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Introduction to a family of Thukral *k*-order method for finding multiple zeros of nonlinear equations

R. Thukral
Padé Research Centre, 39 Deanswood Hill, Leeds, West Yorkshire, LS17 5JS, England

Abstract

A new one-point *k*-order iterative method for finding zeros of nonlinear equations having unknown multiplicity is introduced. In terms of computational cost the new iterative method requires *k*+1 evaluations of functions per iteration. It is shown that the new iterative method has a convergence of order *k*.

Keywords: Newton method; Schroder method; Thukral method; Nonlinear equations; Multiple roots; Order of convergence; Root-finding.

Subject Classifications: AMS (MOS): 65H05, 41A25.

1 Introduction

We propose a new one-point *k*-order iterative method to find multiple roots of the nonlinear equation. The root-solver is of great practical importance since it overcomes theoretical limits of iterative methods concerning convergence order and computational efficiency. In this paper, we are interested in the case that α is a root of multiplicity $m>1$ of a nonlinear equation. Therefore, the purpose of this study is to develop a new class of iterative method for finding multiple roots of nonlinear equations of a higher order than the existing iterative methods [2,5-8] and show further development of the Thukral third and fourth order methods [6-8].

2 Preliminaries

In order to establish the order of convergence of the new *k*-order iterative method, we use the following definitions [2,5-8].

Definition 1 Let $f(x)$ be a real-valued function with a root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α. The order of convergence *p* is given by

$$\lim_{n\to\infty}\frac{x_{n+1}-\alpha}{(x_n-\alpha)^p}=\zeta\neq 0, \tag{1}$$

where $p\in\mathbb{R}^+$ and ζ is the asymptotic error constant [1-9].

Definition 2 Let $e_k=x_k-\alpha$ be the error in the *k*th iteration, then the relation

$$e_{k+1}=\zeta e_k^p+\mathrm{O}\left(e_k^{p+1}\right), \tag{2}$$

is the error equation. If the error equation exists, then *p* is the order of convergence of the iterative method [1-9].

Definition 3 Let *r* be the number of function evaluations of the method. The efficiency of the method is measured by the concept of efficiency index and defined as

$$EI(r,p)=\sqrt[r]{p}, \tag{3}$$

where *p* is the order of convergence of the method [3].

Definition 4 Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of (1). Then the computational order of convergence may be approximated by

$$COC\approx\frac{\ln\left|\delta_n\div\delta_{n-1}\right|}{\ln\left|\delta_{n-1}\div\delta_{n-2}\right|}, \tag{4}$$

where $\delta_i=f(x_i)\div f'(x_i)$, [6-8].

3 Construction of the new *k*-order iterative method

In this section we present a new scheme to find multiple roots of a nonlinear equation. Our aim is to define a new one-point iterative method of *k*-order of convergence and in process we shall demonstrate that three established methods are formed namely, the classical Schroder method, the Thukral third-order method [6], and the Thukral fourth-order method [8].

First we denote the following

$$t_1 = \frac{f'(x_n)}{f(x_n)}, \quad t_2 = \frac{f''(x_n)}{f'(x_n)}, \quad t_3 = \frac{f'''(x_n)}{f''(x_n)}, \tag{5}$$

$$t_4 = \frac{f^{iv}(x_n)}{f'''(x_n)}, \quad t_5 = \frac{f^{v}(x_n)}{f^{iv}(x_n)}, \quad t_6 = \frac{f^{vi}(x_n)}{f^{5}(x_n)}, \tag{6}$$

