



Certain subclass of univalent functions involving fractional q-calculus operator

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Abstract

The main object of the present paper is to introduce certain subclass of univalent function associated with the concept of differential subordination. We studied some geometric properties like coefficient inequality and neighbourhood property, the Hadamard product properties and integral operator mean inequality.

Key words: Differential subordination, Differential superordination, Univalent function, Convex function, Komatu integral operator, Hadamard product.

AMS2010 Mathematics Subject Classification. Primary30C45

1. Introduction and Definitions

Let C be complex plane, let U denote the open unit disc in C ,

$$U = \{z \in C : |z| < 1\}, \tag{1.1}$$

and let S be the class of all analytic and univalent functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, (z \in U) \tag{1.2}$$

For functions f and g in S such that $g(z)$ defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. (z \in U)$$

The Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

Let $H(U)$ be the class of holomorphic functions in U and let $H[a, k]$ be the subclass of $H(U)$ of the form

$$H[a, k] = \{f \in H(U) : f(z) = a + a_k z^k + a_{k+1} z^{k+1} + \dots, (z \in U)\},$$

for $a \in C$ and $k \in N = \{1, 2, \dots\}$, with $H_0 \equiv H[0, 1]$ and $H \equiv H[1, 1]$.

Let K be a subset of S consisting of function f with the following form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \tag{1.4}$$

therefore f are univalent and normalized in the open unit disk U .

Now, we let $f(z)$ and $g(z)$ be members of $H(U)$. The function $f(z)$ is said to be subordinate to a function $g(z)$ or $g(z)$ is said to be superordinate to $f(z)$, if and only if there exists a Schwarz function $w(z)$ analytic in U , with $w(0) = 0$ and $|w(z)| < 1$, ($z \in U$),

such that

$$f(z) = g(w(z)),$$

written as

$$f \prec g \text{ or } f(z) \prec g(z) \quad (z \in U).$$

Furthermore, if the function g is univalent in U , then we get the following equivalence $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ [3], [7].

The linear multiplier fractional q-different integral operator $\mathcal{L}_{\gamma, \tau}^{\alpha, n}$ introduced by [1] defined as follows.

$$\mathcal{L}_{\gamma, \tau}^{\alpha, 0} f(z) = f(z),$$

$$\mathcal{L}_{\gamma, \tau}^{\alpha, 1} f(z) = (1 - \tau) \mathcal{L}_{\gamma, \tau}^{\alpha} f(z) + \tau z \left(\mathcal{L}_{\gamma, \tau}^{\alpha} f(z) \right)' \quad (\tau \geq 0),$$



$$\mathcal{L}_{\gamma,\tau}^{\alpha,2} f(z) = \mathcal{L}_{\gamma,\tau}^{\alpha,1} \left(\mathcal{L}_{\gamma,\tau}^{\alpha,1} f(z) \right),$$

$$\mathcal{L}_{\gamma,\tau}^{\alpha,n} f(z) = \mathcal{L}_{\gamma,\tau}^{\alpha,1} f \left(\mathcal{L}_{\gamma,\tau}^{\alpha,n-1} f(z) \right). \quad (n \in \mathbb{N}) \quad (1.5)$$

If $f(z)$ is given by (1.4) then by (1.5), we get

$$\mathcal{L}_{\gamma,\tau}^{\alpha,n} f(z) = z - \sum_{k=2}^{\infty} \left(\frac{\Gamma_{\gamma}(2-\alpha)\Gamma_{\gamma}(k+1)}{\Gamma_{\gamma}(2)\Gamma_{\gamma}(k+1-\alpha)} [1 - \tau + [k]_{\gamma}\tau] \right)^{\alpha} |a_k| z^k, \quad (1.6)$$

where

$$\mu_k^{\alpha} = \left(\frac{\Gamma_{\gamma}(2-\alpha)\Gamma_{\gamma}(k+1)}{\Gamma_{\gamma}(2)\Gamma_{\gamma}(k+1-\alpha)} [1 - \tau + [k]_{\gamma}\tau] \right)^{\alpha}. \quad (1.7)$$

By (1.6) and (1.7), then we have

$$\mathcal{L}_{\gamma,\tau}^{\alpha,n} f(z) = z - \sum_{k=2}^{\infty} \mu_k^{\alpha} |a_k| z^k. \quad (1.8)$$

Note that, if we put $\alpha = 0$ the operator $\mathcal{L}_{\gamma,\tau}^{\alpha,n}$ reduces to the operator studied by AL-Oboudi [2] and for $\alpha = 0, \tau = 1$, we get the operator introduced by Şal˘agean [9].

