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Restricted Three- Body Problem in Parabolodial Coordinate System

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ABSTRACT

In this paper, the equations of motion for the spatial circular restricted three- body problem in sidereal parabolodial coordinates system were established. Initial value procedure that can be used to compute both the parabolodial and Cartesian sidereal coordinates and velocities was also developed. The application of the procedure was illustrated by numerical example of a hypothetical Trojan asteroid in the Sun Jupiter system and by graphical representations of the variations of the two sidereal coordinate systems.

Keywords

Spatial restricted circular three body problem; regularization; coordinate transformations.

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1.INTRODUCTION

Depending on the application, a certain coordinate system may be simpler to use than the Cartesian coordinate system. As an example, a physical problem with spherical symmetry defined in $\mathbf{R}^3$ (e.g., motion in the field of a point mass), is usually easier to solve in spherical polar coordinates than in Cartesian coordinates. Also, for instance, in the galactic rotation, cylindrical coordinates are usually adopted, while the spherical coordinates are suitable for the dynamics of globular clusters. In fact , the change of the dependent and/or independent variables for the differential equations of motion becomes of the focal point of researches in space dynamics. Some authors proposed successful methods to change of the dependent and/or independent variables so as to regularize the differential equations of motion. Of these , the method established by Stiefel and Scheifele, in 1971, also Sharaf and Alshaery were established new differential equations of motion for the spatial circular restricted three body problem in cylindrical coordinates (Sharaf and Alshaery 2012) and spherical sidereal coordinates(Sharaf and Alshaery 2012) .Many studies on the applications of these devices for some orbital systems were done for the perturbed two body problem [e.g Sharaf ,et-al,1987 ; Sharaf ,et-al,1992; Sharaf and Sharaf 1995; Sharaf ,and Sharaf 1998].

The great success of the these devices in regularizing the equations of motion for the perturbed two body problem ,and on the other hand , the importance of the three body problem in space dynamics (e.g Szebehely 1967) and in stellar dynamics (e.g Binney and Tremaine 1987),tempted us to develop the corresponding deceives for the three body – problem.

The objective of the present paper, is to establish analytically and computationally the equations of motion for spatial restricted circular three body problem in Parabolodial coordinates system. By this paper, we aims at obtaining differential equations which are: (1) regular. (2) Suitable for the geometry to which they referred. (3) Producing slow variations in the coordinates during the orbital motion, a property which produces more stable numerical integration procedures. Also, . Initial value procedure that can be used to compute both the parabolodial and Cartesian sidereal coordinates and velocities was developd. The application of the algorithm was illustrated by numerical example of a hypothetical Trojan asteroid in the Sun Jupiter system and by graphical representations of the variations of the two sidereal coordinate systems.

2.CIRCULAR RESTRICTED THREE-BODY PROBLEM IN SIDEREAL SYSTEM

If two of the bodies, say m_1 and m_2 in the three-body problem move in circular, coplanar orbits about their common center of mass and the mass say m_3 of the third body is too small to affect the motion of the other bodies, the problem of the motion of the third body is called the circular ,restricted ,three body problem. The two revolving bodies are called the primaries*;* their masses are arbitrary but have such internal mass distributions that they may be considered point masses.

The equations of motion of the third body in a dimensionless sidereal (inertial) coordinate (x, y, z) system with the mean motion $n = 1$, are (Szebehely 1967)

$$\ddot{x} = \frac{\partial V}{\partial x}, \qquad (1)$$

$$\ddot{y} = \frac{\partial V}{\partial y}, \qquad (2)$$

$$\ddot{z} = \frac{\partial V}{\partial z}, \qquad (3)$$

where $V = V(x, y, z)$ is given as

$$V = \frac{1-\mu}{r_1} + \frac{\mu}{r_2}, \qquad (4)$$

μ denotes the mass of the smaller primary when the total mass of the primaries has been normalized to unity.

