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NEW OSCILLATION CRITERIA FOR SECOND ORDER NEUTRAL TYPE DIFFERENCE EQUATIONS

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ABSTRACT

In this paper, we present some new oscillation criteria for second order neutral type difference equation of the form

$$\Delta\left(a_n(\Delta z_n)^{\alpha}\right)+q_n f(x_n)=e_n, \quad n\geq n_0>0,$$

where $z_n = x_n - \rho_n x_{n-l}$ and α is ratio of odd positive integers. Examples are provided to illustrate the results.

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INTRODUCTION

In this paper, we study the oscillatory behavior of second order neutral type difference equation of the form

$$\Delta\left(a_n(\Delta z_n)^{\alpha}\right)+q_n f(x_n)=e_n, \quad n\geq n_1>0, \tag{1}$$

where $z_n = x_n - \rho_n x_{n-l}$ and $\alpha>0$ is a ratio of odd positive integers, l is a positive integer.

Subject to the following hypothesis:

(H1) $\{\rho_n\}$, $\{q_n\}$ and $\{e_n\}$ are sequences of real numbers with $0\leq\rho_n\leq\rho<1, q_n>0,\ e_n\geq 0$ and $\{a_n\}$ is a positive real sequence with $\sum_{s=n_0}^{n}\frac{1}{a_s^{1/\alpha}}\to\infty$ as $n\to\infty$.

(H2) $f:R\to R$ such that $uf(u)>0$ for all $u\neq 0$ and there exists a positive constant M such that $\frac{f(u)}{u^{\beta}}\geq M$ for all $u\neq 0$, where β is a positive constant.

By a solution of equation (1), we mean a real sequence $\{x_n\}$ defined for all $n\geq n_0$ and satisfying equation (1). A solution $\{x_n\}$ of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is said to be non-oscillatory.

Recently there has been an increasing interest in the study of the oscillation and non-oscillation of the second order neutral difference equations, see for example [6, 7, 10-14] and the references cited there in. In [14], we see that the oscillation criteria for second order non-positive neutral term of the form

$$\Delta\left(a_n(\Delta z_n)^{\alpha}\right)+q_n f(x_{n-\sigma})=0, \quad n\geq n_0>0, \tag{2}$$

where $z_n = x_n - \rho_n x_{n-l}$ with $\sum_{s=n_0}^{n}\frac{1}{a_s^{1/\alpha}}\to\infty$ as $n\to\infty$ and $\frac{f(u)}{u^{\alpha}}\geq M$.

In [6] the authors studied oscillation criteria for second order neutral difference equation of the form

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$$\Delta\left(r_n(\Delta x_n + x_{n-k})^{\gamma}\right) + q_n x_{n+1}^{\alpha} = e_n. \tag{3}$$

In section 2, we present some oscillation criteria for equation (1) and in section 3, we provide some examples to illustrate the main results.

Definition 1.1 *A solution* $\{x_n\}$ *of equation (1) is said to be almost oscillatory if either* $\{x_n\}$ *is oscillatory or* Δx_n *is oscillatory or* $x_n \to 0$ *as* $n \to \infty$.

We provide two lemmas which are useful in proving the main results.

Lemma 1.1 *Set* $F(x) = ax^{\alpha-\gamma} + \dfrac{b}{x^{\gamma}}$ *for* $x > 0$. *If* $a > 0, b > 0$ *and* $\alpha > \gamma \geq 1$, *then* $F(x)$ *attains its minimum*

$$F_{\min} = \frac{\alpha a^{\gamma/\alpha} b^{1-\gamma/\alpha}}{\gamma^{\gamma/\alpha}(\alpha-\gamma)^{1-\gamma/\alpha}}. \tag{4}$$

Lemma 1.2 *For all* $x \geq y \geq 0$ *and* $\gamma \geq 1$, *we have the following inequality*

$$x^{\gamma} - y^{\gamma} \geq (x-y)^{\gamma}. \tag{5}$$

2. ALMOST OSCILLATION RESULTS

In this section, we establish new almost oscillation criteria for equation (1).

