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Results on Permutation Labeling of Graphs

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ABSTRACT. A $(p,q)-$graph is said to be permutation graph if there exists a bijective function $f : V(G) \rightarrow \{1,2,...,p\}$ such that the induced edge function $g_f : E(G) \rightarrow \mathcal{N}$ is defined as follows

$$g_f(uv) = \begin{cases} {}^{f(u)}P_{f(v)} & if\ f(u) > f(v), \\ {}^{f(v)}P_{f(u)} & if\ f(v) > f(u). \end{cases}$$

In this paper, we investigate the permutation labeling for generalized prism and anti prism, grids, extended sun graph, necklace graph, special comb, comb, Mobius ladder and at the end a very general proof for permutation graphs.

Keywords : permutation, distinct, prism and anti prism, grid, necklace, comb, ladder.

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1. INTRODUCTION AND PRELIMINARY RESULTS

Informally, by a graph labeling we means an assignment of integers to elements of a graph, such as vertices or edges or both, subject to some specified conditions. These conditions are usually expressed on the basis of the values (called weights) of some evaluating function.

In our case, the evaluating function will be simply to produce permutation number of the labeled elements of the graph. One of the situations that we are particularly interested in is when all the edge weights permutation numbers are the different. In such a case we call the labeled graph has a permutation labeling. The study of these graphs was motivated in [4].

The concept of labeling of graphs has gained a lot of popularity in the area of graph theory. This popularity is not only due to mathematical challenges of graph labelling but also to the wide range of applications that graph labeling offer to other branches of science, for instance, x-ray, crystallography, coding theory, cryptography (secret sharing schemes), astronomy, circuit design and communication design.

Hegde and Shetty in [6] define a graph G with p vertices and q edges is said to be permutation graph if there exists a bijective function $f : \mathrm{V}(G) \to \{1, 2, ..., p\}$ such that the induced edge function $g_f : \mathrm{E}(G) \to \mathcal{N}$ is defined as follows

$$g_f(uv) = \begin{cases} {}^{f(u)}P_{f(v)} & \text{if } f(u) > f(v), \\ {}^{f(v)}P_{f(u)} & \text{if } f(v) > f(u). \end{cases}$$

In [3] Baskar and Vishnupriya proved that the complete graph K_n is a permutation graph if and only if $n \le 5$. The edge value in any permutation labelings are large numbers, investigations of suitable additional constraints to control edge values is a scope for further study. Also they strongly believe that all trees admit permutation labelings. In the same paper, they also proved that the following graphs are permutation graphs namely, path P_n cycle C_n star $K_{1,n}$, graphs obtained adding a pendent edge to each edge of a star, graphs obtained by joining the centers of two identical stars with an edge or a path of length 2 and complete binary trees with at least three vertices. In [7] Seoud and Salim determined all permutation graphs of order ≤ 9. They also proved that every bipartite graph of order ≤ 50 has a permutation labeling.

In this paper, we investigate the permutation labelings of some graphs namely prism, anti-prism ,grids, sun, necklace, special comb, comb, Mobius ladder and a very general proof graphs which always contain the permutation labeling.

2. PERMUTATION LABELING OF GENERALIZED PRISM AND GENERALIZED ANTI-PRISM GRAPHS

The generalized prism $\mathbb{D}_n^k$ [1] can be defined as the cartesian product $C_n \times \mathrm{P}_k$ of a cycle of length n with a path on k vertices. Also k denotes the level of generalized prism.

Theorem 2.1. The generalized prism $\mathbb{D}_n^k = C_n \times \mathrm{P}_k$ admits a permutation labeling for every integer $n \ge 4\,; k \ge 1$.

Proof. Denote the vertices of $\mathbb{D}_n^k$ consecutively as $x_1, x_2, ..., x_n, x_{n+1}, ..., x_{nk}$ with $|V| = nk$ and $|V| = 2nk - n$

The Labeling scheme for generalized prism for $n \ge 4\,; k \ge 1$, assign the label to vertices in such a way i.e. $f\left(x_i^j\right) = i + n(j-1)$ for $1 \le i \le n-2$ and $1 \le j \le k$, $f\left(x_{n-1}^j\right) = nj$ for $1 \le j \le k$, and $f\left(x_n^j\right) = nj - 1$ for $1 \le j \le k$. Clearly, the defined labeling is a permutation $gf\,(E) = \{A_1, A_2, A_3, A_4, A_5, A_6\}$ labeling with the edge value set where $A_1 = (nj + i - n + 1)!, A_2 = (nj + i)!,\ A_3 = \dfrac{(nj)!}{2},\ A_4 = (nj)!, A_5 = \dfrac{(nj+i)!}{n!}$ and $A_6 = \dfrac{(nj+n-1)!}{n!}$ are the edge values set is obtained from the edges $\left\{x_i^j x_{i+1}^j : 1 \le i \le n-2, 1 \le j \le k\right\}$,