The Method

In general, we define a new scheme as

$$x_{n+1} = x_n - (k-1)\left[\frac{N_k\left(f(x_n), f'(x_n), f''(x_n), \cdots, f^{(k-1)}(x_n)\right)}{D_k\left(f(x_n), f'(x_n), f''(x_n), \cdots, f^{(k)}(x_n)\right)}\right]\left(\frac{f(x_n)}{f'(x_n)}\right), \tag{7}$$

where

$$N_k\left(f(x_n), f'(x_n), f''(x_n), \cdots, f^{(k-1)}(x_n)_i\right) = \sum_{i=1}^{p} \alpha_i F_i^k, \tag{8}$$

$$D_k\left(f(x_n), f'(x_n), f''(x_n), \cdots, f^{(k)}(x_n)\right) = \sum_{i=1}^{q} \beta_i F_i^k, \tag{9}$$

$$\left.\begin{array}{lllll}
F_1^1 = f'(x_n), & & & & \\
F_1^2 = f'(x_n)^2, & F_2^2 = f(x_n) f''(x_n), & & & \\
F_1^3 = f'(x_n)^3, & F_2^3 = f(x_n) f'(x_n) f''(x_n), & F_3^3 = f(x_n)^2 f'''(x_n), & & \\
F_1^4 = f'(x_n)^4, & F_2^4 = f(x_n) f'(x_n)^2 f''(x_n), & F_3^4 = f(x_n)^2 f'(x_n) f'''(x_n), & F_4^4 = f(x_n)^3 f^{(iv)}(x_n), & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right\} \tag{10}$$

$$\alpha_1 = \beta_1, \quad \left\{\alpha_i, \beta_j\right\} \in \Re, \tag{11}$$

$$\sum_{i=1}^{p} \alpha_i = 0, \quad \sum_{i=1}^{q} \beta_i = 0, \tag{12}$$

$$N_{k+1}\left(f(x_n), f'(x_n), f''(x_n), \cdots, f^{(k-1)}(x_n)\right) = (k-1) f'(x_n) D_k\left(f(x_n), f'(x_n), f''(x_n), \cdots, f^{(k-1)}(x_n)\right), \tag{13}$$

$$\left[\frac{N_k\left(f(x_n), f'(x_n), f''(x_n), \cdots, f^{(k-1)}(x_n)\right)}{D_k\left(f(x_n), f'(x_n), f''(x_n), \cdots, f^{(k)}(x_n)\right)}\right] \approx \hat{m}_k, \tag{14}$$

$$k > 1, \tag{15}$$

$$p < q, \tag{16}$$

$$\{n,k,p,q\}\in\aleph. \tag{17}$$

When $k=1,$ without loss of generality, we consider the classical Newton method as our first scheme and in fact it has first-order convergence given by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}, \tag{18}$$

$$x_{n+1}=x_n-\frac{1}{t_1}. \tag{19}$$

When $k=2,$ the classical Schroder second-order method [4] is obtained and is given as

$$x_{n+1}=x_n-(k-1)\left[\frac{N_2\left(f(x_n),f'(x_n)\right)}{D_2\left(f(x_n),f'(x_n),f''(x_n)\right)}\right]\left(\frac{f(x_n)}{f'(x_n)}\right) \tag{20}$$

$$x_{n+1}=x_n-(k-1)\left[\frac{f(x_n)D_1\left(f(x_n),f'(x_n)\right)}{D_2\left(f(x_n),f'(x_n),f''(x_n)\right)}\right], \tag{21}$$

$$x_{n+1}=x_n-\frac{f(x_n)\,f'(x_n)}{f'(x_n)^2-f(x_n)\,f''(x_n)}, \tag{22}$$

$$x_{n+1}=x_n-\left[\frac{f'(x_n)^2}{f'(x_n)^2-f(x_n)\,f''(x_n)}\right]\left(\frac{f(x_n)}{f'(x_n)}\right), \tag{23}$$

$$x_{n+1}=x_n-\frac{1}{t_1-t_2}, \tag{24}$$

Next $k=3,$ the Thukral third-order method is obtained [6] and is given as,

$$x_{n+1}=x_n-(k-1)\left[\frac{N_3\left(f(x_n),f'(x_n),f''(x_n)\right)}{D_3\left(f(x_n),f'(x_n),f''(x_n),f'''(x_n)\right)}\right]\left(\frac{f(x_n)}{f'(x_n)}\right), \tag{25}$$