Theorem 2.1 *Assume that there exists a sequence* $\{P_n\}$ *such that*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} P_s Q_s^* - \frac{(\Delta P_s)^{\alpha+1}(P_{s+1})^{1/\alpha} a_s}{(\alpha+1)^{\alpha+1} P_s^{\alpha}} = \infty, \tag{6}$$

and

$$\sum_{s=n_0}^{n-1} \sum_{u=n_0}^{s-1} \left(Mb^{\beta} q_u \pm e_u\right)^{\frac{1}{\alpha}} = \infty, \tag{7}$$

where

$$Q_n = \frac{\beta M^{\alpha/\beta} q_n^{\alpha/\beta} e^{1-\alpha/\beta}(1+\rho_n)^{\alpha}}{\alpha^{\alpha/\beta}(\beta-\alpha)^{1-\alpha/\beta}},$$

$$Q_n^* = \min\left\{Q_n, d^{\beta-\alpha}(1+\rho_n)Mq_n - d^{-\alpha}e_n\right\}, \tag{8}$$

$M > 0$ and $d > 0$. Then every solution of equation (1) is almost oscillatory.

Proof. Suppose that sequence $\{x_n\}$ is not almost oscillatory solution of equation (1). There exists a positive solution $\{x_n\}$ of a equation (1) such that $x_{n-l} > 0$ and $x_n > 0$ for all $n \geq n_1 \geq n_0$. Then by definition of almost oscillatory there are two possible cases arise.

Case I: Assume that $\Delta x_n > 0$ for all $n \geq n_1$. Thus $\Delta z_n > 0$ for all $n \geq n_1$. From the definition of z_n, we have $z_n = x_n - \rho_n x_{n-l}$ and $x_n \geq (1+\rho_n)z_n$. Then from equation (1) and (H_2), we have

$$\Delta(a_n(\Delta z_n)^{\alpha}) + Mq_n(1+\rho_n)^{\beta} z_n^{\beta} \leq e_n.$$

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$$\Delta(a_n(\Delta z_n)^\alpha) + Mq_n(1+\rho_n)^\beta z_n^\beta \le e_n. \tag{9}$$

From the above inequality, we conclude that

$$a_n(\Delta z_n)^\alpha \ge a_{n+1}(\Delta z_{n+1})^\alpha. \tag{10}$$

Let us denote the sequence $\{w_n\}$ by the following

$$w_n = \frac{P_n a_n (\Delta z_n)^\alpha}{z_n^\alpha}. \quad n \ge n_1. \tag{11}$$

$$\Delta w_n = \frac{P_n \Delta(a_n(\Delta z_n)^\alpha)}{z_n^\alpha} + a_{n+1}(\Delta z_{n+1})^\alpha \Delta\left(\frac{P_n}{z_n^\alpha}\right),$$

$$= \frac{P_n \Delta(a_n(\Delta z_n)^\alpha)}{z_n^\alpha} + a_{n+1}(\Delta z_{n+1})^\alpha \left[\Delta\left(\frac{1}{z_n^\alpha}\right)P_n + \frac{1}{z_{n+1}^\alpha}\Delta P_n\right],$$

$$= \frac{P_n \Delta(a_n(\Delta z_n)^\alpha)}{z_n^\alpha} + \frac{\Delta P_n a_{n+1}(\Delta z_{n+1})^\alpha}{z_{n+1}^\alpha} - \frac{P_n a_{n+1}(\Delta z_{n+1})^\alpha \Delta(z_n^\alpha)}{z_n^\alpha z_{n+1}^\alpha}.$$