$\{x_i^j x_i^{j+1}: 1 \le i \le n-2, 1 \le j \le k\}$, $\{x_{n-2}^j x_{n-1}^j: 1 \le j \le k\}$, $\{x_{n-1}^j x_n^j: 1 \le j \le k\}$, $\{x_{n-1}^j x_{n-1}^{j+1}: 1 \le j \le k\}$ and $\{x_n^j x_n^{j+1}: 1 \le j \le k\}$ respectively.

It is easy to check that all edge values are distinct. The union of A_i's are the whole induced edge value set of $gf(E)$ and the pairwise intersection of two elements of A_i's have empty intersection i.e.

$$\bigcup_{i=1}^{6} A = gf(E), A_i \cap A_j = \varphi, \forall i \ne j$$

So the generalized prism $\mathbb{D}_n^k$ admits permutation labeling.

The generalized anti-prism A_n^k [1] can be obtained by completing the generalized prism $C_n \times P_j$ by edges say, $A = \{x_i^{j+1} x_{i+1}^j: 1 \le i \le n-1, 1 \le j \le k-1\} \cup \{x_n^{j+1} x_1^j: 1 \le j \le k-1\}$. So the vertex set and edge set of generalized anti-prism is $(A_n^k) = V(\mathbb{D}_n^k)$ and $V(A_n^k) = E(\mathbb{D}_n^k) \cup \{A\}$, respectively.

Theorem 2.2. The generalized anti-prism admits a permutation labeling for every integer $n \ge 4; k \ge 1$.

Proof. The labeling scheme for generalized anti-prism A_n^k is same as the labeling of generalized prism $\mathbb{D}_n^k$. After adding new edges, we have some more edge value set i.e. $\{A_7, A_8, A_9, A_{10}\}$, where $A_7 = \frac{(nj+i+1)!}{(i+n)!}, A_8 = \frac{(nj+n)!}{(n+2)!}, A_9 = \frac{(nj+n+1)!}{(n-1)!}, A_{10} = \frac{(nj+1)!}{2}$ are the edge values set is obtained from of edges $\{x_i^j x_{i+1}^{j+1}: 1 \le i \le n-3, 1 \le j \le k-1\}$, $\{x_{n-2}^j x_{n-1}^j: 1 \le j \le k-1\}$, $\{x_{n-1}^j x_n^j: 1 \le j \le k-1\}$ and $\{x_n^j x_1^j: 1 \le j \le k-1\}$ respectively. Thus the whole edge set for generalized anti-prism is $gf(E) = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}\}$.

We see that all edge values are distinct. The union of A_i's are the whole induced edge value set of $gf(E)$ and the pairwise intersection of two elements of A_i's is have empty intersection. So the generalized anti-prism A_n^k admits permutation labeling. A two dimensional grid graph is a graph that is cartesian product $P_n \times P_j$ of path graphs. The grid graph $G_{n,j} = P_n \times P_j$ can also be constructed after removing the k edges $x_i^j x_{i+1}^j$ from each cycle C_n of generalized prism.

For proving the some results of this paper, we frequently use the lemma which is given as below.

Lemma 2.3. If G ia a permutation graph then $\{\{K_6 | e_1\} | e_2\}$ (= the graph obtained from G by deleting the edge e) is also a permutation graph.

Proof. Since there is no repeated edge labels in G then we use the same labeling of G to label $G|e$, so $G|e$ has no repeated edge labels.

Now using Lemma 2.3 and the labeling of generalized prism we have the following corollary.

Corollary 2.4. The grid graph $G_{n,j}$ admits permutation labeling for every integer $n, j \ge 2$.

An $n-$sun graph is defined as a cycle C_n with an edge terminating a vertex of degree 1attached to each vertex. The extended sun graph S_n^k consist of a cycle of length n with k edges path (the tail) attached to each vertex of the cycle C_n. It can also be obtained from the generalized prism graph by removing all edges of cycle C_n from level two to onwards.

By Lemma 2.3 and using the labeling of generalized prism we have the following corollary.

Corollary 2.5. The extended sun graph S_n^k admits permutation labeling for every k and $n \ge 4$.