$$x_{n+1}=x_n-2\left[\frac{f(x_n)D_2\left(f(x_n),f'(x_n),f''(x_n)\right)}{D_3\left(f(x_n),f'(x_n),f''(x_n),f'''(x_n)\right)}\right], \tag{26}$$

$$x_{n+1}=x_n-\left[\frac{2\left(f'(x_n)^3-f(x_n)\,f'(x_n)\,f''(x_n)\right)}{2\,f'(x_n)^3-3\,f(x_n)\,f'(x_n)\,f''(x_n)+f'''(x_n)\,f(x_n)^2}\right]\left(\frac{f(x_n)}{f'(x_n)}\right), \tag{27}$$

$$x_{n+1}=x_n-\frac{2\left(f(x_n)\,f'(x_n)^2-f(x_n)^2\,f''(x_n)\right)}{2\,f'(x_n)^3-3\,f(x_n)\,f'(x_n)\,f''(x_n)+f'''(x_n)\,f(x_n)^2}, \tag{28}$$

$$x_{n+1}=x_n-\frac{2t_1-2t_2}{2t_1^2-3t_1t_2+t_2t_3}, \tag{29}$$

And next when $k=4,$ we obtain the Thukral fourth-order method [8], given by

$$x_{n+1}=x_n-(k-1)\left[\frac{N_4\left(f(x_n),f'(x_n),f''(x_n),f'''(x_n)\right)}{D_4\left(f(x_n),f'(x_n),f''(x_n),f'''(x_n),f^{(iv)}(x_n)\right)}\right]\left(\frac{f(x_n)}{f'(x_n)}\right), \tag{30}$$

$$x_{n+1}=x_n-3\left[\frac{f(x_n)D_3\left(f(x_n),f'(x_n),f''(x_n),f'''(x_n)\right)}{D_4\left(f(x_n),f'(x_n),f''(x_n),f'''(x_n),f^{(iv)}(x_n)\right)}\right], \tag{31}$$

$$x_{n+1}=x_n-3\left[\frac{\sum_{i=1}^{3}\alpha_i F_i^4}{\sum_{i=1}^{5}\beta_i F_i^4}\right]\left(\frac{f(x_n)}{f'(x_n)}\right), \tag{32}$$

where

$$\left.\begin{array}{l} F_1^4=f'(x_n)^4, \\ F_2^4=f(x_n)f'(x_n)^2 f''(x_n), \\ F_3^4=f(x_n)^2 f'(x_n)f'''(x_n), \\ F_4^4=f(x_n)^3 f^{iv}(x_n), \\ F_5^4=\left(f(x_n)f''(x_n)\right)^2, \end{array}\right\} \tag{33}$$

$$\alpha_1=6,\quad \alpha_2=-9,\quad \alpha_3=3, \tag{34}$$

$$\beta_1=6,\quad \beta_2=-12,\quad \beta_3=4,\quad \beta_4=-1,\quad \beta_5=3, \tag{35}$$

Also, it was shown that the fourth-order method (32) can also be expressed as

$$x_{n+1}=x_n-\frac{6t_1^2-9t_1t_2+3t_2t_3}{6t_1^3-12t_1^2t_2+4t_1t_2t_3-t_2t_3t_4+3t_1t_2^2}, \tag{36}$$