By mean value theorem, there exists $\xi \in (z_n, z_{n+1})$ such that

$$\Delta(z_n^\alpha) = \alpha \xi^{\alpha-1} \Delta z_n,$$

we have

$$\Delta w_n = \frac{P_n \Delta(a_n(\Delta z_n)^\alpha)}{z_n^\alpha} + \Delta p_n \frac{w_{n+1}}{p_{n+1}} - \alpha p_n \frac{\xi^{\alpha-1} a_{n+1}(\Delta z_{n+1})^\alpha \Delta z_n}{z_{n+1}^\alpha z_n^\alpha}.$$

In the view of (9), (10) and (11), we obtain

$$\Delta w_n \le -P_n\left[Mq_n(1+\rho_n)^\beta z_n^{\beta-\alpha} - \frac{e_n}{z_n^\alpha}\right] + \Delta P_n \frac{w_{n+1}}{p_{n+1}} - \frac{\alpha P_n \xi^\alpha a_{n+1}(\Delta z_{n+1})^\alpha \Delta z_n}{\xi z_{n+1}^\alpha z_n^\alpha},$$

and

$$\Delta w_n \le -P_n\left[Mq_n(1+\rho_n)^\beta z_n^{\beta-\alpha} - \frac{e_n}{z_n^\alpha}\right] + \Delta P_n \frac{w_{n+1}}{P_{n+1}} - \frac{\alpha P_n w_{n+1}^{1+1/\alpha}}{P_{n+1}^{1+1/\alpha} a_n^{1/\alpha}}. \tag{12}$$

Set

$$F(u) = Mq_n(1+\rho_n)^\beta u^{\beta-\alpha} - \frac{e_n}{u^\alpha} \tag{13}$$

since u is increased, there is a constant $d > 0$ such that $u \ge d > 0$ and

$$F(u) \ge d^{\beta-\alpha}(1+\rho_n)^\beta Mq_n - d^{-\alpha} e_n.$$

By using the inequality

$$Bu - Au^{1+1/\alpha} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}, \quad A > 0, \tag{14}$$

we have

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$$\frac{\Delta P_n}{P_{n+1}} w_{n+1} - \frac{\alpha}{p_{n+1}^{1+1/\alpha} a_n^{1/\alpha}} w_{n+1}^{1+1/\alpha} \le \frac{(\Delta P_n)^{\alpha+1} P_{n+1}^{1/\alpha} a_n}{(\alpha+1)^{\alpha+1} P_n^{\alpha}}. \tag{15}$$

From inequalities (12), (13) and (14), we have

$$\Delta w_n \le -\left[P_n Q_n^* - \frac{(\Delta p_n)^{\alpha+1} P_{n+1}^{1/\alpha} a_n}{(\alpha+1)^{\alpha+1} P_n^{\alpha}} \right]. \tag{16}$$

Summing the inequality (16) from n_1 to $n-1$, we obtain

$$\sum_{s=n_1}^{n-1} \left[P_s Q_s^* - \frac{(\Delta P_s)^{\alpha+1} P_{s+1}^{1/\alpha} a_s}{(\alpha+1)^{\alpha+1} P_s^{\alpha}} \right] \le w_{n_1} - w_n \le w_{n_1}, \quad for\ all \quad n,$$

which is contradiction to (6).

Next, we assume that $x_n < 0$ for all $n \ge n_1$. We use the transformation $y_n = -x_n$, then we have $\{y_n\}$ is an eventually positive solution of equation

$$\Delta\left(a_n (\Delta z_n)^{\alpha}\right) + q_n f(y_n) = -e_n, \tag{17}$$

where $z_n = y_n - \rho_n y_{n-l} > 0$. Define

$$w_n = \frac{P_n a_n (\Delta z_n)^{\alpha}}{z_n^{\alpha}}, \quad n \ge n_1. \tag{18}$$