Definition 1.1. A function $f(z) \in K$ and $\theta \geq 0$ the θ -neighborhood f is defined as,

$$N_{K,\theta}(f) = \{g(z) = z - \sum_{k=2}^{\infty} |b_k| z^k \in K; \sum_{k=2}^{\infty} k |a_k| - |b_k| \leq \theta\}. \quad (1.9)$$

In particular, for the function $e(z)=z$, we see that,

$$N_{K,\theta}(e) = \{g(z) = z - \sum_{k=2}^{\infty} |b_k| z^k \in K; \sum_{k=2}^{\infty} k |b_k| \leq \theta\}. \quad (1.10)$$

The concept of neighborhoods was investigated by Goodman [4] and then generalized by Ruscheweyh [8], and studied by some authors, A. R. S. Juma and S. R. Kulkarni[5].

Definition 1.2. A function $f(z)$ belonging to K is in the class $K(\alpha, \lambda, n, \gamma, \tau, A, B)$, if it satisfies the following

$$1 + \frac{z \left(\mathcal{L}_{\gamma,\tau}^{\alpha+\lambda,n} f(z) \right)''}{\left(\mathcal{L}_{\gamma,\tau}^{\alpha,n} f(z) \right)'} < \frac{1 + Az}{1 + Bz}, \quad (-1 \leq B < A \leq 1) \quad (1.11)$$

and is equivalent to the following condition:

$$\left| \frac{\frac{z \left(\mathcal{L}_{\gamma,\tau}^{\alpha+\lambda,n} f(z) \right)''}{\left(\mathcal{L}_{\gamma,\tau}^{\alpha,n} f(z) \right)'}}{B \frac{z \left(\mathcal{L}_{\gamma,\tau}^{\alpha+\lambda,n} f(z) \right)''}{\left(\mathcal{L}_{\gamma,\tau}^{\alpha,n} f(z) \right)' } + (B - A)} \right| < 1, \quad (z \in U).$$

Then, we can write

$$K(\alpha, \lambda, n, \gamma, \tau, 1-2\beta, -1) = K(\alpha, \lambda, n, \gamma, \tau, \beta),$$

where $K(\alpha, \lambda, n, \gamma, \tau, \beta)$ denoted the class of functions in K satisfying the convex inequality:

$$\Re \left\{ 1 + \frac{z \left(\mathcal{L}_{\gamma,\tau}^{\alpha+\lambda,n} f(z) \right)''}{\left(\mathcal{L}_{\gamma,\tau}^{\alpha,n} f(z) \right)'} \right\} > \beta, \quad (0 \leq \beta < 1; z \in U).$$

2. Main results

An this section, we derive the coefficient inequality for the class $K(\alpha, \lambda, n, \gamma, \tau, A, B)$.

Theorem 2.1. A function $f \in K$ belong to the class $K(\alpha, \lambda, n, \gamma, \tau, A, B)$ if and only if

$$\sum_{k=2}^{\infty} k(2k-1) \mu_k^{\alpha} (1-B) \mu_k^{\lambda} + (A-B) |a_k| \leq (A-B), \quad (2.1)$$

for $\alpha, \lambda, n \in \mathbb{N}_0, \tau \leq n+1, \gamma \geq 0$ and $-1 \leq B < A \leq 1$.



