in L_1[a,b].\end{cases}$$

Substituting (3.4) in (3.3), we get

$$\left|\int_a^b H(t)\,d(f'(t))\right|$$
$$\leq L\begin{cases}\frac{\|g'\|_p}{(q+1)^{1/q}(b-a)^{1/q}}\int_a^b(t-a)(b-t)^{1+\frac{1}{q}}\left[1+\left(\frac{t-a}{b+t-2a}\right)^q\right]^{1/q}dt,\ g'\in L_p[a,b],\\ \frac{1}{2}\frac{\|g'\|_1}{b-a}\int_a^b(t-a)(b-t)\,dt, \qquad g'\in L_1[a,b],\end{cases}$$
$$\leq L\begin{cases}\frac{\|g'\|_p}{(q+1)^{1/q}(b-a)^{1/q}}\sup_{t\in[a,b]}\left[1+\left(\frac{t-a}{b+t-2a}\right)^q\right]^{1/q}\\ \times\int_a^b(t-a)(b-t)^{1+\frac{1}{q}}dt,\quad g'\in L_p[a,b],\\ \frac{1}{12}(b-a)^2\|g'\|_1, \qquad g'\in L_1[a,b],\end{cases}$$
$$=L\begin{cases}\frac{\|g'\|_p}{(q+1)^{1/q}(b-a)^{1/q}}\left[1+\frac{1}{2^q}\right]^{1/q}(b-a)^{3+\frac{1}{q}}\frac{q^2}{(2q+1)(3q+1)},\ g'\in L_p[a,b],\\ \frac{1}{12}(b-a)^2\|g'\|_1, \qquad g'\in L_1[a,b],\end{cases}$$

where, $\sup_{t\in[a,b]}\left[1+\left(\frac{t-a}{b+t-2a}\right)^q\right]^{1/q}=\left(1+\frac{1}{2^q}\right)^{1/q}$, and

$$\int_a^b(t-a)(b-t)^{1+\frac{1}{q}}dt$$
$$=(b-a)^{3+\frac{1}{q}}\int_0^1(1-t)\,t^{1+\frac{1}{q}}dt=(b-a)^{3+\frac{1}{q}}\frac{q^2}{(2q+1)(3q+1)},$$

which proves the second and the third inequalities in (3.1). ∎

Corollary 2. *In Theorem 11, setting $g(t)=t$, we obtain*

$$\text{(3.5)}\qquad \left|\int_a^b f(t)\,dt-(b-a)\frac{f(a)+f(b)}{2}\right|\leq\frac{1}{12}L(b-a)^3.$$

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Theorem 12. *Let $f, g : [a,b] \to \mathbb{R}$ be such that f' is L–Lipschitzian on $[a,b]$ and g is of r-H–Hölder type on $[a,b]$, then*

$$\left| \int_a^b f(t)\, dg(t) - [G - g(a)] f(a) - [g(b) - G] f(b) \right| \leq LH \frac{(b-a)^{r+2}}{(r+1)^2 (r+2)}. \tag{3.6}$$

Proof. As f' is L–Lipschitzian and g is of r-H–Hölder type on $[a,b]$, by (3.3) and using (1.8) we have

$$\begin{aligned} \left| \int_a^b H(t)\, d(f'(t)) \right| &\leq L \int_a^b \left| (t-a) \left[\frac{1}{t-a} \int_a^t g(u)\, du - \frac{1}{b-a} \int_a^b g(u)\, du \right] \right| dt \\ &\leq \frac{LH}{r+1} \int_a^b (t-a)(b-t)^r\, dt \\ &= LH \frac{(b-a)^{r+2}}{(r+1)^2 (r+2)}. \end{aligned}$$

where, for the last inequality, a simple calculation yields that

$$\int_a^b (t-a)(b-t)^r\, dt = (b-a)^{r+2} \int_0^1 (1-t)\, t^r dt = \frac{(b-a)^{r+2}}{(r+1)(r+2)},$$

which completes the proof. ∎

Corollary 3. *In Theorem 12, if g is M–Lipschitzian on $[a,b]$, then we have*

$$\left| \int_a^b f(t)\, dg(t) - [G - g(a)] f(a) - [g(b) - G] f(b) \right| \leq \frac{1}{12} LM (b-a)^3. \tag{3.7}$$