When $k=5,$ we progress to define a fifth-order iterative method for finding multiple roots of a nonlinear equation. In order to construct the new iterative method we require a total of five function evaluations. Hence the new scheme is given as

$$x_{n+1}=x_n-(k-1)\left[\frac{N_k\left(f(x_n),f'(x_n),f''(x_n),f'''(x_n),f^{(iv)}(x_n)\right)}{D_k\left(f(x_n),f'(x_n),f''(x_n),f'''(x_n),f^{(iv)}(x_n),f^{(v)}(x_n)\right)}\right]\left(\frac{f(x_n)}{f'(x_n)}\right), \tag{37}$$

$$x_{n+1}=x_n-4\left[\frac{N_5\left(f(x_n),f'(x_n),f''(x_n),f'''(x_n),f^{(iv)}(x_n)\right)}{D_5\left(f(x_n),f'(x_n),f''(x_n),f'''(x_n),f^{(iv)}(x_n),f^{(v)}(x_n)\right)}\right]\left(\frac{f(x_n)}{f'(x_n)}\right), \tag{38}$$

$$x_{n+1}=x_n-4\left[\frac{f(x_n)D_4\left(f(x_n),f'(x_n),f''(x_n),f'''(x_n),f^{(iv)}(x_n)\right)}{D_5\left(f(x_n),f'(x_n),f''(x_n),f'''(x_n),f^{(iv)}(x_n),f^{(v)}(x_n)\right)}\right], \tag{39}$$

$$x_{n+1} = x_n - 4\left[\frac{\sum_{i=1}^{5}\alpha_i F_i^5}{\sum_{i=1}^{7}\beta_i F_i^5}\right]\left(\frac{f(x_n)}{f'(x_n)}\right), \tag{40}$$

where

$$\left.\begin{aligned}
F_1^5 &= f'(x_n)^5,\\
F_2^5 &= f(x_n)f'(x_n)^3 f''(x_n),\\
F_3^5 &= f(x_n)^2 f'(x_n)^2 f'''(x_n),\\
F_4^5 &= f(x_n)^3 f'(x_n) f^{(iv)}(x_n),\\
F_5^5 &= \left(f(x_n)f''(x_n)\right)^2 f'(x_n),\\
F_6^5 &= f(x_n)f''(x_n)f'(x_n)^3,\\
F_7^5 &= f(x_n)^4 f^{(v)}(x_n),
\end{aligned}\right\} \tag{41}$$

$$\alpha_1 = 24 \quad \alpha_2 = -48 \quad \alpha_3 = 16 \quad \alpha_4 = -4 \quad \alpha_5 = 12 \tag{42}$$

$$\beta_1 = 24 \quad \beta_2 = -60 \quad \beta_3 = 20 \quad \beta_4 = -5 \quad \beta_5 = 30 \quad \beta_6 = -10 \quad \beta_7 = 1 \tag{43}$$

Also, the new fifth-order method (39) can also be expressed as

$$x_{n+1} = x_n - \frac{24t_1^3 - 48t_1^2t_2 + 16t_1t_2t_3 - 4t_2t_3t_4 + 12t_1t_2^2}{24t_1^4 - 60t_1^3t_2 + 20t_1^2t_2t_3 - 5t_1t_2t_3t_4 + 30t_1^2t_2^2 - 10t_1t_2^2t_3 + t_2t_3t_4t_5}, \tag{44}$$

It is essential to analyse the order of convergence of the new iterative method.

Theorem 1

Let $f : I \subset \mathbb{R}$ be a function for an open interval $I \subset \mathbb{R}$. Let $f(x_n)$ has a multiple root, $x = \alpha \in I$ with multiplicity $m > 1$ and x_0 is the initial guess of the multiple root. Assume that $f(x_n)$ is a sufficiently differentiable function in *I*, then iteration defined by the new scheme (39) has fifth-order convergence and satisfies the error equation

$$e_{n+1} = \left(\frac{\omega_1 T_1^4 - \omega_2 T_1^2 T_2 + \omega_3 T_1 T_3 + \omega_4 T_2^2}{\omega_5}\right)e_n^5. \tag{45}$$

where

$$\left.\begin{aligned}
&\omega_1 = (m+1)(m+2)(m+3), \quad \omega_2 = 2m(2m+3)(m+3),\\
&\omega_3 = 2m^2(2m+3), \quad \omega_4 = 2m^2(m+3),\\
&\omega_5 = m^4(m+1)^2(m+2)(m+3).
\end{aligned}\right\} \tag{46}$$