$$r_i^2 = (x - x_i)^2 + (y - y_i)^2 + z^2 \ ; \ i = 1,2 \qquad (5)$$

and $r_i ; i = 1,2$ are the distances of the third body from the primaries which are located at $(x_i, y_i, 0); i = 1,2$,these coordinates are functions of the time t and are given as

$$x_1 = \mu \cos t \ ; \ x_2 = -(1-\mu)\cos t \ ; \ y_1 = \mu \sin t \ ; \ y_2 = -(1-\mu)\sin t \ . \qquad (6)$$

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3. THE EQUATIONS OF MOTION IN SIDEREAL PARABOLODIAL COORDINATE SYSTEM

Corresponding to the Cartesian sidereal coordinate system (x, y, z), the coordinate system related to the system (x, y, z) by certain transformation ,is also called sidereal coordinate system. In this respect the system (u_1, u_2, u_3) of Equations (7) is called sidereal parabolodial coordinate system.

In what follows we shall establish, the differential equations for the spatial circular restricted three body-problem in sidereal parabolodial coordinate system.

3.1 Coordinate, Velocity Transformations and The Scale Factors

$$x = u_1 u_2 \cos u_3 \;; \qquad y = u_1 u_2 \sin u_3 \;; \qquad z = \frac{1}{2}(u_1^2 - u_2^2), \tag{7}$$

$$\dot{x} = \cos u_3 (u_2 \dot{u}_1 + u_1 \dot{u}_2) - u_1 u_2 \dot{u}_3 \sin u_3, \tag{8.1}$$

$$\dot{y} = \sin u_3 (u_2 \dot{u}_1 + u_1 \dot{u}_2) + u_1 u_2 \dot{u}_3 \cos u_3 , \tag{8.2}$$

$$\dot{z} = u_1 \dot{u}_1 - u_2 \dot{u}_2 . \tag{8.3}$$

where

$$0 \le u_1 < \infty \qquad 0 \le u_2 < \infty , \qquad -\pi \le u_3 \le \pi$$

3.2 Inverse Transformations

Since

$$r = (x^2 + y^2 + z^2)^{1/2} \tag{9}$$

then we get from Equations (7) that:

$$u_1^2 + u_2^2 = 2r . \tag{10}$$

Since u_1 and u_2 are both non-negative, then we get from Equations(7) and (10) that

$$u_1 = (r+z)^{1/2} \quad ; \quad u_2 = (r-z)^{1/2} \quad ; \quad u_3 = \tan^{-1}\left(\frac{y}{x}\right). \tag{11}$$

Differentiating the last of Equations (7) and Equations (9) and (10) with respective to the time t we get:

$$\dot{u}_1 = \frac{[x\dot{x} + y\dot{y} + \dot{z}(z+r)]}{2ru_1} \;; \qquad \dot{u}_2 = \frac{[x\dot{x} + y\dot{y} + \dot{z}(z-r)]}{2ru_2} \;; \qquad \dot{u}_3 = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2} \tag{12}$$

where r (hence u_1 and u_2) is given in terms of (x, y, z) from Equation (9),

3.3 The Equations Of Motion

The kinetic energy of a particle of unit mass in the parabolodial coordinate system is

$$T = \frac{1}{2}\left((u_1^2 + u_2^2)(\dot{u}_1^2 + \dot{u}_2^2) + u_1^2 u_2^2 \dot{u}_3^2\right). \tag{13}$$

By using the transformation equations (Equations(7)), the gravitational potential V could be expressed in term of (u_1, u_2, u_3).

Using Lagrange's dynamical equations, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_j}\right) - \frac{\partial T}{\partial u_j} = \frac{\partial V}{\partial u_j}; \; j = 1,2,3.$$

Consequently, we deduce for the equations of motion in sidereal parabolodial coordinate system, the forms

$$\ddot{u}_1 = \frac{1}{u_1^2 + u_2^2}\left(-\dot{u}_1^2 u_1 + u_1\dot{u}_2^2 - 2u_2\dot{u}_1\dot{u}_2 + u_1 u_2^2 \dot{u}_3^2 + \frac{\partial V}{\partial u_1}\right), \qquad (14.1)$$

$$\ddot{u}_2 = \frac{1}{u_1^2 + u_2^2}\left(\dot{u}_1^2 u_2 - u_2\dot{u}_2^2 - 2u_1\dot{u}_1\dot{u}_2 + u_2 u_1^2 \dot{u}_3^2 + \frac{\partial V}{\partial u_2}\right), \qquad (14.2)$$

