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Solving the oscillation equation with fractional order damping term using a new Fourier transform method

Ozlem Ozturk Mizrak, Nuri Ozalp
Karabuk University, Department of Mathematics, Baliklarkayasi Mevkii 78050, Karabuk, Turkey
oomizrak@ankara.edu.tr
Ankara University, Department of Mathematics, 06100 Besevler, Ankara, Turkey
nozalp@science.ankara.edu.tr

ABSTRACT

We propose an adapted Fourier transform method that gives the solution of an oscillation equation with a fractional damping term in ordinary domain. After we mention a transformation of cosmic time to individual time (CTIT), we explain how it can reduce the problem from fractional form to ordinary form when it is used with Fourier transformation, via an example for 1<α<2, where α is the order of fractional derivative. Then, we give an application of the results.

Keywords: Fourier transform, oscillation equation, damping term, the CTIT transformation, fractional derivative.

Mathematics Subject Classification: 26A33, 42A38, 42B10

1. INTRODUCTION

Fourier analysis is one of those areas of mathematics that is very useful for modelling physical phenomena, including acoustics, heat transport, electricity and magnetism, quantum mechanics and many others [1]. On the other side, in recent decades, non integer (fractional) differentiation has become a more and more popular tool for modeling physical systems from diverse areas such as heat flow [2], electrical circuits [3]-[5], control [6]-[8] and medicine [9]. Thus the hug of Fourier analysis and fractional analysis was inevitable and in the literature, one can find so many fractional calculus applications that use Fourier transform as a solution method [10]-[11]. While some of them introduce an α-th order Fourier transform operator ($\mathcal{F}_\alpha$) in different forms [12]-[13], some use conventional Fourier transform operator ($\mathcal{F}$) directly [10],[14] or implicitly [11] but as far as we know, in the current literature, there is not any approach that gives Fourier transform of the α-th derivative of a function in terms of Fourier transform of another function's integer order derivative. Such an approach means reducing the process of working with Fourier transform from fractional domain to integer domain that gives the final solution in a simpler form. The main objective of this paper is to test if such an alternative treatment is possible rather than implementing Fourier transform method directly while finding the solution of an oscillation equation with a fractional damping term.

In this context, the article is organized as follows: In Section 2, some preliminaries about fractional calculus are presented. In Section 3, for the convenience of the reader, we mention about the motivation under the proposed Fourier transform method. Some main results are shown in Section 4. Section 5 is devoted to an example to show the adapted Fourier transform method works properly making comparison with the solution in [15]. Finally, some conclusions are given in Section 6.

2. PRELIMINARIES

Definition 1. The Riemann-Liouville type fractional derivative of order $\alpha > 0$ of a function $f:(-1,1) \to R$ is defined by

$$D_+^\alpha f(t) = \frac{d^n}{dt^n}\frac{1}{\Gamma(n-\alpha)}\int_{-\infty}^{t}(t-\tau)^{n-\alpha-1}f(\tau)d\tau \quad (2.1)$$

where $n-1$ is the integer part of α.

Definition 2. The Caputo type fractional derivative of order $\alpha > 0$ of a function $f:(-1,1) \to R$ is defined by

$$^{C}D_+^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{-\infty}^{t}(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau \quad (2.2)$$

where $n-1$ is the integer part of α.

We note that because of the lower terminal of the integrals in the above denitions is taken as infinity, actually these denitions coincide with each other (see 2.271 in [16]).

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3. MOTIVATION UNDER THE PROPOSED METHOD

It is generally known that as conventional calculus includes just integer order differential and integral operators, it significantly simplify its use for solving applied problems in various fields of science. However, in case of fractional calculus, it is not so, even if it represents a rapidly growing field both in theory and in applications to real world problems. Because of this, if a link between the ordinary domain and the fractional domain was provided, it would be ideal to facilitate the fractional order applications. Such a relation might be constructed on using two kinds of time: the individual time, τ, (which is considered as flowing equably) and the cosmic time, T, (which flows non-equably). Transformation from Cosmic Time to Individual Time (CTIT) can be described by the equation

$$T = g_t(\tau) = \frac{1}{\Gamma(\alpha+1)}(t^\alpha - (t-\tau)^\alpha) \quad (3.1)$$

or equivalently

$$\tau = t - (t^\alpha - T\Gamma(\alpha+1))^{1/\alpha} \quad (3.2).$$

This relationship between two time scales is used for giving a meaningful geometric and physical interpretation of fractional integration and fractional differentiation by the paper of Igor Podlubny and interested readers can find a detailed information on these two kinds of time in [17]. By using above equality, a transformation method that gives the exact solution of a fractional differential equation in terms of the solution of the corresponding integer order differential equation is presented in [18]. q extension of this method is also presented in [19]. Also, we have recently found a fractional Laplace transform method in the framework of this transformation [20]. Following the same methodology we generate a new adapted Fourier transform method which can be used to solve oscillation equations with a damping term in the form:

$$x''(t) + aD_+^\alpha x(t) + bx(t) = f(t) \quad (3.3)$$

where D_+^α represents any of the mentioned fractional differentiations (Riemann- Liouville D_+^α or Caputo $^cD_+^\alpha$), $\alpha \in (0,2)$, $a, b \in \mathbb{R}$ are the damping and the stifness constants per mass, x (the displacement at time t of a mass) and f (the external force per mass) are square-summable over $\mathbb{R}$, which implies that $x(t)$ and $f(t)$ approach 0 in the mean for $t \to \pm 1$.

4. MAIN RESULTS

Fourier transform of a functions α-th derivative is given by the following theorem in [16].

Theorem 1.

$$\mathcal{F}(D_+^\alpha x(t)) = (-i\omega)^\alpha \mathcal{F}(x(t)) \quad (4.1)$$

where $\alpha > 0$, $n-1 < \alpha < n (n \in \mathbb{N})$, $\omega \in \mathbb{R}$, $x(t) \in C^n$ and $x^{(k)}(t)(k = 0,1,2,\ n-1)$ vanishes for $t \to \pm 1$.

Theorem 2.

$$\mathcal{F}(D_+^\alpha x(t)) = (-i\omega)^{\alpha-n} \mathcal{F}(x_*^{(n)}(t^\alpha/\Gamma(\alpha+1)) \quad (4.2)$$

where $\alpha > 0$, $n-1 < \alpha < n$, $n = [\alpha]+1$, $\omega \in \mathbb{R}$, $x(t) \in C^n$ $x^{(k)}(t)(k = 0,1,2,\ n-1)$ vanishes for $t \to \pm 1$ and $x_*(T) = x(t - (t^\alpha - T\Gamma(\alpha+1))^{1/\alpha})$.

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Proof. We use the induction method for the proof: First we should check the validity of the theorem for $n = 1$. Let us start with the Laplace transform of the function

$$y_n(t) = (t^\alpha/\Gamma(\alpha+1))^{\frac{n-\alpha-1}{\alpha}}$$

which can be written as

$$\mathcal{L}(y_n(t)) = (\Gamma(\alpha+1))^{\frac{\alpha+1-n}{\alpha}}\Gamma(n-\alpha)s^{\alpha-n}$$

where $\alpha > 0, n-1 < \alpha < n, n =$[α]+1. Then the Fourier transform of the function

$$y_n^+(t) = \begin{Bmatrix} (t^\alpha/\Gamma(\alpha \quad +1))^{\frac{n-\alpha-1}{\alpha}} & t \geq 0 \\ 0 & t < 0 \end{Bmatrix}$$

can be written immediately as

$$\mathcal{F}(y_n^+(t)) = (\Gamma(\alpha+1))^{\frac{\alpha+1-n}{\alpha}}\Gamma(n-\alpha)(-i\omega)^{\alpha-n}.$$

When one substitutes the equality (3.2) into the denition of $\mathcal{F}(D_+^\alpha x(t))$, the equality (4.2) is easily obtained:

$$\mathcal{F}(D_+^\alpha x(t)) = \int_{-\infty}^{\infty} e^{i\omega t}\,(\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{t}(t-\tau)^{-\alpha}x'(\tau)d\tau)dt$$

$$= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)}\int_{-\infty}^{\infty} e^{i\omega t}\int_{-\infty}^{t^\alpha/\Gamma(\alpha+1)} x_*(T)((t^\alpha/\Gamma(\alpha+1)) - T)^{-1}dTdt$$

$$= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)}\int_{-\infty}^{\infty} e^{i\omega t}\{x_*(t^\alpha/\Gamma(\alpha+1)) * y_1^+(t)\}dt$$

$$= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)}\mathcal{F}(x_*'(t^\alpha/\Gamma(\alpha+1)))\mathcal{F}(y_1^+(t))$$

$$= (-i\omega)^{\alpha-1}\mathcal{F}\left(x_*(t^\alpha/\Gamma(\alpha+1))\right)$$

Now assume that the equality (4.2) is provided for $n = k$ where $k =$ 2,3,4,…Then is the equality (4.2) vericated for $n = k+1$ where $k = 2,3,4,\ldots$ or is the equality

$$\mathcal{F}(D_+^\alpha x(t)) = (-i\omega)^{\alpha-k-1}\mathcal{F}\left(x_*^{(k+1)}(t^\alpha/\Gamma(\alpha+1))\right) \quad (4.3)$$

true for $k < \alpha < k+1$?