Theorem 13. *Let $f, g : [a,b] \to \mathbb{R}$ be such that f' is of bounded variation on $[a,b]$ and g is monotonic nondecreasing on $[a,b]$, then*

$$\left| \int_a^b f(t)\, dg(t) - [G - g(a)] f(a) - [g(b) - G] f(b) \right| \leq \frac{1}{6} L (b-a)^2 [g(b) - g(a)]. \tag{3.8}$$

Proof. As f' is L–Lipschitzian on $[a,b]$ and g is monotonic nondecreasing on $[a,b]$, by (3.2) and using (3.3) we have

$$\left| \int_a^b H(t)\, d(f'(t)) \right| \leq L \int_a^b \left| (t-a) \left[\frac{1}{t-a} \int_a^t g(u)\, du - \frac{1}{b-a} \int_a^b g(u)\, du \right] \right| dt \tag{3.9}$$

In the third part of inequality (1.9), setting $d = t$ and $c = a$, we get

$$\left| \frac{1}{t-a} \int_a^t g(s)\, ds - \frac{1}{b-a} \int_a^b g(s)\, ds \right| \leq \frac{b-t}{b-a} [g(b) - g(a)]. \tag{3.10}$$

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Substituting (3.10) in (3.9), we get

$$\left|\int_a^b H(t)\,d(f'(t))\right| \le L\frac{g(b)-g(a)}{b-a}\int_a^b (t-a)(b-t)\,dt$$
$$= \frac{1}{6}L(b-a)^2[g(b)-g(a)],$$

which completes the proof.

4. More Inequalities

In this section we give other related results:

Theorem 12. *Let* $f, g: [a,b] \to \mathbb{R}$ *be such that* f' *and* g *are of bounded variation on* $[a,b]$.

then

$$(4.1)\qquad \left|\int_a^b f(t)\,dg(t) - [G-g(a)]f(a) - [g(b)-G]f(b)\right|$$
$$\le \frac{1}{4}(b-a)\bigvee_a^b(g)\bigvee_a^b(f').$$

Proof. As f' and g is of bounded variation on $[a,b]$, by (2.3) and using (2.4), we have

$$(4.2)\qquad \left|\int_a^b H(t)\,d(f'(t))\right|$$
$$\le \sup_{t\in[a,b]}\left|(t-a)\left[\frac{1}{t-a}\int_a^t g(u)\,du - \frac{1}{b-a}\int_a^b g(u)\,du\right]\right|\bigvee_a^b(f').$$

In the first inequality of (1.9), setting $d=t$ and $c=a$, we get

Substituting (4.3) in (4.2), we get

$$\left|\int_a^b H(t)\,d(f'(t))\right| \le \frac{1}{b-a}\sup_{t\in[a,b]}\{(t-a)(b-t)\}\bigvee_a^b(g)\bigvee_a^b(f')$$
$$= \frac{1}{4}(b-a)\bigvee_a^b(g)\bigvee_a^b(f'),$$

which completes the proof.

Corollary 4. *In Theorem 14, let $g(t)=t$, then*

$$\left|\int_a^b f(t)\,dt - (b-a)\frac{f(a)+f(b)}{2}\right| \le \frac{1}{4}(b-a)^2\bigvee_a^b(f').$$

When the integrator is of bounded variation we have:

Theorem 15. *Let $f,g:[a,b]\to\mathbb{R}$ be such that f' is L–Lipschitzian on $[a,b]$ and g is of bounded variation on $[a,b]$, then*

$$\left|\int_a^b f(t)\,dg(t)-[G-g(a)]\,f(a)-[g(b)-G]\,f(b)\right|\le\frac{1}{6}L(b-a)^2\bigvee_a^b(g),$$

Proof. As f' is L–Lipschitzian on $[a,b]$ and g is of bounded variation on $[a,b]$, by (2.3) and using (2.4), we have

$$\left|\int_a^b H(t)\,d(f'(t))\right| \le L\int_a^b\left|(t-a)\left[\frac{1}{t-a}\int_a^t g(u)\,du-\frac{1}{b-a}\int_a^b g(u)\,du\right]\right|dt. \tag{4.4}$$

In the second inequality of (1.9), setting $d=t$ and $c=a$, we get

$$\left|\frac{1}{t-a}\int_a^t g(s)\,ds-\frac{1}{b-a}\int_a^b g(s)\,ds\right|\le\frac{b-t}{b-a}\bigvee_a^b(g). \tag{4.5}$$

Substituting (4.5) in (4.4), we get

$$\left|\int_a^b H(t)\,d(f'(t))\right| \le \frac{L}{b-a}\int_a^b(t-a)(b-t)\,dt\bigvee_a^b(g) = \frac{1}{6}L(b-a)^2\bigvee_a^b(g),$$

and the proof is completed.

