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Oscillatory and Asymptotic Behavior of Third Order Half-linear Neutral Differential Equation with "Maxima"

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Abstract

In this paper we study the oscillation and asymptotic behavior of the third order neutral differential equation with "maxima" of the form

$$\left(a(t)\left(\left(x(t)+p(t)x(\tau(t))\right)'\right)^{\alpha}\right)'+q(t)\max_{[\sigma(t),t]}x^{\alpha}(s)=0,\ t\geq t_0,$$

where α is the quotient of odd positive integers. Some examples are given to illustrate the main results.

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Introduction

In this paper, we study the oscillation and asymptotic behavior of the third order neutral differential equation with "maxima" of the form

$$\left(a(t)\left(\left(x(t)+p(t)x(\tau(t))\right)'\right)^{\alpha}\right)'+q(t)\max_{[\sigma(t),t]}x^{\alpha}(s)=0,\ t\geq t_0, \tag{1.1}$$

where α is the quotient of odd positive integers, subject to the following conditions:

(H1) $a(t), p(t), q(t), \tau(t),$ and $\sigma(t)\in C([t_0,\infty))$;

(H2) $a(t)>0$, and $\int_{t_0}^{\infty}\frac{1}{a^{1/\alpha}(t)}dt=\infty$;

(H3) $0\leq p(t)\leq p<1, q(t)\geq 0$ and $q(t)$ is not identically zero on any ray of $q(t)\in[t_x,\infty)$ for any $t_x\geq t_0$;

(H4) $\tau(t)\leq t, \tau(t)\geq\tau_0>0, \sigma\circ\tau=\tau\circ\sigma$ with $\sigma(\tau(t))\leq t$ and $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty$.

Set $z(t)=x(t)+p(t)x(\tau(t))$. By a solution of equation (1.1), we mean a continuous function $x(t)\in C([T_x,\infty)), T_x\geq t_0$, which has the properties $z(t)\in C^2([T_x,\infty)), a(t)(z''(t))^{\alpha}\in C^1([T_x,\infty))$ and satisfies equation (1.1) on $[T_x,\infty)$. We consider only those solutions $x(t)$ of equation (1.1) which satisfy $sup\{|x(t)|: t\geq T\}>0$ for all $T\geq t_0$. We assume that equation (1.1) possesses such a solution. A solution of equation (1.1) is called oscillatory if it has infinitely many zeros on $[T_x,\infty)$ and called nonoscillatory otherwise.

In recent years there has been great attention denoted to the oscillatory and asymptotic behavior of third order neutral differential equations without maxima, see [1, 7, 8, 9, 13, 14], and the references contained therein. However few results are available on the oscillatory behavior of third order differential equation with "maxima", see [2, 3, 4, 5, 6].

Motivated by these observations, in this paper, we present several sufficient conditions for the oscillatory behavior of solutions of equation (1.1). The results obtained in this paper extend that of in [1, 8, 12, 13, 14] for the equation without maxima.

In Section 2, we present preliminary lemmas which will be used to prove that main results. Section 3 deals with oscillation results and in Section 4, we provide some examples to illustrate the main results.

Preliminary Lemmas

In this section we present some useful lemmas, which will be used to prove the main results.

Lemma 2.1 Let $a,b\in[0,\infty)$. Then

$$a^{\alpha}+b^{\alpha}\geq\frac{1}{2^{\alpha-1}}(a+b)^{\alpha},\ \alpha\geq 1, \tag{2.1}$$

and

$$a^{\alpha}+b^{\alpha}\geq(a+b)^{\alpha},0<\alpha\leq 1. \tag{2.2}$$

Proof. The proof can be found in [13].

Lemma 2.2 Let f and $g\in C([t_0,\infty),\mathrm{R})$ and $\delta\in C([t_0,\infty),\mathrm{R})$ satisfies $\lim_{t\to\infty}\delta(t)=\infty$ and $\delta(t)\leq t$ for all $t\in[t_0,\infty)$; further suppose that there exists $h\in C([t_{-1},\infty),\mathrm{R}^{+})$ where $t_{-1}=\min_{t\in[t_0,\infty)}\{\delta(t)\}$, such that $f(t)=g(t)h(\delta(t))$ holds for all $t\in[t_0,\infty)$. Suppose that $\lim_{t\to\infty}f(t)$ exists and $\lim_{t\to\infty}infg(t)>-1$. Then $\lim_{t\to\infty}suph(t)>0$ implies $\lim_{t\to\infty}f(t)>0$.

Proof. The proof can be found in [6].

Lemma 2.3 Let $x(t)$ be a positive solution of equation (1.1), then the corresponding function $z(t)$ satisfies one of the following two cases:

(I) $z(t)>0,z'(t)>0,z''(t)>0,\left(a(t)(z''(t))^{\alpha}\right)<0$;

(II) $z(t)>0,z'(t)<0,z''(t)>0,\left(a(t)(z''(t))^{\alpha}\right)<0$,

for all $t\geq T$, where T is sufficiently large.

Proof. The proof can be found in [2], and hence omitted.

Lemma 2.4 Let $x(t)$ be a negative solution of equation (1.1), then the corresponding function $z(t)$ satisfies one of the following two cases:

(I) $z(t)<0,z'(t)<0,z''(t)<0,\left(a(t)(z''(t))^{\alpha}\right)>0$;

(II) $z(t)<0,z'(t)>0,z''(t)<0,\left(a(t)(z''(t))^{\alpha}\right)>0$,

for all $t\geq T$, where T is sufficiently large.