As $D^{\alpha+m}(x(t)) = D^\alpha(D^m x(t))$ where $m = 0,1,2,..$ and $n-1 < \alpha < n$ (see (2.142), (2.143) in [16]), $\mathcal{F}(x^{(\alpha)}(t))$ can be written as $\mathcal{F}((x^{(k)})^{(\alpha-k)}(t))$ where $k = 2,3,4,\ldots$ and $k < \alpha < k+1$. So

$$\mathcal{F}(x^{(\alpha)}(t)) = \mathcal{F}((x^{(k)})^{(\alpha-k)}(t))$$

$$= (-i\omega)^{\alpha-k-1}\mathcal{F}\left((x_*^{(k)}(t^\alpha/\Gamma(\alpha+1)))'\right)$$

$$= (-i\omega)^{\alpha-k-1}\mathcal{F}\left(x_*^{(k+1)}(t^\alpha/\Gamma(\alpha+1))\right)$$

and the proof is completed.

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Theorem 3. *Assume that* $x \in C^2[(-1,1)]$ *and* $x(t), x'(t)$ *vanishes for* $t \to \pm\infty$.

$$\mathcal{F}(x(t)) = \Gamma(2-\alpha)\Gamma(\alpha)(-i\omega)^{\alpha-1}\mathcal{F}\left(x_*\left(t^\alpha/\Gamma(\alpha+1)\right)\right) \quad (4.4)$$

where $1 < \alpha < 2, \omega \in \mathbb{R}$ and $x_*(T) = x(t-(t^\alpha - T\Gamma(\alpha+1))^{1/\alpha})$.

Proof. First assume that

$$x'(t) = g(t). \quad (4.5)$$

This is equivalent to the equality

$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau. \quad (4.6)$$

Let us use the CTIT transformation (3.2). (4.6) can be written as

$$x(t) = \Gamma(\alpha)\int_{-\infty}^{t^\alpha/\Gamma(\alpha+1)} g\left(t-(t^\alpha - T\Gamma(\alpha+1))^{1/\alpha}\right)(t^\alpha - T\Gamma(\alpha+1))^{1/\alpha-1}dT$$

$$= \Gamma(\alpha)\int_{-\infty}^{t^\alpha/\Gamma(\alpha+1)} g_*(T)(t^\alpha - T\Gamma(\alpha+1))^{1/\alpha-1}dT.$$

So taking the equality $g_*(T) = x_*'(T)$ into consideration $\mathcal{F}(x(t))$ can be written as follows:

$$\mathcal{F}(x(t)) = \Gamma(\alpha)\int_{-\infty}^{\infty} e^{i\omega t}\int_{-\infty}^{t^\alpha/\Gamma(\alpha+1)} g_*(T)(t^\alpha - T\Gamma(\alpha+1))^{1/\alpha-1}dT$$

$$= \Gamma(\alpha)(\Gamma(\alpha+1))^{1/\alpha-1}\int_{-\infty}^{\infty} e^{i\omega t}\{g_*(t^\alpha/\Gamma(\alpha+1)) * (t^\alpha/\Gamma(\alpha+1))dt$$

$$= \Gamma(2-\alpha)\Gamma(\alpha)(-i\omega)^{\alpha-2}\mathcal{F}(g_*\left(t^\alpha/\Gamma(\alpha+1)\right))$$

$$= \Gamma(2-\alpha)\Gamma(\alpha)(-i\omega)^{\alpha-2}\mathcal{F}(x_*'\left(t^\alpha/\Gamma(\alpha+1)\right))$$

$$= \Gamma(2-\alpha)\Gamma(\alpha)(-i\omega)^{\alpha-1}\mathcal{F}(x_*\left(t^\alpha/\Gamma(\alpha+1)\right)).$$

5. EXAMPLE

Example 1. The generalized damping equation [15] is given by (3.3). Let us take $a = b = 1, \alpha = 3/2, \omega \in \mathbb{R}$ and $f(t) = 1/t$. Applying Fourier transform (4.2) to (3.3) and taking (4.4) into account, we find the solution of the given fractional differential equation as

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\frac{i\sqrt{\frac{\pi}{2}}}{k(\omega)}d\omega \quad (5.1)$$

where $k(\omega) = \frac{\pi}{2}((-i\omega)^{5/2} + (-i\omega)^{1/2}) + (-i\omega)^{3/2}$.

Whereas, for the same equation, solution in [15] is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\tau\omega}\frac{1}{p(\omega)(t-\tau)}d\omega d\tau \quad (5.2)$$

where $p(\omega) = (i\omega)^2 + (i\omega)^{3/2} + 1$.