Remark 2.6. If G is a permutation graph, then $G|v$ (= the graph obtained from G by deleting the vertex v) has may or may not be the permutation labeling. As in Fig 1, we see that the graph has a permutation labeling because there are not pair of same permutation $\left(P_4^5, P_3^6\right)$ and $\left(P_1^6, P_2^3\right)$ while after deleting the pendent vertex (vertices), the graph may or may not has the permutation labeling. After the deleting of one pendent vertex, we can rearrange the label to make a graph is a permutation graph but after deleting the second pendent vertex it has not permutation labeling, because one of the above pair occur in new graph. Also [7] Seoud and Salim showed that $\left\{\left\{K_6|e_1\right\}|e_2\right\}$ has a permutation labeling. It means that K_6 itself has not permutation labeling. But the original graph has a permutation labeling.

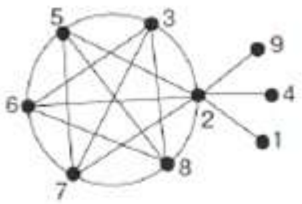

Fig.1. Permutation graph

After removing the vertices from S_n^j we have a kite graph $(n, j)-$kite consists of a cycle of length n with $j-$edge path (the tail) attached to one vertex of C_n.

Corollary 2.7. The kite graph $k(n, \mathrm{m})$ for every $n > 4, \mathrm{m} > 1$ admits a permutation labeling.

3. PERMUTATION LABELING OF NECKLACE AND MOBIUS LADDER GRAPHS

Let P_n be a path of length $\mathrm{n}-1$ with vertices labeled from 1 to n along P_n. The comb Cb_n is the tree consisting of P_n together with edges $\left\{x_i\, y_{i-1} : i \in \{2,\ldots,\mathrm{n}-1\}\right\}$. The special comb Cb_n^* is obtained from the comb Cb_n by joining a new edges $x_1 y_1$ and $x_n y_{n-2}$. Also the necklace graph Ne_n is obtained from Cb_n^* by adding the edge $x_1 x_n$.

Theorem 3.1. The necklace graph Ne_n admits permutation labeling.

Proof. For our convenience, we defined the vertex set and edge set of necklace graph Ne_n as $(Ne_n) = \{u, v\} \cup \{x_i : 1 \le i \le n\} \cup \{y_i : 1 \le i \le n\}$ and $E(Ne_n) = \{ux_1, uy_1, vy_n, uv\} \cup \{x_i x_{i+1}, \mathrm{y}_i\, y_{i+1} : 1 \le i \le n-1\} \cup \{x_i y_i : 1 \le i \le n\}$. We can define the required labeling for necklace graph Ne_n as that, we assign label to vertices in such a way that $\left\{x_1 x_{\frac{n}{2}+1}\right\}, \dfrac{\left(\frac{n}{2}+2\right)!}{\left(\frac{n}{2}+1\right)!}$ for $2 \le i \le n$ and $f(y_i) = 2n+3-i$ for $1 \le i \le n$. Clearly thedescribed labeling is the permutation labeling. The edge value $(i+1)!$ is obtained from the edge $\{x_i x_{i+1} : 2 \le i \le n-1\}, (2n+3-i)!$ from $\{y_i y_{i+1} : 2 \le i \le n-1\}, \dfrac{(2n+3-i)!}{(2n+2-2i)!}$ from $\{x_i x_i : 2 \le i \le n\}, \{(n+2)!, (n+3)!\}$ from $\{x_n v, \mathrm{y}_n\, v\}$ and $\{2, 3, 2(n+1), 2(n+1)(2n+1)\}$ from $\{ux_1, x_1x_2, x_1y_1, uy_1\}$ respectively.

One can easily compute that all edge values are distinct. The union of A_i's are the whole induced edge value set of $gf(E)$, the pairwise intersection of two elements of A_i's have empty intersection i.e. $A_i \bigcap A_j = \varphi, \forall i \neq j$.The necklace graph Ne_n admits permutation labeling.

From Theorem 3.1 and Lemma 2.3 we have the following corollaries.

Corollary 3.2. The special comb Cb_n graph admits a permutation labeling.

Corollary 3.3. The comb graph Cb_n admits a permutation labeling.

The Mobius ladder M_n [2] is defined as a cubic circulant graph with an even number of vertices, formed from an $n-$ cycle by adding edges(called "rungs") connecting opposite pairs of vertices in the cycle.