Proof. If $f(z) \in K(\alpha, \lambda, n, \gamma, \tau, A, B)$, then by (1.11), we obtain

$$1 + \frac{z \left(\mathcal{L}_{\gamma, \tau}^{\alpha + \lambda, n} f(z) \right)''}{\left(\mathcal{L}_{\gamma, \tau}^{\alpha, n} f(z) \right)''} < \frac{1 + Az}{1 + Bz},$$

thus, there exists an analytic function $h(z)$ defined by

$$h(z) = \frac{z \left(\mathcal{L}_{\gamma, \tau}^{\alpha + \lambda, n} f(z) \right)''}{Bz \left(\mathcal{L}_{\gamma, \tau}^{\alpha + \lambda, n} f(z) \right)'' + (B - A) \left(\mathcal{L}_{\gamma, \tau}^{\alpha, n} f(z) \right)''}. \tag{2.2}$$

Therefore,

$$|h(z)| = \left| \frac{z \left(\mathcal{L}_{\gamma, \tau}^{\alpha + \lambda, n} f(z) \right)''}{Bz \left(\mathcal{L}_{\gamma, \tau}^{\alpha + \lambda, n} f(z) \right)'' + (B - A) \left(\mathcal{L}_{\gamma, \tau}^{\alpha, n} f(z) \right)''} \right|$$

$$\left| \frac{-\sum_{k=2}^{\infty} k(k-1) \mu_k^{\alpha + \lambda} |a_k| z^k}{(B - A)z - \sum_{k=2}^{\infty} k^2 \mu_k^{\alpha} (B\mu_k^{\lambda} + (B - A)) |a_k| z^k} \right| < 1.$$

Hence,

$$\left| \frac{\sum_{k=2}^{\infty} k(k-1) \mu_k^{\alpha + \lambda} |a_k| z^k}{(A - B)z + \sum_{k=2}^{\infty} k^2 \mu_k^{\alpha} (B\mu_k^{\lambda} + (B - A)) |a_k| z^k} \right| < 1.$$

Thus, we have

$$\Re e \left\{ \frac{\sum_{k=2}^{\infty} k(k-1) \mu_k^{\alpha + \lambda} |a_k| z^k}{(A - B)z + \sum_{k=2}^{\infty} k^2 \mu_k^{\alpha} (B\mu_k^{\lambda} + (B - A)) |a_k| z^k} \right\} < 1. \tag{2.3}$$

Let $|z| = r$ and $0 < r < 1$, since $h(z)$ is analytic function for $|z| = 1$. Then by (2.3), we obtain

$$\sum_{k=2}^{\infty} k(k-1) \mu_k^{\alpha + \lambda} |a_k| r^k < (A - B)r + \sum_{k=2}^{\infty} k^2 \mu_k^{\alpha} (B\mu_k^{\lambda} + (B - A)) |a_k| r^k,$$

implies that

$$\sum_{k=2}^{\infty} k(2k-1) \mu_k^{\alpha} \left((1 - B)\mu_k^{\lambda} + (A - B) \right) |a_k| r^k \leq (A - B)r,$$

the required result follows as $r \rightarrow 1$.

Conversely, for $|z| = r$ and $0 < r < 1$, we get $r^k < r$. That is,

$$\sum_{k=2}^{\infty} k(2k-1) \mu_k^{\alpha} \left((1 - B)\mu_k^{\lambda} + (A - B) \right) |a_k| r^k \leq \sum_{k=2}^{\infty} k(2k-1) \mu_k^{\alpha} \left((1 - B)\mu_k^{\lambda} + (A - B) \right) |a_k| r^k$$

$$\leq (A - B)r.$$

By (2.1), we get

$$\left| \sum_{k=2}^{\infty} k(k-1) \mu_k^{\alpha + \lambda} |a_k| z^k \right| \leq \sum_{k=2}^{\infty} k(k-1) \mu_k^{\alpha + \lambda} |a_k| r^k$$

$$< (A - B)r + \sum_{k=2}^{\infty} k^2 \mu_k^{\alpha} (B\mu_k^{\lambda} + (B - A)) |a_k| r^k < \left| (A - B)z + \sum_{k=2}^{\infty} k^2 \mu_k^{\alpha} (B\mu_k^{\lambda} + (B - A)) |a_k| z^k \right|.$$

Then, we obtain



$$1 + \frac{z \left(\mathcal{L}_{\gamma, \tau}^{\alpha+\lambda, n} f(z) \right)''}{\left(\mathcal{L}_{\gamma, \tau}^{\alpha, n} f(z) \right)'} < \frac{1 + Az}{1 + Bz}, \quad (z \in U)$$

and $f(z) \in K(\alpha, \lambda, n, \gamma, \tau, A, B)$.