6. CONCLUSION

In this paper, we propose an alternative way including conventional Fourier transform to solve the generalized damping equations. Our proposed methodology produces the solution just by applying a transformation, namely the CTIT transformation, to the definition of a function's α-th order derivative and reduces the problem into ordinary domain, so it yields the final solution in a simpler form. Also, an illustrative example is given to provide a comparison with the solution given in [15].

REFERENCES

1. B. J. West, M. Bologna and P. Grigolini, Physics of Fractal Operators, New York, 2003.

10.24297/jam.v13i5.6483

2. Y. Zhang, H. M. Srivastava and M. C. Baleanu, Local fractional variational iteration algorithm II for non-homogeneous model associated with the non-differentiable heat flow, Advances in Mechanical Engineering, 2015, Vol. 7(10) 15.
3. A. Alsaedi, J. J. Nieto and V. Venktesh, Fractional electrical circuits, Advances in Mechanical Engineering, 2015, Vol. 7(12) 17.
4. A. Atangana, J. J. Nieto, Numerical solution for the model of RLC circuit via the fractional derivative without singular kernel, Advances in Mechanical Engineering, 2015, Vol. 7(10) 17.
5. A. Atangana, B. S. Alkahtani, Extension of the resistance, inductance, capacitance electrical circuit to fractional derivative without singular kernel, Advances in Mechanical Engineering, 2015, Vol. 7(6) 16.
6. Q. Gao, Y. Hou, T. Deng, C. Wang and R. Hou, Extended state observer-based fractional order proportional-integral-derivative controller for a novel electro-hydraulic servo system with iso-actuation balancing and positioning, Advances in Mechanical Engineering, 2015, Vol. 7(12) 111.
7. Q. Gao, K. Li, Y. Hou, R. Hou and C. Wang, Balancing and positioning for a gun control system based on fuzzy fractional order proportional-integral-derivative strategy, Advances in Mechanical Engineering, 2016, Vol. 8(3) 19.
8. A. M. Lopes, J. T. Machado, Visualizing control systems performance: A fractional perspective, Advances in Mechanical Engineering, 2015, Vol. 7(12) 18.
9. Y. Zhang, S. Wang, G. Liu and J.Yang, Computer aided diagnosis of abnormal breasts in mammogram images by weighted type fractional Fourier transform, Advances in Mechanical Engineering, 2016, Vol. 8(2) 111.
10. S. Kempe, L. Gaul, Global solutions of fractional linear dierential equations, Proc. of ICIAM95, Zeitschrift Angew. Math. Mech., vol76, suppl. 2, 1996, pp. 571-572.
11. R. R. Nigmatullin, Y. E. Ryabov, ColeDavidson dielectric relaxation as a self-similar relaxation process, Phys. Solid State, vol. 39, no.1, 1997, pp. 87-90.
12. Y. F. Luchko, H. Matrinez, J. J. Trujillo, Fractional Fourier transform and some of its applications, Fractional Calculus and Applied Analysis, Volume 11, Number 4 (2008).
13. G. Jumarie, Fouriers transform of fractional order via Mittag-Leffler function and modified Riemann-Liouville derivative, J. Appl. Math. & Informatics, Vol. 26 (2008), No. 5-6, pp. 1101-1121.
14. L. Gaul, P.Klein, S. Kempe, Damping description involving fractional operators, Mech. Systems and Signal Processings 5 (1991) 2, 81-88.
15. H. Beyer, S. Kempe, Denition of Physically Consistent Damping Laws with Fractional Derivatives, Z. angew. Math. Mech. 75 (1995) 8, 623-635.
16. I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.
17. I. Podlubny, Geometric and physical interpretation of fractional integration and fractional differentiation, Fractional Calculus and Applied Analysis 5 (4) (2002) 367386.
18. E. Demirci, N. Özalp, A method for solving differential equations of fractional order, J. Comput. Appl. Math. 236, 2754-2762 (2012).
19. I. Koca, A method for solving differential equations of q-fractional order, Applied Mathematics and Computation 266 (2015) 1-5.
20. N. Ozalp, O. Ozturk Mizrak, Fractional Laplace transform method in the framework of the CTIT transformation, Journal of Computational and Applied Mathematics 317 (2017), 90-99.

BIOGRAPHY

Ozlem Ozturk Mizrak received the B.Sc. and M.Sc. degrees in 2010 and 2013 from the Department of Mathematics from Gazi University, Turkey respectively. She is still doing doctorate at Ankara University with the thesis title “On Some Fractional Dynamic Models”. She is also working as a research assistant at Department of Mathematics of Karabuk University. Her research interests are fractional differential equations, mathematical modelling and mathematical oncology.

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