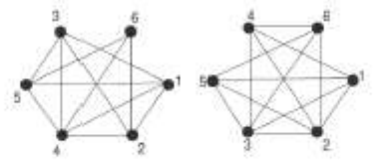

Fig. 2. Permutation labeling of $\{K_6|\{e_1, e_2\}\}$

It is normal because (with the exception of $M_6 = K_{3,3}$) M_n has exactly $\frac{n}{2} 4-$ cycles which link together by their shared edges to form a topological Mobius strip. Mobius ladders were named and first studied by Guy and Harary in [5].

Theorem 3.4. The Mobius ladder M_n admits permutation labeling for $n \geq 6$.

Proof. Define the required permutation labeling for Mobius ladder in the following way that we assign label to vertices $f(x_1) = 2, f(x_2) = 2$ and $f(x_i) = i$ for $3 \leq i \leq n.$ The above defined labeling is a permutation labeling for Mobius ladder. The edge values are distinct. We obtain the edge value $(i+1)!$ from $\{x_i x_{i+1}\}$ for $3 \leq i \leq n, \frac{(\frac{n}{2}+1)!}{(\frac{n}{2}-1)!}$ from $\{x_2 x_{\frac{n}{2}+2}\}, \frac{(\frac{n}{2}+i)!}{(\frac{n}{2})!}$ from $\{x_i x_{\frac{n}{2}+i}\}$ for $1 \leq i \leq \frac{n}{2}$ and $\{2, 3, n(n+1)\}$ from $\{x_1 x_2, x_2 x_3, x_n x_1\}$ respectively.

Obviously the above graph has a permutation labeling under the above labeling scheme with edge values as described above also. One can easily compute that union of A_i's are the whole edge value, the pairwise intersection of A_i's are empty, $A_i \bigcap A_j = \varphi, \forall i \neq j.$ So the Mobius ladder admits permutation labeling.

4. PERMUTATION LABELING OF GENERAL GRAPH

In general the complete graph K_n is not permutation graph for all n. However it is known for some n that if we omit some number of edges from complete graph K_n, it become a permutation graph. For any n, the number of edges in K_n are $\binom{n}{2}$. There must exist an integer $p \in \mathbb{N}$ such that the subgraph of K_n for $t \geq p,$ we have $K_n | t = \{K_n | \{e_1, e_2, ..., e_t\}\}$ is a permutation graph. For instance, K_6 is not a permutation graph. However, if we remove two or more edges from K_6, then there are two possibilities to remove the edges from K_6 i.e. either they are adjacent or not. But the resulting graph admits permutation labeling(See Fig. 2).

Theorem 4.1. A $(p,q)-$graph is a permutation graph if and only if there is η_p such that for all $r \geq \eta_p$ with $q \leq \binom{p}{2} - r$

Proof. If then the graph $q = \binom{p}{2} - r$, then the graph $G \cong K_p|\eta_p$ and η_p denotes the number of same permutation as shown in Table 1. If $p \leq 5$, there exist no two pairs of number having the same permutation value i.e $\eta_p = 0$ for $p \leq 5$. For $p \geq 6$ we have η_p number of same permutations. If we delete η_p edges from K_p, we obtain the distinct permutation label for each edge. So the graph $G \cong K_p|\eta_p$ is a permutation graph. Also for some $r > \eta_p$ if $G \cong K_p|r$ then G is a proper subgraph of $K_p|\eta_p$, which is by Lemma 2.3 a permutation graph. Thus for all $r > \eta_p$ we obtained $G \cong K_p|r$ is a permutation graph. It implies that $E(G) \leq \binom{p}{2} - r$, shows G has a permutation labeling.

Conversely, Let G be a permutation graph. Suppose contrary $q \geq \binom{p}{2} - r$ for all $r \geq \eta_p$.This implies that there exist $s \leq \eta_p$ such that $G \cong K_p|s$.

Since $s \leq \eta_p$, so we must have at least two pair of numbers whose permutation values are equal. Labeling those numbers to adjacent vertices makes sure that $K_p|s$ is not a permutation graph. Which contradicts our supposition.

Finally a $(p,q)-$graph with $q \leq \binom{p}{2} - r$ admits a permutation labeling.

Parameter p	No. of same permutations η_p
$p \leq 5$	0
$6 \leq p \leq 9$	2
$10 \leq p \leq 11$	4
$12 \leq p \leq 14$	6
$15 \leq p \leq 19$	8
$20 \leq p \leq 23$	10
$24 \leq p \leq 29$	12
$30 \leq p \leq 41$	14
$42 \leq p \leq 50$	16

Table 1

We can find an η_p for every value of p.

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