Theorem 2.2. If

$$\theta = \frac{A-B}{3 \left(\frac{\Gamma_{\gamma}(2-\alpha)\Gamma_{\gamma}(k+1)}{\Gamma_{\gamma}(2)\Gamma_{\gamma}(k+1-\alpha)} [1-\tau+[k]_{\gamma}\tau] \right)^{\alpha} \left[(1-B) \left(\frac{\Gamma_{\gamma}(2-\alpha)\Gamma_{\gamma}(k+1)}{\Gamma_{\gamma}(2)\Gamma_{\gamma}(k+1-\alpha)} [1-\tau+[k]_{\gamma}\tau] \right)^{\lambda} + (A-B) \right]}, \quad (2.4)$$

then

$$K(\alpha, \lambda, n, \gamma, \tau, A, B) \subset N_{K, \theta}(e).$$

Proof. By from (2.1), and $f(z) \in K(\alpha, \lambda, n, \gamma, \tau, A, B)$, then

$$3\mu_2^{\alpha} \left((1-B)\mu_2^{\lambda} + (A-B) \right) \sum_{k=2}^{\infty} k |a_k| \leq (A-B).$$

Therefore,

$$3 \left(\frac{\Gamma_{\gamma}(2-\alpha)\Gamma_{\gamma}(k+1)}{\Gamma_{\gamma}(2)\Gamma_{\gamma}(k+1-\alpha)} [1-\tau+[k]_{\gamma}\tau] \right)^{\alpha} \left[(1-B) \left(\frac{\Gamma_{\gamma}(2-\alpha)\Gamma_{\gamma}(k+1)}{\Gamma_{\gamma}(2)\Gamma_{\gamma}(k+1-\alpha)} [1-\tau+[k]_{\gamma}\tau] \right)^{\lambda} + (A-B) \right]$$

$$\times \sum_{k=2}^{\infty} k |a_k| \leq (A-B), \quad (2.5)$$

implies that

$$\sum_{k=2}^{\infty} k |a_k| \leq \frac{A-B}{3 \left(\frac{\Gamma_{\gamma}(2-\alpha)\Gamma_{\gamma}(k+1)}{\Gamma_{\gamma}(2)\Gamma_{\gamma}(k+1-\alpha)} [1-\tau+[k]_{\gamma}\tau] \right)^{\alpha} \left[(1-B) \left(\frac{\Gamma_{\gamma}(2-\alpha)\Gamma_{\gamma}(k+1)}{\Gamma_{\gamma}(2)\Gamma_{\gamma}(k+1-\alpha)} [1-\tau+[k]_{\gamma}\tau] \right)^{\lambda} + (A-B) \right]} = \theta \quad (2.6)$$

From (1.10), we have the following result.

$$K(\alpha, \lambda, n, \gamma, \tau, A, B) \subset N_{K, \theta}(e).$$

Definition 2.3. Let $g(z)$ is analytic and defined by $g(z) = z - \sum_{k=2}^{\infty} |b_k| z^k$ is said to be a member of the class $K_c(\alpha, \lambda, n, \gamma, \tau, A, B)$ if there exist a function $f(z) \in K(\alpha, \lambda, n, \gamma, \tau, A, B)$ such that

$$\left| \frac{g(z)}{f(z)} - 1 \right| \leq 1 - c, \quad (0 \leq c < 1, \quad z \in U) \quad (2.7)$$

Theorem 2.4. If $f(z) \in K(\alpha, \lambda, n, \gamma, \tau, A, B)$ and

$$c = 1 - \frac{3\theta\mu_2^{\alpha} \left((1-B)\mu_2^{\lambda} + (A-B) \right)}{6\mu_2^{\alpha} \left((1-B)\mu_2^{\lambda} + (A-B) \right) - (A-B)}, \quad (2.8)$$

then $N_{K, \theta}(f) \subset K_c(\alpha, \lambda, n, \gamma, \tau, A, B)$.

