



Improved Newton-Raphson methods for Solving Nonlinear Equations

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ABSTRACT

In this paper, we mainly study the numerical algorithms for simple root of nonlinear equations based on Newton-Raphson method. Two modified Newton-Raphson methods for solving nonlinear equations are suggested. Both of the methods are free from second derivatives. Numerical examples are made to show the performance of the presented methods, and to compare with other ones. The numerical results illustrate that the proposed methods are more efficient and performs better than Newton-Raphson method.

Keywords

Newton-Raphson method; Nonlinear equations; Iterative method; Order of convergence; Efficiency index.

SUBJECT CLASSIFICATION

Mathematics Subject Classification: 41A25, 65H05, 65D99

Introduction

Nonlinear problems is an important direction of research in the field of numerical calculation. Solving nonlinear equations is a classical problem which has interesting applications in various branches of science and engineering, and the problem of solving nonlinear equations by numerical methods has gained more importance than before.

To solve nonlinear equations, we can use iterative methods such as Newton's method (NM), Newton-Raphson method (NRM) or their variants.

In this paper, to improve the convergence properties, we consider iterative method to find a simple root x^* of a nonlinear equation $f(x) = 0$, where $f : \Gamma \subseteq \mathbb{R} \rightarrow \mathbb{R}$ for an open interval Γ is a scalar function and it is sufficiently differentiable in a neighborhood of x^* .

It is well known that Newton-Raphson method is a basic and important method for solving nonlinear equation [1] using the iterative expression

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

which is quadratically convergent in the neighborhood of x^* .

To improve the convergence properties, many variants of Newton-Raphson method has been proposed in the last years. In [2], Potra and Pták presented the following iterative method (PPM)

$$x_{n+1} = x_n - \frac{f(x_n) + f(y_n)}{f'(x_n)} \tag{2}$$

which is cubically convergent. In [3], Changbum Chun presented the following iterative scheme (CM)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(y_n)}{f'(x_n)} + \frac{f(y_n)f'(y_n)}{f'^2(x_n)} \tag{3}$$

which is fourth-order convergent.

In recent years, much attention has been given to develop iterative methods for solving nonlinear equations and a vast literature has been produced [3-10].

Motivated and inspired by the on going activities in this direction, the aim of this paper is to construct new efficient iterative methods to solve nonlinear equations. Based on Newton-Raphson method, we present two improved Newton-Raphson methods, and the convergence of the new scheme is proved. The proposed methods do require the computation of only first-order derivatives unlike other methods requiring the computation of second or higher derivatives. The order of convergence of the proposed algorithms is established. Several examples are presented and compared to other methods, showing the accuracy and fast convergence of the new algorithms.

The Improved Newton-Raphson Methods And Their Convergence

Let us consider the following iterative method.

Algorithm 1. For given x_0 , we consider the following iteration scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{4}$$



$$w_n = y_n - \frac{2f(y_n)}{f'(x_n)} + \frac{f(y_n)f'(y_n)}{f'^2(x_n)}, \quad (5)$$

$$x_{n+1} = w_n - \frac{3f'^2(x_n) + f'^2(y_n)}{2f'(x_n)f'(y_n) + 2f'^2(y_n)} \frac{f(w_n)}{f'(x_n)}. \quad (6)$$

Now, we are in the position to give the convergence of Algorithm 1.

Theorem 1. Assume that the function $f : \Gamma \subseteq \mathbb{R} \rightarrow \mathbb{R}$ has a single root $x^* \in \Gamma$, where Γ is an open interval. If $f(x)$ has first, second and third derivatives in Γ , then Algorithm 1 is sixth-order convergent in a neighborhood of x^* and it satisfies error equation

$$e_{n+1} = 5c_2^5 e_n^6 + O(e_n^7), \quad (7)$$

where

$$e_n = x_n - x^*, \quad c_k = \frac{f^{(k)}(x^*)}{k!f'(x^*)}, \quad k = 1, 2, \dots. \quad (8)$$