The proof of Lemma 2.4 is analogous to that of Lemma 2.3.

Lemma 2.5 The function $x(t)$ is a negative solution of equation (1.1) if and only if $-x(t)$ is a positive solution of the equation

$$\left(a(t)(z''(t))^{\alpha}\right)+q(t)\min_{[\sigma(t),t]}x^{\alpha}(s)=0. \tag{2.3}$$

Proof. The assertion can be verified easily and hence omitted.

Lemma 2.6 Let $x(t)$ be positive function defined for all $t\geq t_0$. Then

$$\max_{[\sigma(t),t]}x^{\alpha}(s)+\max_{[\sigma(\tau(t)),t]}x^{\alpha}(s)\geq\max_{[\sigma(t),t]}\left(x^{\alpha}(s)+x^{\alpha}(\tau(s))\right). \tag{2.4}$$

Proof. The proof can be found in [11].

Lemma 2.7 Let $x(t)$ be a positive solution of equation (1.1), and let the corresponding function $z(t)$ satisfy Case (II) of Lemma 2.3. If

$$\int_{t_0}^{\infty}\int_{s}^{\infty}\left(\frac{1}{a(\tau(u))}\int_{u}^{\infty}Q(v)dv\right)^{\frac{1}{\alpha}}dsdt=\infty \tag{2.5}$$

where

$$Q(t)=\min\{q(t),q(\tau(t))\} \tag{2.6}$$

then $\lim_{t\to\infty}x(t)=\lim_{t\to\infty}z(t)=0$.

Proof. Let $x(t)$ be a positive solution of equation (1.1). We may only prove the case when $\alpha\geq 1$, since the case when $0<\alpha\leq 1$ is similar. From equation (1.1) and Case(II) of Lemma 2.3, we obtain

$$\left(a(t)(z''(t))^{\alpha}\right)+p_0^{\alpha}\frac{[\sigma(\tau(t))(z''(\tau(t)))^{\alpha}]'}{\tau_0}+q(t)\max_{[\sigma(t),t]}x^{\alpha}(s)+p_0^{\alpha}q(\tau(t))\max_{[\sigma(\tau(t)),\tau(t)]}x^{\alpha}(s)\leq 0.$$

which follows from (2.1) and (2.6), $\sigma\circ\tau=\tau\circ\sigma$ and Lemma 2.6 that

$$\left(a(t)(z''(t))^{\alpha}\right)+\frac{p_0^{\alpha}}{\tau_0}\left(a(\tau(t))(z''(\tau(t)))^{\alpha}\right)+\frac{Q(t)}{2^{\alpha-1}}\max_{[\sigma(t),t]}z^{\alpha}(s)\leq 0. \tag{2.7}$$

Since $z(t)$ is positive and decreasing, we obtain

$$\left(a(t)(z''(t))^{\alpha}\right)+\frac{p_0^{\alpha}}{\tau_0}\left(a(\tau(t))(z''(\tau(t)))^{\alpha}\right)+\frac{Q(t)}{2^{\alpha-1}}z^{\alpha}(\sigma(t))\leq 0.$$

Integrating the last inequality from t to ∞, we obtain

$$a(t)(z''(t))^{\alpha}+\frac{p_0^{\alpha}}{\tau_0}a(\tau(t))(z''(\tau(t)))^{\alpha}\geq\frac{1}{2^{\alpha-1}}\int_{t}^{\infty}Q(s)z^{\alpha}(\sigma(s))ds.$$

Since $\tau(t)\leq t$, and $a(t)(z''(t))^{\alpha}$ is decreasing, we see that

$$a(t)(z''(t))^{\alpha}\leq a(\tau(t))(z''(\tau(t)))^{\alpha}.$$

Thus

$$a(\tau(t))(z''(\tau(t)))^{\alpha}\geq\frac{1}{2^{\alpha-1}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)}\int_{t}^{\infty}Q(s)z^{\alpha}(\sigma(s))ds.$$

Inview of Case(II) of Lemma 2.3, we have $\lim_{t\to\infty}z(t)=L\geq 0$. Assume $L>0$, then $z^{\alpha}(\sigma(t))\geq L^{\alpha}$. Then, we have

$$z''(\tau(t))\geq L\left(\frac{1}{2^{\alpha-1}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)}\right)^{\frac{1}{\alpha}}\left(\frac{1}{a(\tau(t))}\int_{t}^{\infty}Q(s)ds\right)^{\frac{1}{\alpha}}.$$

Integrating the last inequality from t to ∞, and then integrating the resulting inequality from t_1 to ∞, we obtain

$$\frac{1}{\tau_0^2} z(\tau(t_1)) \geq L\left(\frac{1}{2^{\alpha-1}\left(1+\frac{p_0^\alpha}{\tau_0}\right)}\right)^{\frac{1}{\alpha}} \int_{t_1}^{\infty}\int_{s}^{\infty}\left(\frac{1}{a(\tau(u))}\int_u^{\infty} Q(v)dv\right)^{\frac{1}{\alpha}} duds$$

which contradicts (2.5). Thus $\lim_{t\to\infty} z(t) = 0$. Since $\lim_{t\to\infty} x(t) \leq \lim_{t\to\infty} z(t)$ implies $\lim_{t\to\infty} x(t) = 0$. This completes the proof.