Proof. Let $g(z) \in N_{K, \theta}(f)$. By (1.9), we obtain that

$$\sum_{k=2}^{\infty} k |a_k| - |b_k| \leq \theta,$$

implies that

$$\sum_{k=2}^{\infty} ||a_k| - |b_k|| \leq \frac{\theta}{2}.$$

Since $f(z) \in K(\alpha, \lambda, n, \gamma, \tau, A, B)$, by (2.1), we get



$$\sum_{k=2}^{\infty} |a_k| \leq \frac{A-B}{6\mu_2^\alpha \left((1-B)\mu_2^\lambda + (A-B) \right)},$$

where

$$\mu_k^\alpha = \left(\frac{\Gamma_\gamma(2-\alpha)\Gamma_\gamma(k+1)}{\Gamma_\gamma(2)\Gamma_\gamma(k+1-\alpha)} [1-\tau + [k]_\gamma \tau] \right)^\alpha,$$

$$\mu_k^\lambda = \left(\frac{\Gamma_\gamma(2-\alpha)\Gamma_\gamma(k+1)}{\Gamma_\gamma(2)\Gamma_\gamma(k+1-\alpha)} [1-\tau + [k]_\gamma \tau] \right)^\lambda,$$

thus

$$\left| \frac{g(z)}{f(z)} - 1 \right| = \left| \frac{\sum_{k=2}^{\infty} (|a_k| - |b_k|) z^k}{z - \sum_{k=2}^{\infty} |a_k| z^k} \right| < \frac{\sum_{k=2}^{\infty} |a_k| - |b_k|}{1 - \sum_{k=2}^{\infty} |a_k|}$$

$$\leq \theta \frac{3\mu_2^\alpha \left((1-B)\mu_2^\lambda + (A-B) \right)}{6\mu_2^\alpha \left((1-B)\mu_2^\lambda + (A-B) \right) - (A-B)} = 1 - c.$$

Hence, be Definition 2.3., $g(z) \in K_c(\alpha, \lambda, n, \gamma, \tau, A, B)$ for all c from (2.8).

3. Hadamard product properties

In this section we give some properties of the convolution belongs to the concept.

Theorem 3.1. Let the functions f_j ($j=1,2$) defined by

$$f_j(z) = z - \sum_{k=2}^{\infty} |a_{k,j}| z^k, \quad (j = 1,2). \quad (3.1)$$

Then $f_1 * f_2 \in K(\alpha, \lambda, n, \gamma, \tau, A, \sigma)$, where

$$\sigma \leq \frac{Ak(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right)^2 - (A-B)^2(A + \mu_k^\lambda)}{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right)^2 - (A-B)^2(1 + \mu_k^\lambda)}.$$

Proof. First, we find the largest σ so that

$$\sum_{k=2}^{\infty} \frac{k(2k-1)\mu_k^\alpha \left((1-\sigma)\mu_k^\lambda + (A-\sigma) \right)}{A-\sigma} |a_{k,1}| |a_{k,2}| \leq 1.$$

Since $f_j(z) \in K(\alpha, \lambda, n, \gamma, \tau, A, B)$, we get

$$\sum_{k=2}^{\infty} \frac{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right)}{A-B} |a_{k,j}| \leq 1. \quad (j = 1,2) \quad (3.2)$$

By Cauchy – Schwarz inequality, we have

$$\sum_{k=2}^{\infty} \frac{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right)}{A-B} \sqrt{|a_{k,1}| |a_{k,2}|} \leq 1. \quad (3.3)$$

We show that,

$$\frac{k(2k-1)\mu_k^\alpha \left((1-\sigma)\mu_k^\lambda + (A-\sigma) \right)}{A-\sigma} |a_{k,1}| |a_{k,2}| \leq \frac{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right)}{A-B} \sqrt{|a_{k,1}| |a_{k,2}|}.$$

Thus, implies that



$$\sqrt{|a_{k,1}||a_{k,2}|} \leq \frac{(A - \sigma) \left((1 - B)\mu_k^\lambda + (A - B) \right)}{(A - B) \left((1 - \sigma)\mu_k^\lambda + (A - \sigma) \right)}.$$

By (3.3), we get

$$\sqrt{|a_{k,1}||a_{k,2}|} \leq \frac{A - B}{k(2k - 1)\mu_k^\alpha \left((1 - B)\mu_k^\lambda + (A - B) \right)}.$$