$$\ddot{u}_3 = -\frac{2}{u_1 u_2}(u_2\dot{u}_1 + u_1\dot{u}_2)\dot{u}_3 + \frac{1}{u_1^2 u_2^2}\frac{\partial V}{\partial u_3}, \qquad (14.3)$$

where $\frac{\partial V}{\partial u_j}$ are given as

$$\frac{\partial V}{\partial u_j} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial u_j} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial u_j} + \frac{\partial V}{\partial z}\frac{\partial z}{\partial u_j}; \; j = 1,2.3, \qquad (15)$$

$\partial x/\partial u_j, \partial y/\partial u_j$ and $\partial z/\partial u_j$; $j = 1,2,3$ can be computed from Equations (7), while $\partial V/\partial x$, $\partial V/\partial y$, and $\partial V/\partial z$, can be computed from Equations (1),(2) and(3) ,so we get

$$\frac{\partial V}{\partial u_1} = (u_1(8Q_2(\mu-1) - \mu Q_1) - u_2\cos u_3(Q_5 Q_2 + Q_3 Q_1) - u_2 \sin u_3 (Q_6 Q_2 + Q_4 Q_1))/Q_1 Q_2$$

$$\frac{\partial V}{\partial u_2} = -(u_2(8Q_2(\mu-1) - \mu Q_1) + u_1\cos u_3(Q_5 Q_2 + Q_3 Q_1) + u_1 \sin u_3 (Q_6 Q_2 + Q_4 Q_1))/Q_1 Q_2$$

$$\frac{\partial V}{\partial u_3} = u_1 u_2\,(\mu-1)\,\mu \sin(t - u_3)(Q_1 - 8Q_2)/Q_1 Q_2$$

where

$$Q_1 = \left(4\mu^2 + u_1^4 - 8\,\mu\, u_1 u_2 \cos(t - u_3) + 2u_1^2 u_2^2 + u_2^4\right)^{3/2}$$

$$Q_2 = \frac{1}{8}\left(4(\mu-1)^2 + u_1^4 - 8\,(\mu-1)\,u_1 u_2 \cos(t-u_3) + 2u_1^2 u_2^2 + u_2^4\right)^{3/2}$$

$$Q_3 = \mu(\cos t(1-\mu) + u_1 u_3 \cos u_3)$$

$$Q_4 = \mu(\sin t(1-\mu) + u_1 u_2 \sin u_3)$$

$$Q_5 = 8\,(\mu-1)\,(\mu\cos t - u_1 u_2 \cos u_3)$$

$$Q_6 = 8\,(\mu-1)\,(\mu\sin t - u_1 u_2 \sin u_3)$$

4.COMPUTATIONAL DEVELOPMENTS

4.1 Initial Value Procedure

In what follows, we shall establish a procedure that can be used to compute $\forall\; t_0 \le t \le t_f$ (say) both:

1-the parabolodial sidereal coordinates and velocities $(u_1, u_2, u_3, \dot{u}_1, \dot{u}_2, \dot{u}_3)$,and

2- the Cartesian sidereal coordinates and velocities $(x, y, z, \dot{x}, \dot{y}, \dot{z})$.

So, such procedure is a double usefulness computational algorithm, for which a differential solver can be used for the parabolodial sidereal six order system to obtain $(u_1, u_2, u_3, \dot{u}_1, \dot{u}_2, \dot{u}_3)$. While the Cartesian sidereal coordinates and velocities $(x, y, z, \dot{x}, \dot{y}, \dot{z})$ are obtained by the substitutions in the direct transformation formulae (Equations (7) and (8)), rather than solving the six order system of Equations (1),(2) and (3).By this way, great time can be saved.

This initial value procedure using sidereal parabolodial coordinate system will be described through its basic points, input, output and computational steps.