Lemma 2.8 Assume that $z(t)$ satisfies Case (I) of Lemma 2.3 for $t \geq t_1$. Then

$$z'(t) \geq \left(a^{1/\alpha} z''(t)\right)\beta_1(t,t_1), \tag{2.8}$$

and

$$z(t) \geq \left(a^{1/\alpha}(t) z''(t)\right)\beta_2(t,t_1) \tag{2.9}$$

where $\beta_1(t,t_1) = \int_{t_1}^{t} \frac{1}{a^{1/\alpha}(s)} ds$ and $\beta_2(t,t_1) = \int_{t_1}^{t}\int_{t_1}^{s} \frac{1}{a^{1/\alpha}(u)} duds$.

Proof. The proof can be found in [13].

Oscillation Results

In this section, we present sufficient conditions for the oscillatory behavior of solutions of equation (1.1). Throughout this section, without loss of generality we can deal only with the positive solution of equation (1.1) since the proof for the negative case is similar.

Theorem 3.1 Let $\alpha \geq 1$. If condition (2.5) holds, and assume that there exists a positive nondecreasing differentiable function $\rho(t)$ for $t \geq t_2 \geq t_1$ such that

$$\lim_{t\to\infty} sup \int_{t_2}^{t}\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\rho'(s))^{\alpha+1}}{(\rho(s))^{\alpha}}\left(\frac{1}{(\beta_1(s,t_1))^{\alpha}} + \frac{p_0^\alpha}{\tau_0}\frac{1}{(\beta_1(\tau(s),t_1)\tau'(s))^{\alpha}}\right)\right] ds = \infty, \tag{3.1}$$

then every solution of equation (1.1) is either oscillatory or tends to zero as $t \to \infty$.

Proof. Let $x(t)$ be a positive solution of equation (1.1). Without loss of generality assume that $x(t) > 0, x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1 \geq t_0$. Then the corresponding function $z(t)$ satisfies Case (I) and Case (II) of Lemma 2.3 for all $t \geq t_1$.

Case(I). From the proof of Lemma 2.7, we have (2.7). Since $z(t)$ is positive and increasing, we have from (2.7) that

$$\left(a(t)(z''(t))^{\alpha}\right) + \frac{p_0^\alpha}{\tau_0}\left(a(\tau(t))(z''(\tau(t)))^{\alpha}\right) + Q(t)z^{\alpha}(t) \leq 0. \tag{3.2}$$

Define

$$w(t) = \rho(t)\frac{a(t)(z''(t))^{\alpha}}{z^{\alpha}(t)}, \ t \geq t_1. \tag{3.3}$$

Then $w(t) > 0,$ and

$$w'(t) = \rho'(t)\frac{a(t)(z''(t))^{\alpha}}{z^{\alpha}(t)} + \rho(t)\frac{(a(t)(z''(t))^{\alpha})'}{z^{\alpha}(t)} - \frac{\alpha\rho(t)a(t)(z''(t))^{\alpha}}{z^{\alpha+1}(t)} z'(t)$$

$$\leq \frac{\rho'(t)}{\rho(t)} w(t) + \rho(t) \frac{(a(t)(z''(t))^{\alpha})'}{z^{\alpha}(t)} - \frac{\alpha\beta_1(t,t_1)}{\rho^{1/\alpha}(t)} w^{1+\frac{1}{\alpha}}(t). \tag{3.4}$$

Similarly, define another function v by

$$v(t) = \rho(t) \frac{a(\tau(t))(z''(\tau(t)))^{\alpha}}{z^{\alpha}(\tau(t))}, \ t \geq t_1. \tag{3.5}$$

Then $v(t) > 0$, and

$$v'(t) = \rho'(t) \frac{a(\tau(t))(z''(\tau(t)))^{\alpha}}{z^{\alpha}(\tau(t))} + \rho(t) \frac{(a(\tau(t))(z''(\tau(t)))^{\alpha})'}{z^{\alpha}(\tau(t))} - \frac{\alpha\rho(t)a(\tau(t))(z''(\tau(t)))^{\alpha}}{z^{\alpha+1}(\tau(t))} z'(\tau(t))\tau'(t)$$

$$\leq \frac{\rho'(t)}{\rho(t)} v(t) - \frac{\alpha\beta_1(\tau(t),t_1)}{\rho^{1/\alpha}(t)} \tau'(t) v^{1+\frac{1}{\alpha}}(t) + \rho(t) \frac{(a(\tau(t))(z''(\tau(t)))^{\alpha})'}{z^{\alpha}(\tau(t))}. \tag{3.6}$$