Proof

Let α be a root of multiplicity *m,* that is $f(\alpha) = f'(\alpha) = \cdots f^{(m-1)}(\alpha) = 0,$ and $f^{(m)}(\alpha) \neq 0.$ Since $f(x_n)$ is a sufficiently differentiable function, therefore we expand $f(\alpha)$ about $x = \alpha$ by the Taylor series. Also let $e_n = x_n - \alpha$ and using the Taylor series expansion of $f(x_n), f'(x_n), f''(x_n), f'''(x_n), f^{iv}(x_n), f^{v}(x_n),$ about α , we have

$$f(x_n)=\left(\frac{f^{(m)}(\alpha)}{m!}\right)e_n^m\left[1+A_1e_n+A_2e_n^2+A_3e_n^3+\cdots\right], \tag{47}$$

$$f'(x_n)=\left(\frac{f^{(m)}(\alpha)}{(m-1)!}\right)e_n^{m-1}\left[1+B_1e_n+B_2e_n^2+B_3e_n^3+\cdots\right], \tag{48}$$

$$f''(x_n)=\left(\frac{f^{(m)}(\alpha)}{(m-2)!}\right)e_n^{m-2}\left[1+C_1e_n+C_2e_n^2+C_3e_n^3+\cdots\right], \tag{49}$$

$$f'''(x_n)=\left(\frac{f^{(m)}(\alpha)}{(m-3)!}\right)e_n^{m-3}\left[1+D_1e_n+D_2e_n^2+D_3e_n^3+\cdots\right], \tag{50}$$

$$f^{iv}(x_n)=\left(\frac{f^{(m)}(\alpha)}{(m-4)!}\right)e_n^{m-4}\left[1+E_1e_n+E_2e_n^2+E_3e_n^3+\cdots\right], \tag{51}$$

$$f^{v}(x_n)=\left(\frac{f^{(m)}(\alpha)}{(m-5)!}\right)e_n^{m-5}\left[1+G_1e_n+G_2e_n^2+G_3e_n^3+\cdots\right], \tag{52}$$

where

$$\left.\begin{aligned}&T_k=\frac{f^{(m+k)}(\alpha)}{f^{(m)}(\alpha)},\quad A_k=\frac{m!T_k}{(m+k)!},\quad B_k=\frac{(m-1)!T_k}{(m+k-1)!},\quad C_k=\frac{(m-2)!T_k}{(m+k-2)!},\\&D_k=\frac{(m-3)!T_k}{(m+k-3)!},\quad E_k=\frac{(m-4)!T_k}{(m+k-4)!},\quad G_k=\frac{(m-5)!T_k}{(m+k-5)!},\end{aligned}\right\} \tag{53}$$

From (47)-(52), we get

$$\frac{f(x_n)}{f'(x_n)}=\frac{e_n}{m}-\frac{T_1e_n^2}{m^2(m+1)}+\frac{\left(T_1^2(m+2)-2mT_2\right)e_n^3}{m^3(m+1)(m+2)}+\cdots, \tag{54}$$

$$\frac{f'(x_n)}{f(x_n)}=\frac{m}{e_n}+\frac{T_1}{(m+1)}-\frac{\left(T_1^2(m+2)-(m+1)T_2\right)e_n}{(m+1)^2(m+2)}+\cdots, \tag{55}$$

$$\frac{f''(x_n)}{f'(x_n)}=\frac{m-1}{e_n}+\frac{T_1}{m}-\frac{\left(T_1^2(m+1)-2mT_2\right)e_n}{m^2(m+1)}+\cdots, \tag{56}$$

$$\frac{f'''(x_n)}{f''(x_n)}=\frac{m-2}{e_n}+\frac{T_1}{(m-1)}-\frac{\left(T_1^2m-2(m-1)T_2\right)e_n}{m(m-1)^2}+\cdots, \tag{57}$$