Proof. Let $x^* \in \Gamma$ be the simple root of $f(x)$,

$$c_k = \frac{f^{(k)}(x^*)}{k!f'(x^*)}, \quad k = 1, 2, \dots \text{ and } e_n = x_n - x^*.$$

Consider the iteration function $F(x)$ defined by

$$F(x) = w(x) - \frac{3f'^2(x) + f'^2(y(x))}{2f'(x)f'(y(x)) + 2f'^2(y(x))} \frac{f(w(x))}{f'(x)} \quad (9)$$

where

$$w(x) = y(x) - \frac{2f(y(x))}{f'(x)} + \frac{f(y(x))f'(y(x))}{f'^2(x)}, \quad y(x) = x - \frac{f(x)}{f'(x)}. \quad (10)$$

By some computations using Maple we can obtain

$$F(x^*) = x^*, \quad F^{(i)}(x^*) = 0, \quad i = 1, 2, 3, 4, 5, \quad F^{(6)}(x^*) = \frac{225f^{(2)}(x^*)^5}{2f'(x^*)^5}. \quad (11)$$

Furthermore, from the Taylor expansion of $F(x_n)$ around x^* , we have

$$\begin{aligned} x_{n+1} = F(x_n) = & F(x^*) + F'(x^*)(x_n - x^*) + \frac{F^{(2)}(x^*)}{2!}(x_n - x^*)^2 + \frac{F^{(3)}(x^*)}{3!}(x_n - x^*)^3 \\ & + \frac{F^{(4)}(x^*)}{4!}(x_n - x^*)^4 + \frac{F^{(5)}(x^*)}{5!}(x_n - x^*)^5 + \frac{F^{(6)}(x^*)}{6!}(x_n - x^*)^6 + O((x_n - x^*)^7). \end{aligned} \quad (12)$$

Substituting (11) into (12) yields

$$x_{n+1} = x^* + e_{n+1} = x^* + 5c_2^5 e_n^6 + O(e_n^7).$$

Therefore, we have



$$e_{n+1} = 5c_2^5 e_n^6 + O(e_n^7)$$

which shows that the algorithm 1 is sixth-order convergent.

Now, we give the other further improved Newton-Raphson method.

Algorithm 2. For given x_0 , we consider the following iteration scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{13}$$

$$w_n = y_n - \frac{2f(y_n)}{f'(x_n)} + \frac{f(y_n)f'(y_n)}{f'^2(x_n)}, \tag{14}$$

$$x_{n+1} = w_n + \frac{f'(x_n) + f'(y_n)}{f'(x_n) - 3f'(y_n)} \frac{f(w_n)}{f'(x_n)}. \tag{15}$$

Next, we consider the convergence of Algorithm 2.

Theorem 2. Assume that the function $f : \Gamma \subseteq \mathbb{R} \rightarrow \mathbb{R}$ has a single root $x^* \in \Gamma$, where Γ is an open interval. If $f(x)$ has first, second and third derivatives in Γ , then Algorithm 2 is seventh-order convergent in a neighborhood of x^* and it satisfies error equation

$$e_{n+1} = 10c_2^4(c_2^2 - c_3)e_n^7 + O(e_n^8), \tag{16}$$

where

$$e_n = x_n - x^*, \quad c_k = \frac{f^{(k)}(x^*)}{k!f'(x^*)}, \quad k = 1, 2, \dots$$

Proof. Let $x^* \in \Gamma$ be the simple root of $f(x)$,

$$c_k = \frac{f^{(k)}(x^*)}{k!f'(x^*)}, \quad k = 1, 2, \dots \text{ and } e_n = x_n - x^*.$$

Consider the iteration function $F(x)$ defined by

$$F(x) = w(x) + \frac{f'(x) + f'(y(x))}{f'(x) - 3f'(y(x))} \frac{f(w(x))}{f'(x)} \tag{17}$$

where

$$w(x) = y(x) - \frac{2f(y(x))}{f'(x)} + \frac{f(y(x))f'(y(x))}{f'^2(x)}, \quad y(x) = x - \frac{f(x)}{f'(x)}.$$