From (3.4) and (3.6), we obtain

$$w'(t) + \frac{p_0^{\alpha}}{\tau_0} v'(t) \leq \frac{\rho(t)\left[a(t)(z''(t))^{\alpha}\right] + \frac{p_0^{\alpha}}{\tau_0}\left[a(\tau(t))(z''(\tau(t)))^{\alpha}\right]}{z^{\alpha}(t)} + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\alpha\beta_1(t,t_1)}{\rho^{1/\alpha}(t)} w^{1+\frac{1}{\alpha}}(t)$$

$$+ \frac{p_0^{\alpha}}{\tau_0}\left[\frac{\rho'(t)}{\rho(t)} v(t) - \frac{\alpha\beta_1(\tau(t),t_1)}{\rho^{1/\alpha}(t)} \tau'(t) v^{1+\frac{1}{\alpha}}(t)\right] \tag{3.7}$$

where we have used $(a(t)(z''(t))^{\alpha})'$ is negative and $z(t)$ is increasing. Since $z(t)$ is increasing from (2.7) and (??), we obtain

$$w'(t) + \frac{p_0^{\alpha}}{\tau_0} v'(t) \leq -\frac{\rho(t)Q(t)}{2^{\alpha-1}} + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\alpha\beta_1(t,t_1)}{\rho^{1/\alpha}(t)} w^{1+\frac{1}{\alpha}}(t)$$

$$+ \frac{p_0^{\alpha}}{\tau_0}\left[\frac{\rho'(t)}{\rho(t)} v(t) - \frac{\alpha\beta_1(\tau(t),t_1)}{\rho^{1/\alpha}(t)} \tau'(t) v^{1+\frac{1}{\alpha}}(t)\right]. \tag{3.8}$$

Now using (3.8) and the inequality

$$Bu - Au^{1+1/\alpha} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^{\alpha}}, \ A > 0, \tag{3.9}$$

we have

$$w'(t) + \frac{p_0^{\alpha}}{\tau_0} v'(t) \leq -\frac{\rho(t)Q(t)}{2^{\alpha-1}} + \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\rho'(t))^{\alpha+1}}{(\rho(t)\beta_1(t,t_1))^{\alpha}} + \frac{p_0^{\alpha}}{\tau_0(\alpha+1)^{\alpha+1}} \frac{(\rho'(t))^{\alpha+1}}{(\rho(t)\beta_1(\tau(t),t_1)\tau'(t))^{\alpha}}. \tag{3.10}$$

Integrating (3.10) from $t_2 (t_2 \geq t_1)$ to t, we get

$$\int_{t_2}^{t}\left[\rho(s)\frac{Q(s)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\rho'(s))^{\alpha+1}}{(\rho(s))^{\alpha}}\left(\frac{1}{(\beta_1(s,t_1))^{\alpha}} + \frac{p_0^{\alpha}}{\tau_0} \frac{1}{(\beta_1(\tau(s),t_1)\tau'(s))^{\alpha}}\right)\right] ds \leq w(t_2) + \frac{p_0^{\alpha}}{\tau_0} v(t_2),$$

which contradicts (3.1).

Case(II). From condition (2.5) and the proof of Lemma 2.7, we see that $\lim_{t\to\infty} x(t) = 0$. This completes the proof of the theorem.

By Lemma 2.1, similar to the proof of Theorem 3.1, we have the following theorem.

Theorem 3.2 Let $0<\alpha\le 1$. If condition (2.5) holds, and assume that there exists a positive nondecreasing differentiable function $\rho(t)$ for $t\ge t_2\ge t_1$ such that

$$\lim_{t\to\infty} sup\int_{t_2}^{t}\left[\rho(s)Q(s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\rho'(s))^{\alpha+1}}{(\rho(s))^{\alpha}}\left(\frac{1}{(\beta_1(s,t_1))^{\alpha}}+\frac{p_0^{\alpha}}{\tau_0}\frac{1}{(\beta_1(\tau(s),t_1)\tau'(s))^{\alpha}}\right)\right]ds=\infty, \quad (3.11)$$

then every solution of equation (1.1) is either oscillatory or tends to zero as $t\to\infty$.

Theorem 3.3 Let $\alpha\ge 1$. If condition (2.5) holds, and assume that there exists a positive nondecreasing differentiable function $\rho(t)$ for $t\ge t_2\ge t_1$ such that

$$\lim_{t\to\infty} sup\int_{t_2}^{t}\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}}-\frac{(\rho'(s))^2}{4\alpha\rho(s)}\left(\frac{(\beta_2(s,t_1))^{1-\alpha}}{\beta_1(s,t_1)}+\frac{p_0^{\alpha}}{\tau_0}\frac{(\beta_2(\tau(s),t_1))^{1-\alpha}}{\beta_1(\tau(s),t_1)\tau'(s)}\right)\right]ds=\infty, \quad (3.12)$$

then every solution of equation (1.1) is either oscillatory or tends to zero as $t\to\infty$.

Proof. Let $x(t)$ be a positive solution of equation (1.1). Without loss of generality assume that $x(t)>0, x(\tau(t))>0$, and $x(\sigma(t))>0$ for all $t\ge t_1\ge t_0$. Then the corresponding function $z(t)$ satisfies Case (I) and Case (II) of Lemma 2.3 for all $t\ge t_1$.

Case(I). From the proof of Lemma 2.7, we have (2.7). From Lemma 2.8, we get (2.8) and (2.9).