When both functions are Lipschitzian we have:

Theorem 16. *Let $f,g:[a,b]\to\mathbb{R}$ be respectively such that f' and g are L_1–, L_2–Lipschitzian on $[a,b]$, then*

$$\left|\int_a^b f(t)\,dg(t)-[G-g(a)]\,f(a)-[g(b)-G]\,f(b)\right|\le\frac{1}{12}L_1L_2(b-a)^3.$$

Proof. As f' and g are L_1–, L_2–Lipschitzian on $[a,b]$, respectively; by (3.2) and using (3.3), we have

$$\left|\int_a^b H(t)\,d(f'(t))\right| \le L_1\int_a^b\left|(t-a)\left[\frac{1}{t-a}\int_a^t g(u)\,du-\frac{1}{b-a}\int_a^b g(u)\,du\right]\right|dt. \tag{4.6}$$

In the second inequality of (1.9), setting $d=t$ and $c=a$, we get

$$\left|\frac{1}{t-a}\int_a^t g(s)\,ds-\frac{1}{b-a}\int_a^b g(s)\,ds\right|\le\frac{1}{2}L_2(b-t). \tag{4.7}$$

Substituting (4.7) in (4.6), we get

$$\left|\int_a^b H(t)\,d(f'(t))\right|\le\frac{1}{2}L_1L_2\int_a^b(t-a)(b-t)\,dt=\frac{1}{12}L_1L_2(b-a)^3,$$

which completes the proof.

Corollary 5. *In Theorem 16, let $g(t) = t$, then*

$$\left| \int_a^b f(t)\, dt - (b-a) \frac{f(a) + f(b)}{2} \right| \leq \frac{1}{12} L_1 (b-a)^3 .$$

Remark 1. Let g be as in Theorems 8–16. By applying the same techniques used in the corresponding proofs of each theorem, we may obtain several inequalities for monotonic non-decreasing integrator f' using the fact that for a monotonic nondecreasing function $\nu: [a,b] \to \mathbb{R}$ and continuous function $p: [a,b] \to \mathbb{R}$ one has the inequality

$$\left| \int_a^b p(t)\, d\nu(t) \right| \leq \int_a^b |p(t)|\, d\nu(t).$$

We leave the details to the interested reader.

Remark 2. We also observe that the inequalities in the previous sections hold for $|\mathcal{D}(g, f)|$, which therefore give new bounds for the functional (1.1). The details are omitted.

5. Applications To Quadrature Rules

Let $I_n : a = x_0 < x_1 < \cdots < x_n = b$, be a division of the interval $[a, b]$. Define the *Mercer–Trapezoid Quadrature rule* as

$$S(f, g, I_n) = \sum_{i=0}^{n-1} [G_i - g(x_i)]\, f(x_i) + [g(x_{i+1}) - G_i]\, f(x_{i+1}), \tag{5.1}$$

where,

$$G_i := \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} g(s)\, ds.$$

In the following, we establish an upper bound for the error approximation of the Riemann-Stieltjes $\int_a^b f(t) dg(t)$ integral by its Riemann-Stieltjes sum $S(f, g, I_n)$. As a sample we use only the inequality (4.1).

Theorem 17. *Under the assumptions of Theorem 14, we have*

$$\int_a^b f(t)\, dg(t) = S(f, g, I_n) + R(f, g, I_n)$$

where, $S(f, g, I_n)$ is given in (5.1) and the remainder $R(f, g, I_n)$ satisfies the bound

$$|R(f, g, I_n)| \leq \frac{1}{4} (b-a) \bigvee_a^b (g) \bigvee_a^b (f').$$

Proof. Applying Theorem 14 on the intervals $[x_i, x_{i+1}]$, we may state that

$$\left| \int_{x_i}^{x_{i+1}} f(t)\, dg(t) - [G_i - g(x_i)]\, f(x_i) - [g(x_{i+1}) - G_i]\, f(x_{i+1}) \right|$$
$$\leq \frac{1}{4} (x_{i+1} - x_i) \bigvee_{x_i}^{x_{i+1}} (g) \bigvee_{x_i}^{x_{i+1}} (f'),$$

for all $i \in \{0, 1, 2, \cdots, n-1\}$.