Input:

(1) $x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0$ at $t = t_0$,

(2) the final time $t = t_f$;

(3) $\frac{\partial V}{\partial x} = F_1(x, y, z); \quad \frac{\partial V}{\partial y} = F_2(x, y, z); \quad \frac{\partial V}{\partial z} = F_3(x, y, z);$

Output: (1) $u_j; \dot{u}_j; j = 1,2,3 \quad \forall \quad t_0 \le t \le t_f$

(2) $x, y, z; \dot{x}, \dot{y}, \dot{z} \quad \forall \quad t_0 \le t \le t_f$

Computational steps

1-Using the given values $x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0$ at $t = t_0$ and the inverse transformations to compute the initial values $u_{0j}; j = 1,2,3,4,5,6.$

2-Using the partial derivatives $\frac{\partial V}{\partial u_1}; \frac{\partial V}{\partial u_2}; \frac{\partial V}{\partial u_3}$ (functions of $u_j; j = 1,2,3$) to construct the analytical forms of equations of motion as first order system .

3- Using the initial conditions $u_{0j}; j = 1,2,3,4,5,6$ from step 1 to solve numerically the above differential system of step 2 for $u_j; j = 1,2 \cdots, 6 \quad \forall \quad t_0 \le t \le t_f$, (note that $u_4 \equiv \dot{u}_1, u_5 \equiv \dot{u}_2, u_6 \equiv \dot{u}_3$).

4- Using $u_j; \dot{u}_j; j = 1,2,3$ from step 3 and the direct transformations of Equations (7) and (8) to compute numerically x, y, z and $\dot{x}, \dot{y}, \dot{z} \quad \forall \quad t_0 \le t \le t_f$.

5-End

4.2 Numerical Example

Consider hypothetical Trojan asteroid (Hellings 1994) in the Sun Jupiter system. Since we considered circular motion ,it is therefore necessary to ignore the eccentricity of the orbit of Jupiter(0.048417). In addition to circular motion we shall assume(as in the above reference) that motion is also planner, that is to say, the asteroid moves exactly in the Jupiter's orbital plane. The initial conditions are

$z_0 = 0.0 \qquad y_0 = 0.883345912 \qquad x_0 = -0.509046125$

$w_0 = 0.0 \qquad v_0 = 0.0149272418 \qquad u_0 = 0.0258975212$

$t_f = 8.0 \qquad t_0 = 0.0 \qquad \mu = 0.000953875$

applying the above procedure we get the results as displayed in Tables I and II

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Table I: The values of sidereal parabolodial coordinates and velocities

t	$u_1(t)$	$u_2(t)$	$u_3(t)$	$u_4(t)$	$u_5(t)$	$u_6(t)$
0.	1.00971	1.00971	2.09358	1.40136 □10^□6	1.40136 □10^□6	□0.0293191
0.4	0.930374	1.01113	2.0812	□0.417503	0.0192519	□0.0345867
0.8	0.639829	1.06086	2.06272	□1.08771	0.34353	□0.0672527
1.2	0.150741	1.3923	1.97971	□1.07097	1.21282	□0.709588
1.6	0.171981	1.87728	□0.978504	0.596866	1.14226	□0.299887
2.	0.368162	2.29833	□1.02417	0.4098	0.970926	□0.0436589
2.4	0.512263	2.66076	□1.03501	0.31993	0.848419	□0.0168258
2.8	0.628994	2.98165	□1.0399	0.267798	0.760558	□0.00888699
3.2	0.728848	3.27209	□1.04269	0.233598	0.694525	□0.0054958
3.6	0.817172	3.53917	□1.0445	0.209264	0.642827	□0.00373702
4.	0.897054	3.78767	□1.04578	0.190944	0.601028	□0.00270755
4.4	0.970449	4.02094	□1.04672	0.176569	0.566366	□0.00205286
4.8	1.03867	4.24147	□1.04745	0.164931	0.537037	□0.00161055
5.2	1.10266	4.45112	□1.04802	0.155277	0.511812	□0.00129761
5.6	1.16309	4.65135	□1.0485	0.147109	0.489819	□0.00106802
6.	1.2205	4.84332	□1.04889	0.140088	0.470427	□0.000894548
6.4	1.27528	5.02797	□1.04922	0.133971	0.453162	□0.000760265
6.8	1.32777	5.20608	□1.0495	0.128583	0.437663	□0.000654176
7.2	1.37823	5.3783	□1.04974	0.12379	0.42365	□0.000568894
7.6	1.42687	5.54517	□1.04996	0.119493	0.410899	□0.000499304
8.	1.47388	5.70716	□1.05014	0.115611	0.399232	□0.000441775