Define the function w and v by (3.3) and (3.5), respectively. Proceeding as in the proof of Theorem 3.1, we have (3.4) and (3.6). It follows from (2.8) and (3.3), we get

$$w^{1+\frac{1}{\alpha}}(t)\le\frac{\beta_2^{\alpha-1}(t,t_1)}{\rho^{1-\frac{1}{\alpha}}(t)}w^2,\ t\ge t_1. \quad (3.13)$$

Using (3.13) in (3.4), we obtain

$$w'(t)\le\frac{\rho(t)(a(t)(z''(t))^{\alpha})'}{z^{\alpha}(t)}+\frac{\rho'(t)}{\rho(t)}w(t)-\frac{\alpha(\beta_2(t,t_n))^{\alpha-1}}{\rho(t)}\beta_1(t,t_1)w^2(t). \quad (3.14)$$

Similarly from (2.9) and (3.5), we get

$$v^{1+\frac{1}{\alpha}}(t)\le\frac{(\beta_2(\tau(t),t))^{\alpha-1}}{\rho^{1-\frac{1}{\alpha}}(t)}v^2(t),\ t\ge t_1. \quad (3.15)$$

Using (3.15) in (3.6), we have

$$v'(t)\le\frac{\rho(t)(a(\tau(t))(z''(\tau(t)))^{\alpha})'}{z^{\alpha}(\tau(t))}+\frac{\rho'(t)}{\rho(t)}v(t)-\frac{\alpha(\beta_2(\tau(t),t_1))^{\alpha-1}\beta_1(\tau(t),t_1)\tau'(t)}{\rho(t)}v^2(t). \quad (3.16)$$

From (3.14) and (3.16), we obtain

$$\begin{aligned}w'(t)+\frac{p_0^{\alpha}}{\tau_0}v'(t)\le\rho(t)&\frac{\left[a(t)(z''(t))^{\alpha}\right]+\frac{p_0^{\alpha}}{\tau_0}\left[a(\tau(t))(z''(\tau(t)))^{\alpha}\right]}{z^{\alpha}(t)}\\&+\frac{\rho'(t)}{\rho(t)}w(t)-\frac{\alpha(\beta_2(t,t_1))^{\alpha-1}\beta_1(t,t_1)}{\rho(t)}w^2(t)\\&+\frac{p_0^{\alpha}}{\tau_0}\left[\frac{\rho'(t)}{\rho(t)}v(t)-\frac{\alpha(\beta_2(\tau(t),t_1))^{\alpha-1}\beta_1(\tau(t),t_1)}{\rho(t)}\tau'(t)v^2(t)\right]\end{aligned} \quad (3.17)$$

where we have used $\tau(t) \le t, z(t)$ is increasing and $(a(t)(z''(t))^{\alpha})'$ is negative. Since $z(t)$ is positive and increasing, we have from (2.7) and (3.17), we obtain

$$w'(t) + \frac{p_0^{\alpha}}{\tau_0} v'(t) \le -\frac{\rho(t)Q(t)}{2^{\alpha-1}} + \frac{(\rho'(t))^2}{\alpha\rho(t)} \left[\frac{(\beta_2(t,t_1))^{1-\alpha}}{\beta_1(t,t_1)} + \frac{p_0^{\alpha}}{\tau_0} \frac{(\beta_2(\tau(t),t_1))^{1-\alpha}}{\beta_1(\tau(t),t_1)\tau'(t)} \right]$$

where we have used the inequality $Bu - Au^2 \le \frac{B^2}{4A}$, $A > 0$. Integrating the above inequality from $t_2 (t_2 \ge t_1)$ to t, we obtain

$$\int_{t_2}^{t} \left[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{(\rho'(s))^2}{4\alpha\rho(s)} \left(\frac{(\beta_2(s,t_1))^{1-\alpha}}{\beta_1(s,t_1)} + \frac{p_0^{\alpha}}{\tau_0} \frac{(\beta_2(\tau(s),t_1))^{1-\alpha}}{\beta_1(\tau(s),t_1)\tau'(s)} \right) \right] ds \le w(t_2) + \frac{p_0^{\alpha}}{\tau_0} v(t_2),$$

which contradicts (3.12).

Case(II). The proof is similar to that of Case (II) of Theorem 3.1. The proof is now complete.

From (2.2), similar to that of proof of Theorem 3.3, we obtain the following results.

Theorem 3.4 Let $0 < \alpha \le 1$. If condition (2.5) holds, and assume that there exists a positive, nondecreasing differentiable function $\rho(t)$ for $t \ge t_2 \ge t_1$ such that

$$\lim_{t\to\infty} sup \int_{t_2}^{t} \left[\rho(s)Q(s) - \frac{(\rho'(s))^2}{4\alpha\rho(s)} \left(\frac{(\beta_2(s,t_1))^{1-\alpha}}{\beta_1(s,t_1)} + \frac{p_0^{\alpha}}{\tau_0} \frac{(\beta_2(\tau(s),t_1))^{1-\alpha}}{\beta_1(\tau(s),t_1)\tau'(s)} \right) \right] ds = \infty \tag{3.18}$$

then every solution of equation (1.1) is either oscillatory or tends to zero as $t \to \infty$.

Remark 3.1 From Theorem 3.1 to 3.4, we can get several oscillation criteria for equation (1.1) with different choices of the function ρ.

Next, we establish some Philos type oscillation results for equation (1.1).

Definition 5.3.1[10]

Consider the sets $\mathrm{D}_0 = \{(t,s) : t > s \ge t_0\}$ and $\mathrm{D} = \{(t,s) : t \ge s \ge t_0\}$. Assume that $H \in C(\mathrm{D},\mathrm{R})$ satisfies the following conditions:

(A1) $H(t,t) = 0, t \ge t_0; H(t,s) > 0, (t,s) \in \mathrm{D}_0;$

(A2) H has a non-positive continuous partial derivative with respect to the second variable in D_0.