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Summing the above inequality over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce

$$\begin{aligned}
&|R(f,g,I_n)| \\
&= \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(t)\,dg(t) - [G_i - g(x_i)]\,f(x_i) - [g(x_{i+1}) - G_i]\,f(x_{i+1})\right| \\
&\le \frac{1}{4}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\bigvee_{x_i}^{x_{i+1}}(g)\bigvee_{x_i}^{x_{i+1}}(f') \\
&\le \frac{1}{4}\sup_{i=0,1,\dots,n-1}(x_{i+1}-x_i)\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(g)\bigvee_{x_i}^{x_{i+1}}(f') \\
&\le \frac{1}{4}(b-a)\sup_{i=0,1,\dots,n-1}\bigvee_{x_i}^{x_{i+1}}(g)\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f') \\
&\le \frac{1}{4}(b-a)\bigvee_a^b(g)\bigvee_a^b(f'),
\end{aligned}$$

which completes the proof.

Remark 3. One may use the remaining inequalities in the previous sections to obtain other bounds for $R(f,g,I_n)$. We omit the details.

6. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A\in\mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda\in\mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \le \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda\in\mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A) \tag{6.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [29, p. 256]:

Theorem 18 (Spectral Representation Theorem). *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m=\min\{\lambda\,|\,\lambda\in Sp(A)\}=:\min Sp(A)$ and $M=\max\{\lambda\,|\,\lambda\in Sp(A)\}=:\max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda \le E_{\lambda'}$ *for* $\lambda\le\lambda'$;
b) $E_{m-0}=0, E_M=I$ *and* $E_{\lambda+0}=E_\lambda$ *for all* $\lambda\in\mathbb{R}$;
c) *We have the representation*

$$A = \int_{m-0}^{M}\lambda\,dE_\lambda. \tag{6.2}$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda'_k)\left(E_{\lambda_k} - E_{\lambda_{k-1}}\right) \right\| \leq \varepsilon \tag{6.3}$$

whenever

$$\begin{cases} \lambda_0 < m = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = M, \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \lambda'_k \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases} \tag{6.4}$$

this means that

$$\varphi(A) = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda, \tag{6.5}$$

where the integral is of Riemann-Stieltjes type.

Corollary 6. *With the assumptions of Theorem 18 for A, E_λ and φ we have the representations*

$$\varphi(A)x = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda x \quad \text{for all } x \in H \tag{6.6}$$

and

$$\langle \varphi(A)x, y\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \quad \text{for all } x, y \in H. \tag{6.7}$$

In particular,

$$\langle \varphi(A)x, x\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \quad \text{for all } x \in H. \tag{6.8}$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \quad \text{for all } x \in H. \tag{6.9}$$

We recall the following result, see [24] that provides an upper bound for the total variation of the function $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda x, y\rangle \in \mathbb{C}$ on an interval $[\alpha, \beta]$:

Theorem 19. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A and let $m = \min Sp(A)$ and $M = \max Sp(A)$. Then for any $x, y \in H$ and $\alpha < \beta$ we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta}\left(\langle E_{(\cdot)}x, y\rangle\right)\right]^2 \leq \langle (E_\beta - E_\alpha)x, x\rangle \langle (E_\beta - E_\alpha)y, y\rangle, \tag{6.10}$$

where $\bigvee_{\alpha}^{\beta}\left(\langle E_{(\cdot)}x, y\rangle\right)$ denotes the total variation of the function $\langle E_{(\cdot)}x, y\rangle$ on $[\alpha, \beta]$.

Remark 4. *For $\alpha = m - \varepsilon$ with $\varepsilon > 0$ and $\beta = M$ we get from (6.10) the inequality*

$$\bigvee_{m-\varepsilon}^{M} \left(\langle E_{(\cdot)}x, y\rangle\right) \leq \langle (1_H - E_{m-\varepsilon})\, x, x\rangle^{1/2} \langle (1_H - E_{m-\varepsilon})\, y, y\rangle^{1/2} \tag{6.11}$$

for any $x, y \in H$.

This implies, for any $x, y \in H$, that

$$\bigvee_{m-0}^{M} \left(\langle E_{(\cdot)}x, y\rangle\right) \leq \|x\|\, \|y\|, \tag{6.12}$$

where $\bigvee_{m-0}^{M} \left(\langle E_{(\cdot)}x, y\rangle\right)$ *denotes the limit* $\lim_{\varepsilon \to 0+} \left[\bigvee_{m-\varepsilon}^{M} \left(\langle E_{(\cdot)}x, y\rangle\right)\right]$.