$$\frac{f^{iv}(x_n)}{f'''(x_n)}=\frac{m-3}{e_n}+\frac{T_1}{(m-2)}-\frac{\left(T_1^2(m-1)-2T_2(m-2)\right)e_n}{(m-2)^2(m-1)}+\cdots, \tag{58}$$

$$\frac{f^{v}(x_n)}{f^{iv}(x_n)}=\frac{m-4}{e_n}+\frac{T_1}{(m-2)}-\frac{\left(T_1^2(m-2)-2T_2(m-3)\right)e_n}{(m-3)^2(m-2)}+\cdots, \tag{59}$$

Substituting appropriate expressions in (39) and simplifying, we have

$$e_{n+1}=\left(\frac{\omega_1 T_1^4-\omega_2 T_1^2 T_2+\omega_3 T_1 T_3+\omega_4 T_2^2}{\omega_5}\right)e_n^5+\mathrm{O}\left(e_n^6\right). \tag{60}$$

where

$$\left.\begin{array}{l}\omega_1=(m+1)(m+2)(m+3), \quad \omega_2=2m(2m+3)(m+3),\\ \omega_3=2m^2(2m+3), \quad \omega_4=2m^2(m+3),\\ \omega_5=m^4(m+1)^2(m+2)(m+3).\end{array}\right\} \tag{61}$$

The expression (60) establishes the asymptotic error constant for the fifth-order of convergence for the new iterative method defined by (39). This completes the proof. □

The new one-point *k*-order method requires (*k*+1) function evaluations and has the order of convergence *k*. To determine the efficiency index of the new method, definition 3 shall be used. Hence, the efficiency index of the new iterative method given by (7)

$$EI(k+1,k)=\sqrt[(k+1)]{k}. \tag{62}$$

4 Application of the new one-point *k*-order iterative method

The proposed one-point *k*-order method given by (7) is employed to solve nonlinear equation with multiple roots. The difference between the multiple root α and the approximation x_n for test function with initial guess x_0 is displayed in table. Furthermore, the computational order of convergence approximations are displayed in table and we observe that this perfectly coincides with the theoretical result. In addition, the difference between the multiplicity m and the approximation $\widehat{m}$ is also displayed in table. The numerical computations listed in the table was performed on an algebraic system called Maple and the errors displayed is of absolute value.

We will demonstrate the convergence of the new one-point *k*-order method for the following nonlinear equation

$$f(x)=\left[e^x+x-2\right]^6, \tag{63}$$

having multiplicity $m=6$ and the exact value of the multiple roots of (63) is $\alpha=0.442854\ldots$. In Table 1 the errors obtained by the new method described, is based on the initial value $x_0=2^{-2}$. We observe that the new one-point *k*-order method is converging to the expected order.

Table 1 Errors occurring in the estimates of the root of (63) by the method described

method	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	$\lvert x_4-\alpha\rvert$	$\lvert m-\hat{m}_k\rvert$	COC
(18)	0.159	0.131	0.108	0.090	-	0.9928
(23)	0.983e-2	0.292e-4	0.260e-9	0.206e-19	0.950e-9	2.0000
(28)	0.740e-3	0.446e-10	0.979e-32	0.103e-96	0.179e-31	3.0000
(32)	0.181e-4	0.124e-20	0.279e-85	0.702e-344	0.509e-85	4.0000
(39)	0.411e-6	0.255e-34	0.231e-175	0.143e-880	0.422e-175	4.9996

5 Remarks and conclusion

A new one-point *k*-order iterative method for solving nonlinear equations with multiple roots has been introduced. Empirically, we have found that the new *k*-order iterative method contains product of *k* function and (*k*+1) function evaluations. We observe that the computational order of convergence approximations perfectly coincides with the theoretical result. The drawback of the proposed method is that we need to evaluate higher order derivatives of a given function, hence further improvement is necessary. Finally, we conjecture that the new scheme (7) can be constructed to produce any higher order of convergence.

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