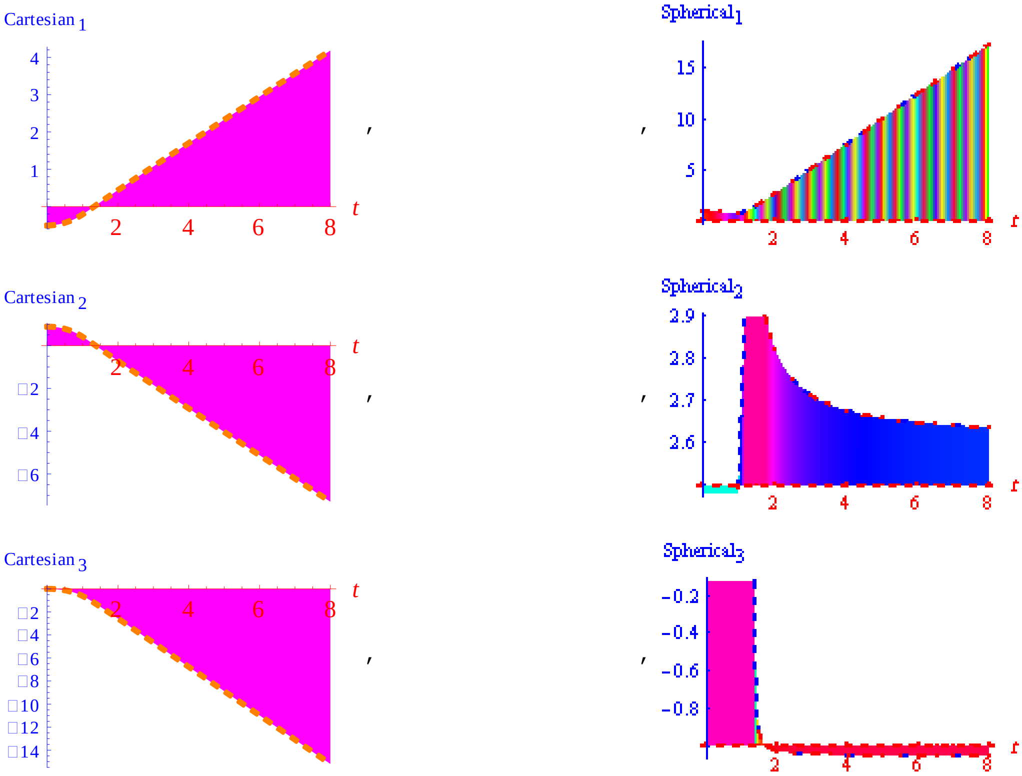
Table II: The values of sidereal Cartesian coordinates and velocities

t	x	y	z	$\dot{x}$	$\dot{y}$	$\dot{z}$
0.	□0.509046	0.883346	0.	0.02589752	0.01492724	0. □10^□21
0.4	□0.459572	0.820836	□0.0783984	0.225871657	□0.33682436	□0.40790011
0.8	□0.320598	0.598283	□0.35802	0.48143596	□0.801783085	□1.06038392
1.2	□0.0834494	0.192573	□0.957888	0.65683986	□1.141211731	□1.85005272
1.6	0.180239	□0.267863	□1.7473	0.65486728	□1.146661321	□2.041689889
2.	0.439836	□0.72286	□2.57339	0.64382950	□1.129187522	□2.08063439
2.4	0.695844	□1.17201	□3.40861	0.63674253	□1.11738416	□2.093551114
2.8	0.949555	□1.61729	□4.24731	0.63211715	□1.10954735	□2.099276478
3.2	1.20172	□2.05995	□5.08768	0.62890938	□1.10406545	□2.102292260
3.6	1.4528	□2.50074	□5.92899	0.62656696	□1.10004158	□2.10407048
4.	1.70305	□2.94011	□6.77086	0.62478577	□1.09697115	□2.10520543
4.4	1.95268	□3.3784	□7.6131	0.62338755	□1.09455477	□2.105973545
4.8	2.2018	□3.81582	□8.45561	0.62226158	□1.09260512	□2.10651730
5.2	2.45051	□4.25253	□9.2983	0.62133589	□1.09099982	□2.106916354
5.6	2.69889	□4.68865	□10.1411	0.620561571	□1.08965535	□2.107217725
6.	2.94698	□5.12428	□10.9841	0.61990443	□1.088513192	□2.10745086
6.4	3.19482	□5.55949	□11.8271	0.61933988	□1.08753114	□2.10763504
6.8	3.44246	□5.99432	□12.6702	0.61884964	□1.08667778	□2.10778302
7.2	3.68991	□6.42884	□13.5133	0.61841997	□1.08592938	□2.10790364
7.6	3.9372	□6.86308	□14.3565	0.61804030	□1.085267732	□2.10800322
8.	4.18435	□7.29707	□15.1997	0.61770248	□1.08467875	□2.10808657