Thus $w_n > 0$ and satisfies

$$\Delta w_n \le -P_n \left[M q_n (1+\rho_n)^{\beta} z_n^{\beta-\alpha} + \frac{e_n}{z_n^{\alpha}} \right] + \Delta P_n \frac{w_{n+1}}{P_{n+1}} - \frac{\alpha P_n w_{n+1}^{1+1/\alpha}}{P_{n+1}^{1+1/\alpha} a_n^{1/\alpha}}. \tag{19}$$

Set

$$F(u) = M q_n (1+\rho_n)^{\beta} u^{\beta-\alpha} + \frac{e_n}{u^{\alpha}}.$$

Using Lemma 1.1, we see that

$$F(u) \ge \frac{\beta q_n^{\alpha/\beta} M^{\alpha/\beta} (1+\rho_n)^{\alpha} e_n^{1-\alpha/\beta}}{\alpha^{\alpha/\beta} (\beta-\alpha)^{1-\alpha/\beta}} \tag{20}$$

and also (14) holds. Then the rest of the proof is similar to that of the above and hence is omitted.

Case II: Assume that $\Delta x_n < 0$ for all $n \ge n_1$, then $\Delta z_n < 0$ for all $n \ge n_1$. From $x_n > 0$ and $\Delta x_n < 0$, we obtain

$$\lim_{n\to\infty} x_n = b > 0. \tag{21}$$

Hence there exists $n_2 \ge n_1$ such that $x_n^{\beta} \ge b^{\beta}$ for $n \ge n_2$. Therefore we have

$$\Delta\left(a_n (\Delta z_n)^{\alpha}\right) \le -M q_n b^{\beta} + e_n. \tag{22}$$

Summing the last inequality from n_2 to $n-1$, we obtain

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$$a_n(\Delta z_n)^\alpha < a_n(\Delta z_n)^\alpha - a_{n_2}(\Delta z_{n_2})^\alpha \le -\left[\sum_{s=n_2}^{n-1} Mb^\beta q_s - e_s\right]$$

and

$$\Delta z_n \le -\left[\sum_{s=n_2}^{n-1} Mb^\beta q_s - e_s\right]^{1/\alpha} a_n^{-1/\alpha}. \tag{23}$$

Again summing the above inequality from n_2 to $n-1$, we obtain

$$z_n \le z_{n_2} - \left[\sum_{s=n_2}^{n-1}\left[\sum_{u=n_2}^{s-1} Mb^\beta q_u - e_u\right]^{1/\alpha} a_s^{-1/\alpha}\right]. \tag{24}$$

Letting $n \to \infty$, from condition (7) implies that z_n is negative for all $n \ge n_2$, a contradiction.

Finally, we assume that $\{x_n\}$ is an eventually negative sequence. It means that there exists $n_3 \in N$ such that $x_n < 0$ for all $n \ge n_3$. We use the transformation $y_n = -x_n$ in equation (1). Then y_n is an eventually positive solution of the equation

$$\Delta\left(a_n(\Delta z_n)^\alpha\right) + q_n f(y_n) = -e_n \tag{25}$$

where $z_n = y_n - \rho_n y_{n-l}$. The rest of the proof is similar to the above and hence omitted. The proof is now complete.

Corollary 2.1. *Assume that all the conditions of Theorem 2.1 hold except the condition (6) is replaced by*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} P_s Q_s^* = \infty, \tag{26}$$

and

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} a_s \frac{(\Delta P_s)^{\alpha+1}}{P_s^\alpha} P_{s+1}^{1/\alpha} < \infty. \tag{27}$$

Then every solution of equation (1) is almost oscillatory.

Theorem 2.2. *Assume that condition (7) holds. Furthermore, assume that there exist a positive sequence $\{p_n\}$ and a double sequence $\{H_{m,n} : m \ge n \ge 0\}$ such that $H_{m,n} = 0$ for $m \ge 0$, $H_{m,n} > 0$ for $m > n > 0$ and $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \le 0$ for $m \ge n \ge 0$. If*

$$\frac{1}{H_{m,n}} \sum \left[H_{m,s} P_s Q_s^* - \frac{P_{s+1}^{1+\alpha} a_s \left(\Delta_2 H_{m,s} + H_{m,s}\dfrac{\Delta P_s}{P_{s+1}}\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1} H_{m,s}^\alpha P_s^\alpha}\right] = \infty. \tag{28}$$

Then every solution of equation (1) is almost oscillatory.