Then the function H has the property P.

Theorem 3.5 Let $\alpha \ge 1$. Assume condition (2.5) holds. Further assume that $H \in C(\mathrm{D},\mathrm{R})$ has the property P and there exists a function $\rho \in C^1([t_0,\infty),(0,\infty))$ for all sufficiently large $t_1 \ge t_0$, there is a $t_2 \ge t_1$ such that

$$-\frac{\partial}{\partial s} H(t,s) - \frac{\rho'(s)}{\rho(s)} H(t,s) = \frac{h(t,s)(H(t,s))^{\frac{\alpha}{\alpha+1}}}{\rho(s)}, (t,s) \in \mathrm{D}_0 \tag{3.19}$$

and

$$\lim_{t\to\infty} sup \frac{1}{H(t,t_2)} \int_{t_2}^{t} G_1(t,s) ds = \infty \tag{3.20}$$

where

$$G_1(t,s) = H(t,s) \frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{(h_-(t,s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \left(\frac{1}{(\beta_1(s,t_1))^{\alpha}} + \frac{p_0^{\alpha}}{\tau_0} \frac{1}{(\beta_1(\tau(s),t_1)\tau'(s))^{\alpha}} \right),$$

$h_{-}(t,s)=\max\{0,-h(t,s)\}$. Then every solution of equation (1.1) is either oscillatory or tends to zero as $t\to\infty$.

Proof. Let $x(t)$ be a positive solution of equation (1.1). Then the corresponding function $z(t)$ satisfies Case(I) and Case(II) of Lemma 2.3 for $t\geq t_1$.

Case(I). Define w and v as in the proof of Theorem 3.1. Then we obtain (3.8). Replace t by s and then multiply both sides of (??) by $H(t,s)$, integrate with respect to s from $t_2(t_2\geq t_1)$ to t, we have

$$\int_{t_2}^{t}H(t,s)\frac{\rho(s)Q(s)}{2^{\alpha-1}}ds\leq-\int_{t_2}^{t}H(t,s)w'(s)ds+\int_{t_2}^{t}H(t,s)\frac{\rho'(s)}{\rho(s)}w'(s)ds-\int_{t_2}^{t}H(t,s)\frac{\alpha\beta_1(s,t_1)}{\rho^{1/\alpha}(s)}w^{1+\frac{1}{\alpha}}(s)ds$$

$$-\frac{p_0^{\alpha}}{\tau_0}\int_{t_2}^{t}H(t,s)v'(s)ds+\frac{p_0^{\alpha}}{\tau_0}\int_{t_2}^{t}H(t,s)\frac{\rho'(s)}{\rho(s)}v'(s)ds$$

$$-\frac{p_0^{\alpha}}{\tau_0}\int_{t_2}^{t}H(t,s)\frac{\alpha\beta_1(\tau(s),t_1)\tau'(s)}{\rho^{1/\alpha}(s)}v^{1+\frac{1}{\alpha}}(s)ds.$$

Then, we obtain

$$\int_{t_2}^{t}H(t,s)\frac{\rho(s)Q(s)}{2^{\alpha-1}}ds\leq H(t,t_2)w(t_2)-\int_{t_2}^{t}\left[-\frac{\partial}{\partial s}H(t,s)-\frac{\rho'(s)}{\rho(s)}H(t,s)\right]w(s)ds$$

$$-\int_{t_2}^{t}H(t,s)\frac{\alpha\beta_1(s,t_1)}{\rho^{1/\alpha}(s)}w^{1+\frac{1}{\alpha}}(s)ds+\frac{p_0^{\alpha}}{\tau_0}H(t,t_2)v'(t_2)$$

$$-\frac{p_0^{\alpha}}{\tau_0}\int_{t_2}^{t}\left[-\frac{\partial}{\partial s}H(t,s)-\frac{\rho'(s)}{\rho(s)}H(t,s)\right]v(s)ds$$

$$-\frac{p_0^{\alpha}}{\tau_0}\int_{t_2}^{t}H(t,s)\frac{\alpha\beta_1(\tau(s),t_1)\tau'(s)}{\rho^{1/\alpha}(s)}v^{1+\frac{1}{\alpha}}(s)ds.$$

Then

$$\int_{t_2}^{t}H(t,s)\frac{\rho(s)Q(s)}{2^{\alpha-1}}ds\leq H(t,t_2)w(t_2)+\frac{p_0^{\alpha}}{\tau_0}H(t,t_2)v'(t_2)$$

$$+\int_{t_2}^{t}\left[\frac{h_{-}(t,s)(H(t,s))^{\frac{\alpha}{\alpha+1}}}{\rho(s)}w(s)-H(t,s)\frac{\alpha\beta_1(s,t_1)}{\rho^{1/\alpha}(s)}w^{1+\frac{1}{\alpha}}(s)\right]ds$$

$$+\frac{p_0^{\alpha}}{\tau_0}\int_{t_2}^{t}\left[\frac{h_{-}(t,s)(H(t,s))^{\frac{\alpha}{\alpha+1}}}{\rho(s)}v(s)-H(t,s)\frac{\alpha\beta_1(\tau(s),t_1)\tau'(s)}{\rho^{1/\alpha}(s)}v^{1+\frac{1}{\alpha}}(s)\right]ds$$

using the inequality (3.9) in the last inequality, we obtain

$$\frac{1}{H(t,t_1)}\int_{t_2}^{t}\left[H(t,s)\frac{\rho(s)Q(s)}{2^{\alpha-1}}-\frac{(h_{-}(t,s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(s)}\left(\frac{1}{(\beta_1(s,t_1))^{\alpha}}+\frac{1}{(\beta_1(\tau(s),t_1)\tau'(s))^{\alpha}}\right)\right]ds\leq w(t_2)+\frac{p_0^{\alpha}}{\tau_0}v(t_2),$$

which contradicts (3.20).