By some computations we can obtain

$$F(x^*) = x^*, \quad F^{(i)}(x^*) = 0, \quad i = 1, 2, 3, 4, 5, 6, \\ F^{(7)}(x^*) = -\frac{225f^{(4)}(x^*)[-3(f''(x^*))^2 + 2f^{(3)}(x^*)f'(x^*)]}{2f'(x^*)^6}. \tag{18}$$

Furthermore, from the Taylor expansion of $F(x_n)$ around x^* , we have



$$\begin{aligned}
 x_{n+1} = F(x_n) = & F(x^*) + F'(x^*)(x_n - x^*) + \frac{F^{(2)}(x^*)}{2!}(x_n - x^*)^2 + \frac{F^{(3)}(x^*)}{3!}(x_n - x^*)^3 \\
 & + \frac{F^{(4)}(x^*)}{4!}(x_n - x^*)^4 + \frac{F^{(5)}(x^*)}{5!}(x_n - x^*)^5 + \frac{F^{(6)}(x^*)}{6!}(x_n - x^*)^6 \\
 & + \frac{F^{(7)}(x^*)}{7!}(x_n - x^*)^7 + O((x_n - x^*)^8).
 \end{aligned} \tag{19}$$

Substituting (18) into (19) yields

$$x_{n+1} = x^* + e_{n+1} = x^* + 10c_2^4(c_2^2 - c_3)e_n^7 + O(e_n^8).$$

Therefore, we have

$$e_{n+1} = 10c_2^4(c_2^2 - c_3)e_n^7 + O(e_n^8)$$

which shows that the algorithm 2 is seventh-order convergent.

Numerical Results

Now, we employ Algorithm 1 and Algorithm 2 presented in this paper to solve some nonlinear equations and compare them with NRM and PPM. Displayed in Table 1 are the number of iterations (ITs) required such that $|f(x_n)| < 1.E-14$.

Table 1. Comparison of Algorithm 1, Algorithm 2, NRM PPM and CM

Functions	x_0	NRM	PPM	CM	Algorithm 1	Algorithm 2
f_1	2.4	7	4	3	3	3
	1.6	6	12	5	4	4
f_2	0.6	4	3	3	3	2
	3.1	7	3	3	2	3
f_3	1.4	6	4	3	3	3
	2.1	5	4	4	3	3
f_4	3.8	16	12	8	7	6
	3.3	10	7	7	6	6
f_5	7.7	13	9	6	5	4
	-0.8	6	5	4	4	3

In table 1, we use the following functions.

$$f_1(x) = (x-1)^3 - 1, \quad x^* = 2.$$

$$f_2(x) = \cos x - x, \quad \alpha = 0.73908513321516.$$

$$f_3(x) = \sin^2(x) - x^2 + 1, \quad x^* = 1.40449164885154.$$

$$f_4(x) = e^{x^2+7x-30} - 1, \quad x^* = 3.$$



$$f_5(x) = (x+2)e^x - 1, x^* = -0.44285440096708.$$

The computational results in Table 1 show that Algorithm 1 and Algorithm 2 require less ITs than NRM and PPM. Therefore, the present algorithms are of practical interest and can compete with NRM, PPM, CM and some existing algorithms.

Conclusions

With the wide development of science and technology, the problem of solving nonlinear equations by numerical methods has gained more importance than before. In order to obtain efficient algorithm for the nonlinear equations which come from the practical problems, in this paper, we present and analyze two improved Newton-Raphson methods for solving nonlinear equations. The methods are free from second derivatives. Several numerical results illustrate the convergence behavior and computational efficiency of the proposed methods. Computational results demonstrate that they are more efficient and perform better than Newton-Raphson method and some existing methods.

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