Proof. Proceeding as in Theorem 2.1, we have two cases to consider.

Case I: Assume that $\Delta x_n > 0$ for all $n \geq n_1$. Define w_n by (11), then $w_n > 0$ and satisfies

$$\Delta w_n \leq -P_n Q_n^* + \frac{\Delta P_n w_{n+1}}{P_{n+1}} - \frac{\alpha P_n w_{n+1}^{1+1/\alpha}}{P_{n+1}^{1+1/\alpha} a_n^{1/\alpha}}. \tag{29}$$

Multiply both side by $H_{m,n}$ and summing from n_1 to $n-1$, we have

$$\sum_{s=n_1}^{n-1} H_{m,s} \Delta w_s \leq -\sum_{s=n_1}^{n-1} H_{m,s} P_s Q_s^* + \sum_{s=n_1}^{n-1} H_{m,s} \Delta P_s \frac{w_{s+1}}{P_{s+1}} - \alpha \sum_{s=n_1}^{n-1} H_{m,s} \frac{P_s w_{s+1}^{1+1/\alpha}}{P_{s+1}^{1+1/\alpha} a_s^{1/\alpha}}$$

$$\sum_{s=n_1}^{n-1} H_{m,s} P_s Q_s^* \leq -\sum_{s=n_1}^{n-1} H_{m,s} \Delta w_s + \sum_{s=n_1}^{n-1} H_{m,s} \Delta P_s \frac{w_{s+1}}{P_{s+1}} - \alpha \sum_{s=n_1}^{n-1} H_{m,s} \frac{P_s w_{s+1}^{1+1/\alpha}}{P_{s+1}^{1+1/\alpha} a_s^{1/\alpha}} \tag{30}$$

Using summation by parts, we obtain

$$\sum_{s=n_1}^{n-1} H_{m,s} P_s Q_s^* \leq H_{m,n} w_{n_1} + \sum_{s=n_1}^{n-1} \left[\left[\Delta_2 H_{m,s} + H_{m,s} \frac{\Delta P_s}{P_{s+1}} \right] w_{s+1} - \alpha H_{m,s} \frac{P_s w_{s+1}^{1+1/\alpha}}{P_{s+1}^{1+1/\alpha} a_s^{1/\alpha}} \right]. \tag{31}$$

Setting $B = \Delta_2 H_{m,s} + H_{m,s} \frac{\Delta P_s}{P_{s+1}},\ A = \frac{\alpha H_{m,s} P_s}{P_{s+1}^{1+1/\alpha} a_s^{1/\alpha}}\ \ and\ \ u = w_{s+1}$

From inequalities (31) and (14), we obtain

$$\sum_{s=n_1}^{n-1} \left[H_{m,s} P_s Q_s^* - \frac{P_{s+1}^{1+\alpha} \left(\Delta_2 H_{m,s} + H_{m,s} \frac{\Delta P_s}{P_{s+1}} \right)^{\alpha+1} a_s}{(\alpha+1)^{\alpha+1} H_{m,s}^{\alpha} P_s^{\alpha}} \right] \leq H_{m,n} w_{n_1},$$

$$\frac{1}{H_{m,n}} \sum_{s=n_1}^{n-1} \left[H_{m,s} P_s Q_s^* - \frac{P_{s+1}^{1+\alpha} \left(\Delta_2 H_{m,s} + H_{m,s} \frac{\Delta P_s}{P_{s+1}} \right)^{\alpha+1} a_s}{(\alpha+1)^{\alpha+1} H_{m,s}^{\alpha} P_s^{\alpha}} \right] \leq w_{n_1}, \tag{32}$$

which contradicts the assumption (28).