Case (II). The proof is similar to that of Theorem 3.1. The proof is now complete.

From Theorem 3.2, similar to that proof of Theorem 3.5, we derive the following theorem.

Theorem 3.6 Let $0<\alpha\le 1$. Assume that condition (2.5) holds. Moreover assume that $H\in C(\mathrm{D},\mathrm{R})$ has the property P, and there exists a function $\rho\in C^1([t_0,\infty),(0,\infty))$ for all sufficiently large $t_1\ge t_0$, there is a $t_2\ge t_1$ such that (3.19) holds, and

$$\lim_{t\to\infty} sup\frac{1}{H(t,t_2)}\int_{t_2}^{t}F_1(t,s)ds=\infty, \tag{3.21}$$

where

$$F_1(t,s)=H(t,s)\rho(s)Q(s)-\frac{(h\_(t,s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(s)}\left(\frac{1}{(\beta_1(s,t_1))^{\alpha}}+\frac{1}{(\beta_1(\tau(s),t_1)\tau'(s))^{\alpha}}\right),$$

$h\_(t,s)=\max\{0,-h(t,s)\}$. Then every solution of equation (1.1) is either oscillatory or tends to zero as $t\to\infty$.

From (3.17) in Theorem 3.3, similar to that proof of Theorem 3.5, we obtain the following results.

Theorem 3.7 Let $\alpha\ge 1$. Assume that condition (2.5) holds. Further, assume that $H\in C(\mathrm{D},\mathrm{R})$ has the property P, and there exists a function $\rho\in C^1([t_0,\infty),(0,\infty))$, for all sufficiently large $t_1\ge t_0$, there is a $t_2>t_1$ such that

$$-\frac{\partial}{\partial s}H(t,s)-\frac{\rho'(s)}{\rho(s)}H(t,s)=\frac{h(t,s)(H(t,s))^{1/2}}{\rho(s)},\ (t,s)\in\mathrm{D}_0, \tag{3.22}$$

and

$$\lim_{t\to\infty} sup\frac{1}{H(t,t_2)}\int_{t_2}^{t}G_2(t,s)ds=\infty, \tag{3.23}$$

where

$$G_2(t,s)=H(t,s)\frac{\rho(s)Q(s)}{2^{\alpha-1}}-\frac{(h\_(t,s))^2}{4\alpha\rho(s)}\left(\frac{(\beta_2(s,t_1))^{1-\alpha}}{\beta_1(s,t_1)}+\frac{p_0^{\alpha}}{\tau_0}\frac{(\beta_2(\tau(s),t_1))^{1-\alpha}}{\beta_1(\tau(s),t_1)\tau'(s)}\right),$$

$h\_(t,s)=\max\{0,-h(t,s)\}$. Then every solution of equation (1.1) is either oscillatory or tends to zero as $t\to\infty$.

By Theorem 3.5, similar to the proof of Theorem 3.5, we establish the following criteria.

Theorem 3.8 Let $0<\alpha\le 1$. Assume that condition (2.5) holds. Further, assume that $H\in C(\mathrm{D},\mathrm{R})$ has the property P, and there exists a function $\rho\in C^1([t_0,\infty),(0,\infty))$, for all sufficiently large $t_1\ge t_0$, there is a $t_2>t_1$ such that (3.22) holds and

$$\lim_{t\to\infty} sup\int_{t_1}^{t}F_2(t,s)ds=\infty, \tag{3.24}$$

where

$$F_2(t,s)=H(t,s)\rho(s)Q(s)-\frac{(h\_(t,s))^2}{4\alpha\rho(s)}\left(\frac{(\beta_2(s,t_1))^{1-\alpha}}{\beta_1(s,t_1)}+\frac{p_0^{\alpha}}{\tau_0}\frac{(\beta_2(\tau(s),t_1))^{1-\alpha}}{\beta_1(\tau(s),t_1)\tau'(s)}\right),$$

$h\_(t,s)=\max\{0,-h(t,s)\}$. Then every solution of equation (1.1) is either oscillatory or tends to zero as $t\to\infty$.

Remark 3.2 From Theorem 3.5-3.8, we can obtain several oscillation criteria for equation (1.1) with different choices of function ρ and H.

Examples

In this section, we will present three examples to illustrate the main results.