The inequality (6.12) was also proved in the recent monographs [25] and [26] and will be utilized in the following.

After these preparations we can state and prove the following trapezoidal type inequality for functions of selfadjoint operators on Hilbert spaces:

Theorem 20. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda | \lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda | \lambda \in Sp(A)\} =: \max Sp(A)$. If $f : [m, M] \to \mathbb{C}$ is such that its derivative f' is of bounded variation on $[m, M]$, then we have the inequality*

$$\begin{aligned}
&\left|\left\langle \left[f(A) - \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m}\right] x, y\right\rangle\right| \\
&\leq \frac{1}{4}(M - m) \bigvee_{m}^{M}(f') \bigvee_{m-0}^{M} \left(\langle E_{(\cdot)}x, y\rangle\right) \\
&\leq \frac{1}{4}(M - m) \bigvee_{m}^{M}(f')\, \|x\|\, \|y\|
\end{aligned} \tag{6.13}$$

for any $x, y \in H$.

Proof. Utilising the inequality (4.1) for the function of bounded variation $g(\lambda) = \langle E_\lambda x, y\rangle$ and the continuous function f we have

$$\begin{aligned}
&\left|\int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y\rangle - f(m) \frac{\int_{m-0}^{M} \langle E_\lambda x, y\rangle\, d\lambda}{M - m}\right. \\
&\left. - f(M)\left(\langle x, y\rangle - \frac{\int_{m-0}^{M} \langle E_\lambda x, y\rangle\, d\lambda}{M - m}\right)\right| \\
&\leq \frac{1}{4}(M - m) \bigvee_{m}^{M}(f') \bigvee_{m-0}^{M} \left(\langle E_{(\cdot)}x, y\rangle\right) \leq \frac{1}{4}(M - m) \bigvee_{m}^{M}(f')\, \|x\|\, \|y\|
\end{aligned} \tag{6.14}$$

for any $x, y \in H$.

By the Spectral Representation Theorem we have

$$\int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y\rangle = \langle f(A)\, x, y\rangle$$

and

$$\begin{aligned}\int_{m-0}^{M}\langle E_\lambda x,y\rangle\,d\lambda &= \langle E_\lambda x,y\rangle\,\lambda|_{m-0}^{M}-\int_{m-0}^{M}\lambda d\langle E_\lambda x,y\rangle\\ &= M\langle x,y\rangle-\langle Ax,y\rangle=\langle(M1_H-A)\,x,y\rangle\end{aligned}$$

for any $x, y \in H$.

If we take $f(t) = t^p$ with $p \geq 1$, then for any positive operator A with $Sp(A) \subset [m, M] \subset [0, \infty)$ we have the inequality

$$\begin{aligned}(6.15)\qquad &\left|\left\langle\left[A^p-\frac{m^p(M1_H-A)+M^p(A-m1_H)}{M-m}\right]x,y\right\rangle\right|\\ &\leq\frac{1}{4}p(M-m)\left(M^{p-1}-m^{p-1}\right)\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)\\ &\leq\frac{1}{4}p(M-m)\left(M^{p-1}-m^{p-1}\right)\|x\|\,\|y\|\end{aligned}$$

for any $x, y \in H$.

If we take the function $f(t) = \ln t$, then for any positive definite operator A with $Sp(A) \subset [m, M] \subset (0, \infty)$ we have the inequality

$$\begin{aligned}(6.16)\qquad &\left|\left\langle\left[\ln A-\frac{\ln m(M1_H-A)+\ln M(A-m1_H)}{M-m}\right]x,y\right\rangle\right|\\ &\leq\frac{1}{4}\frac{(M-m)^2}{mM}\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)\leq\frac{1}{4}\frac{(M-m)^2}{mM}\|x\|\,\|y\|\end{aligned}$$

for any $x, y \in H$.

Finally, if we take $f(t) = \exp(t)$, then we have for any selfadjoint operator A with $Sp(A) \subset [m, M] \subset \mathbb{R}$ the inequality

$$\begin{aligned}(6.17)\qquad &\left|\left\langle\left[\exp(A)-\frac{\exp(m)(M1_H-A)+\exp(M)(A-m1_H)}{M-m}\right]x,y\right\rangle\right|\\ &\leq\frac{1}{4}(M-m)(\exp M-\exp m)\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)\\ &\leq\frac{1}{4}(M-m)(\exp M-\exp m)\|x\|\,\|y\|\end{aligned}$$

for any $x, y \in H$.

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