Hence,

$$\frac{A - B}{k(2k - 1)\mu_k^\alpha \left((1 - B)\mu_k^\lambda + (A - B) \right)} \leq \frac{(A - \sigma) \left((1 - B)\mu_k^\lambda + (A - B) \right)}{(A - B) \left((1 - \sigma)\mu_k^\lambda + (A - \sigma) \right)},$$

this equivalently to

$$\sigma \leq \frac{Ak(2k - 1)\mu_k^\alpha \left((1 - B)\mu_k^\lambda + (A - B) \right)^2 - (A - B)^2(A + \mu_k^\lambda)}{k(2k - 1)\mu_k^\alpha \left((1 - B)\mu_k^\lambda + (A - B) \right)^2 - (A - B)^2(1 + \mu_k^\lambda)}.$$

Theorem3.2. Let the functions f_j ($j=1, 2$) defined by (3.1) be in the class $K(\alpha, \lambda, n, \gamma, \tau, A, B)$. Then the function p defined by

$$p(z) = z - \sum_{k=2}^{\infty} \left(|a_{k,1}|^2 + |a_{k,2}|^2 \right) z^k, \quad (3.4)$$

belong to the class $K(\alpha, \lambda, n, \gamma, \tau, A, \epsilon)$, where

$$\epsilon \leq \frac{k(2k - 1)\mu_k^\alpha \left((1 - B)\mu_k^\lambda + (A - B) \right)^2 - 2\mu_k^\lambda(A - B)^2 - 2(A - B)^2}{Ak(2k - 1)\mu_k^\alpha \left((1 - B)\mu_k^\lambda + (A - B) \right)^2 + 2A(A - B)^2 + 2\mu_k^\lambda(A - B)^2}.$$

Proof. First, we find the largest ϵ so that

$$\sum_{k=2}^{\infty} \frac{k(2k - 1)\mu_k^\alpha \left((1 - \epsilon)\mu_k^\lambda + (A - \epsilon) \right)}{A - \epsilon} \left(|a_{k,1}|^2 + |a_{k,2}|^2 \right) \leq 1.$$

Since $f_j(z) \in K(\alpha, \lambda, n, \gamma, \tau, A, B)$, we have

$$\sum_{k=2}^{\infty} \left(\frac{k(2k - 1)\mu_k^\alpha \left((1 - B)\mu_k^\lambda + (A - B) \right)}{A - B} \right)^2 |a_{k,2}|^2 \leq \left(\sum_{k=2}^{\infty} \frac{k(2k - 1)\mu_k^\alpha \left((1 - B)\mu_k^\lambda + (A - B) \right)}{A - B} |a_{k,1}|^2 \right)^2 \leq 1, \quad (3.5)$$

and

$$\sum_{k=2}^{\infty} \left(\frac{k(2k - 1)\mu_k^\alpha \left((1 - B)\mu_k^\lambda + (A - B) \right)}{A - B} \right)^2 |a_{k,2}|^2 \leq \left(\sum_{k=2}^{\infty} \frac{k(2k - 1)\mu_k^\alpha \left((1 - B)\mu_k^\lambda + (A - B) \right)}{A - B} |a_{k,2}|^2 \right)^2 \leq 1. \quad (3.6)$$

Thus, by (3.5) and (3.6), we give

$$\sum_{k=2}^{\infty} \frac{1}{2} \left(\frac{k(2k - 1)\mu_k^\alpha \left((1 - B)\mu_k^\lambda + (A - B) \right)}{A - B} \right)^2 \left(|a_{k,1}|^2 + |a_{k,2}|^2 \right) \leq 1. \quad (3.7)$$

But $p(z) \in K(\alpha, \lambda, n, \gamma, \tau, A, \epsilon)$ if and only if

$$\sum_{k=2}^{\infty} \left(\frac{k(2k - 1)\mu_k^\alpha \left((1 - \epsilon)\mu_k^\lambda + (A - \epsilon) \right)}{A - \epsilon} \right) \left(|a_{k,1}|^2 + |a_{k,2}|^2 \right) \leq 1. \quad (3.8)$$

By (3.8), satisfies the following inequality



$$\frac{k(2k-1)\mu_k^\alpha \left((1-\epsilon)\mu_k^\lambda + (A-\epsilon) \right)}{A-\epsilon} \leq \frac{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right)^2}{2(A-B)^2}. \quad (3.9)$$

Hence,

$$\epsilon \leq \frac{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right)^2 - 2\mu_k^\lambda (A-B)^2 - 2(A-B)^2}{Ak(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right)^2 + 2A(A-B)^2 + 2\mu_k^\lambda (A-B)^2}.$$

4. Integral Mean Inequalities

In this section we study the integral mean inequality by introduce the following definition.