Example 4.1 Consider the third order half-linear neutral differential equation with "maxima"

$$\left[t\left((x(t)+p_0x(t/2))''\right)^3\right]+\frac{\lambda}{t^6}\max_{[t/2,t]}x^3(s)=0,\ t\geq 1, \tag{4.1}$$

where $\lambda>0$. Here $a(t)=t,\ p(t)=p_0>0,\tau(t)=\sigma(t)=t/2,\alpha=3$, $q(t)=\frac{\lambda}{t^6}$, and $\tau_0=\frac{1}{2}$. Then $Q(t)=\frac{\lambda}{t^6}$, and

$$\beta_1(t,t_1)=\int_{t_1}^{t}\frac{1}{a^{1/\alpha}(s)}ds=\int_{t_1}^{t}\frac{1}{s^{1/3}}ds\geq t^{2/3},$$

for t sufficiently large. It is easy to see that condition (2.5) holds. Set $\rho(t)=t^5$. We obtain

$$\lim_{t\to\infty}sup\int_{t_2}^{t}\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}}-\frac{(\rho'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\rho(s))^{\alpha}}\left(\frac{1}{(\beta_1(s,t_1))^{\alpha}}+\frac{\frac{p_0^{\alpha}}{\tau_0}}{(\beta_1(\tau(s),t_1)\tau'(s))^{\alpha}}\right)\right]ds$$

$$\geq\left[\frac{\lambda}{4}-\frac{5^4(1+64p_0^3)}{4^4}\right]\lim_{t\to\infty}sup\int_{t_2}^{t}\frac{1}{s}ds=\infty$$

provided $\lambda>\frac{5^4(1+64p_0^3)}{4^3}$. Hence by Theorem 3.1, every solution of equation (4.1) is either oscillatory or tends to zero as $t\to\infty$ when $\lambda>\frac{5^4(1+64p_2^3)}{4^3}$.

Example 4.2 Consider the third order neutral differential equation with "maxima"

$$\left(t\left((x(t)+2x(t/2))''\right)^3\right)+\frac{\lambda}{t^6}\max_{[t/2,t]}x^3(s)=0,\ t\geq 1, \tag{4.2}$$

where $\lambda>0$. Here $a(t)=t,\ p(t)=p_0>0,\tau(t)=\sigma(t)=t/2,\alpha=3$, $q(t)=\frac{\lambda}{t^6}$, and $\tau_0=\frac{1}{2}$. Then $Q(t)=\frac{\lambda}{t^6}$, and $\beta_1(t,t_1)\geq t^{2/3}$ and $\beta_2(t,t_1)\geq\frac{2}{5}t^{5/3}$. Clearly condition (2.5) holds. Set $\rho(t)=t^5$. We obtain

$$\lim_{t\to\infty}sup\int_{t_2}^{t}\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}}-\frac{(\rho'(s))^2}{4\alpha\rho(s)}\left(\frac{(\beta_2(s,t_1))^{1-\alpha}}{\beta_1(s,t_1)}+\frac{p_0^{\alpha}}{\tau_0}\frac{(\beta_2(\tau(s),t_1))^{1-\alpha}}{\beta_1(s,t_1)\tau'(s)}\right)\right]ds\geq\left[\frac{\lambda}{4}-\frac{5^4(1+2^6p_0^3)}{48}\right]\lim_{t\to\infty}sup\int_{t_2}^{t}\frac{1}{s}ds=\infty,$$

provided $\lambda>\frac{5^4(1+2^6p_0^3)}{12}$. Hence by Theorem 3.3, every solution of equation (4.2) is either oscillatory or tends to zero as $t\to\infty$.

Example 4.3 Consider the third order half-linear neutral differential equation with "maxima"

$$\left(t^{1/9}\left((x(t)+p_0x(t/2))''\right)^{1/3}\right)+\frac{\lambda}{t^{14/9}}\max_{[t/2,t]}x^{1/3}(s)=0,\ t\geq 1, \tag{4.3}$$

where $\lambda > 0$. Here $a(t) = t^{1/9}, p(t) = p_0 > 0, \tau(t) = \sigma(t) = t/2, \alpha = 1/3$, $q(t) = \frac{\lambda}{t^{14/9}}$, and $\tau_0 = \frac{1}{2}$. Then $Q(t) = \frac{\lambda}{t^{14/9}}$, and $\beta_1(t,t_1) \geq t^{2/3}$ for t sufficiently large. It is easy to see that conditio (2.5) holds. Set $\rho(t) = t^{4/3}$. We obtain

$$\lim_{t\to\infty} sup\int_{t_2}^{t}\left[\rho(s)Q(s) - \frac{(\rho'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(s)}\left(\frac{1}{(\beta_1(s,t_1))^{\alpha}} + \frac{\frac{p_0^{\alpha}}{\tau_0}}{(\beta_1(\tau(s),t_1)\tau'(s))^{\alpha}}\right)\right]ds \geq \left[\lambda - (1+2^{5/3}p_0^{1/3})\right]\lim_{t\to\infty} sup\int_{t_2}^{t}\frac{1}{s^{2/9}}ds = \infty,$$

if $\lambda > (1+2^{5/3}p_0^{1/3})$. Hence by Theorem 3.2, every solution of equation (4.3) is either oscillatory or tends to zero as $t \to \infty$. We conclude this paper with the following remark.

Remark 4.1 In this paper, we have established some new oscillation theorems for the equation (1.1) for the case $0 \leq p(t) \leq p_0 < \infty$. It would be interesting to investigate equation (1.1) under the case when $p(t) < -1$, $\lim_{t\to\infty} p(t) = \infty$ or $p(t)$ is an oscillatory function. Further it is interesting to find different method to remove the condition $\sigma \circ \tau = \tau \circ \sigma$.

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