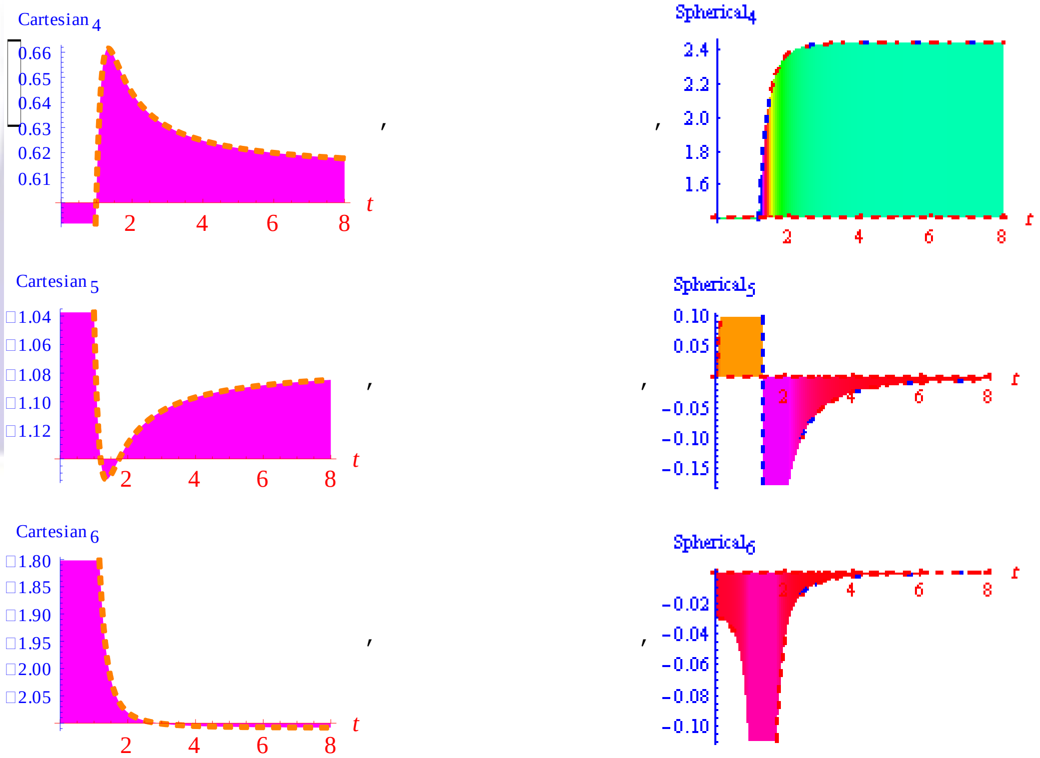
4.3 Graphical Representations

The following figures illustrate the time variations of the two sidereal coordinate systems (x, y, z) (left) and (u_1, u_2, u_3) (right).

Graphical representions for the time variations of position vector ;j□1,2,3



Graphical representions for the time variations of velocity vectors ;j□4,5,6



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5 CONCLUSION

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