Definition 4.1.[10]: The fractional integral of order s ($s > 0$) is defined for a function f by:

$$D_z^{-s} f(z) = \frac{1}{\Gamma(s)} \int_0^z \frac{f(t)}{(z-t)^{1-s}} dt,$$

where the function f is an analytic in a simply connected region of the complex z -plane containing the origin, and multiplicity of $(z-t)^{1-s}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t) > 0$.

In 1925, Littlewood [6] proved the following subordination theorem:-

Theorem 4.2. (Littlewood [6]): If f and g are analytic in U with $f \prec g$, then for $\eta > 0$

and $z = r e^{i\theta}$ ($0 < r < 1$)

$$\int_0^{2\pi} |f(z)|^\eta d\theta \leq \int_0^{2\pi} |g(z)|^\eta d\theta.$$

Theorem 4.3. Let $f(z) \in K(\alpha, \lambda, n, \gamma, \tau, A, B)$ and suppose that f_k is defined by

$$f_k(z) = z - \frac{A-B}{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right)} z^k, \quad (k \geq 2) \quad (4.1)$$

and, we let

$$\sum_{k=2}^{\infty} (i-\nu)_{\nu+1} |a_i| \leq \frac{(A-B)\Gamma(k+1)\Gamma(s+\nu+3)}{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right) \Gamma(k+s+\nu+1)\Gamma(2-\nu)}, \quad (4.2)$$

for $0 \leq \nu \leq i, s > 0$, where $(i-\nu)_{\nu+1}$ defined by

$$(i-\nu)_{\nu+1} = (i-\nu)(i-\nu+1) \dots i,$$

thus, if there exists an analytic function q defined by

$$(q(z))^{k-1} = \frac{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right) \Gamma(k+s+\nu+1)}{(A-B)\Gamma(k+1)} \times \sum_{i=2}^{\infty} (i-\nu)_{\nu+1} H(i) |a_i| z^{i-1}, \quad (4.3)$$

where $i \geq \nu$ and

$$H(i) = \frac{\Gamma(i-\nu)}{\Gamma(i+s+\nu+1)}, \quad (s > 0, i \geq 2) \quad (4.4)$$

then, for $z = r e^{i\theta}$ and $(0 < r < 1)$

$$\int_0^{2\pi} |D_z^{-s-\nu} f(z)|^\eta d\theta \leq \int_0^{2\pi} |D_z^{-s-\nu} f_k(z)|^\eta d\theta. \quad (s > 0, \eta > 0) \quad (4.5)$$



Proof. Let $f(z) = z - \sum_{i=2}^{\infty} |a_i| z^i$. By Definition 4.1, we have

$$\begin{aligned} D_z^{s-\nu} f(z) &= \frac{\Gamma(2)z^{s+\nu+1}}{\Gamma(s+\nu+2)} \left(1 - \sum_{i=2}^{\infty} \frac{\Gamma(i-1)\Gamma(s+\nu+2)}{\Gamma(2)\Gamma(i+s+\nu+1)} |a_i| z^{i-1} \right) \\ &= \frac{\Gamma(2)z^{s+\nu+1}}{\Gamma(s+\nu+2)} \left(1 - \sum_{i=2}^{\infty} \frac{\Gamma(s+\nu+2)}{\Gamma(2)} (i-\nu)_{\nu+1} H(i) |a_i| z^{i-1} \right), \end{aligned}$$

where

$$H(i) = \frac{\Gamma(i-\nu)}{\Gamma(i+s+\nu+1)}, \quad (s > 0, i \geq 2)$$

we get

$$0 < H(i) \leq H(2) = \frac{\Gamma(2-\nu)}{\Gamma(s+\nu+3)}.$$

By (4.1) and Definition 4.1., we have

$$D_z^{s-\nu} f(z) = \frac{\Gamma(2)z^{s+\nu+1}}{\Gamma(s+\nu+2)} \left(1 - \frac{(A-B)\Gamma(k+1)\Gamma(s+\nu+2)}{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right) \Gamma(2)\Gamma(i+s+\nu+1)} z^{k-1} \right),$$