Next we consider the case when $x_n < 0$ for all $n \geq n_1$. We use the transformation $y_n = -x_n$ then y_n is a positive solution of equation

$$\Delta(a_n (\Delta z_n)^{\alpha}) + q_n f(x_n) = -e_n \tag{33}$$

when $z_n = y_n - \rho_n y_{n-l}$. Define w_n by (18) and (20) hold. The remaing of the proof is similar to that of first case of Theorem 2.1 and hence omitted. The proof of the case II is similar to that of second case of Theorem 2.1. The proof is now complete.

Corollary 2.2. *Assume that all the conditions of Theorem 2.2 hold except the condition (28) is replace by*

$$\limsup_{n\to\infty} \frac{1}{H_{m,n}} \sum_{s=n_0}^{n-1} H_{m,s} P_s Q_s^* = \infty,$$

and

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$$\limsup_{n\to\infty}\frac{1}{H_{m,n}}\sum_{s=n_0}^{n-1}\frac{P_{s+1}^{1+\alpha}\left(\Delta_2 H_{m,s}+H_{m,s}\frac{\Delta P_s}{P_{s+1}}\right)^{\alpha+1}a_s}{H_{m,s}^{\alpha}P_s^{\alpha}}<\infty.$$

Then every solution of equation (1) is almost oscillatory.

Examples

In this section, we provide three examples.

Example 3.1. *Let us consider the second order neutral difference equation of the form*

$$\Delta\left(\left(2-\frac{(-1)^n}{n}\right)\Delta\left(x_n-\frac{1}{2}x_{n-1}\right)\right)+x_n^3=\frac{3}{n(n+1)}-13(-1)^n,\ n\geq 2. \quad (34)$$

Here $a_n=2-\frac{(-1)^n}{n}$, $\rho_n=\frac{1}{2}$, $\alpha=1$, $l=1$, $q_n=1$, $f(u)=u^3$ and $e_n=\frac{3}{n(n+1)}-13(-1)^n$. All the conditions of Theorem 2.1 are satisfied. Hence every solution of the equation (34) is almost oscillatory. In fact one such solution is $(-1)^{n+1}$. Here $\{x_n\}$ is oscillatory.

Example 3.2. *Let us consider the second order neutral difference equation of the form*

$$\Delta\left(2+(-1)^n\left(\Delta(x_n-\frac{1}{2}x_{n-1})\right)\right)+x_n^3=14+25(-1)^n,\quad n\geq 2. \quad (35)$$

Here $a_n=2+(-1)^n$, $\rho_n=\frac{1}{2}$, $\alpha=1$, $l=1$, $q_n=1$, $f(u)=u^3$ and $e_n=14+25(-1)^n$. All the conditions of Theorem 2.1 are satisfied. Hence every solution of the equation (35) is almost oscillatory. In fact one such solution is $x_n=2-(-1)^{n+1}$. Here $\{x_n\}$ is nonoscillatory but Δx_n is oscillatory.

Example 3.3. *Let us consider the second order neutral difference equation of the form*

$$\Delta\left(\frac{1}{n+3}\Delta(x_n-2x_{n-1})\right)+n^2x_n^3=\frac{(n-1)(n+1)(n+2)-3}{(n-1)n(n+1)(n+2)},\ n\geq 2. \quad (36)$$

Here $a_n=\frac{1}{n+3}$, $\rho_n=2$, $\alpha=1$, $l=1$, $q_n=n^2$, $f(u)=u^3$ and $e_n=\frac{(n-1)(n+1)(n+2)-3}{(n-1)n(n+1)(n+2)}$. All the conditions of Theorem 2.1 are satisfied. Hence every solution of the equation (36) is almost oscillatory. In fact one such solution is $x_n=\frac{1}{n}$. Here $\{x_n\}$ tends to zero as $n\to\infty$.

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