therefore, we show that

$$\begin{aligned} & \int_0^{2\pi} \left| 1 - \sum_{i=2}^{\infty} \frac{\Gamma(s+\nu+2)}{\Gamma(2)} (i-\nu)_{\nu+1} H(i) |a_i| z^{i-1} \right| d\varphi \\ & \leq \int_0^{2\pi} \left| 1 - \frac{(A-B)\Gamma(k+1)\Gamma(s+\nu+2)}{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right) \Gamma(2)\Gamma(i+s+\nu+1)} z^{k-1} \right| d\varphi. \end{aligned}$$

in virtue of Theorem 4.2, we have

$$\begin{aligned} & 1 - \sum_{i=2}^{\infty} \frac{\Gamma(s+\nu+2)}{\Gamma(2)} (i-\nu)_{\nu+1} H(i) |a_i| z^{i-1} \\ & < 1 - \frac{(A-B)\Gamma(k+1)\Gamma(s+\nu+2)}{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right) \Gamma(2)\Gamma(i+s+\nu+1)} z^{k-1}. \end{aligned}$$

Hence,

$$\begin{aligned} & 1 - \sum_{i=2}^{\infty} \frac{\Gamma(s+\nu+2)}{\Gamma(2)} (i-\nu)_{\nu+1} H(i) |a_i| z^{i-1} \\ & = 1 - \frac{(A-B)\Gamma(k+1)\Gamma(s+\nu+2)}{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right) \Gamma(2)\Gamma(i+s+\nu+1)} (q(z))^{k-1}, \end{aligned}$$

such that

$$(q(z))^{k-1} = \frac{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right) \Gamma(k+s+\nu+1)}{(A-B)\Gamma(k+1)} \times \sum_{i=2}^{\infty} (i-\nu)_{\nu+1} H(i) |a_i| z^{i-1}$$

$q(0) = 0$, then we obtain

$$(q(z))^{k-1} \leq \frac{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right) \Gamma(k+s+\nu+1)}{(A-B)\Gamma(k+1)} \times \sum_{i=2}^{\infty} (i-\nu)_{\nu+1} H(i) |a_i| |z|^{i-1}$$



$$\begin{aligned} &\leq \frac{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right) \Gamma(k+s+\nu+1)}{(A-B)\Gamma(k+1)} \times H(2)|z| \sum_{i=2}^{\infty} (i-\nu)_{\nu+1} |a_i| \\ &= |z| \frac{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right) \Gamma(k+s+\nu+1)}{(A-B)\Gamma(s+\nu+3)\Gamma(k+1)} \\ &\quad \times \Gamma(2-\nu) \sum_{i=2}^{\infty} (i-\nu)_{\nu+1} |a_i| \leq |z| < 1. \end{aligned}$$

If $\nu = 0$, in the Theorem 4.3, we get the following corollary:

Corollary 4.4. Let the functions $f(z)$ in the class $\in K(\alpha, \lambda, n, \gamma, \tau, A, B)$, and suppose that f_k is defined by (4.1), and let

$$\sum_{i=2}^{\infty} i |a_i| \leq \frac{(A-B)\Gamma(k+1)\Gamma(s+3)}{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right) \Gamma(2)\Gamma(k+s+1)}, \quad k \geq 2$$

if q defined by

$$(q(z))^{k-1} = \frac{k(2k-1)\mu_k^\alpha \left((1-B)\mu_k^\lambda + (A-B) \right) \Gamma(k+s+1)}{(A-B)\Gamma(k+1)} \times \sum_{i=2}^{\infty} iH(i) |a_i| z^{i-1},$$

where

$$H(i) = \frac{\Gamma(i)}{\Gamma(i+s+1)}, \quad (s > 0, i \geq 2)$$

for $z = r e^{i\theta}$ and $(0 < r < 1)$

$$\int_0^{2\pi} |D_z^{-s} f(z)|^\eta d\varphi \leq \int_0^{2\pi} |D_z^{-s} f_k(z)|^\eta d\varphi. \quad (s > 0, \eta > 0)$$

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