ISSN 2347-1921
Volume 12 Number 1
Journal of Advances in Mathematics

On Ray's theorem for weak firmly nonexpansive mappings in Hilbert Spaces

Tamer Nabil[1] and Ahmed H. Soliman[2]
[1]Suez Canal University, Faculty of Computers and Informatics, Department of Basic Science, Ismailia, Egypt
[1]t_3bdelsadek@yahoo.com
[2]Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt
[2]ahsolimanm@gmail.com

ABSTRACT

In this work, we introduce notions of generalized firmly nonexpansive (G-firmly nonexpansive) and fundamentally firmly nonexpansive (F-firmly nonexpansive) mappings and utilize to the same to prove Ray's theorem for G-firmly and F-firmly nonexpansive mappings in Hilbert Spaces. Our results extend the result due to F. Kohsaka [Ray's theorem revisited: a fixed point free firmly nonexpansive mapping in Hilbert spaces, Journal of Inequalities and Applications (2015) 2015:86].

Keywords. Ray's theorem; generalized firmly nonexpansive mapping; fundamentally firmly nonexpansive mapping; fixed point; Hilbert space.

Mathematics Subject Classification: 47H10, 54H10, 54H25.

1. INTRODUCTION and PRELIMINARIES

Let H be a real Hilbert space. The inner product and the induced norm on H are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. The dual space of a Banach space X is denoted X^*. Consider K is nonempty closed convex subset of H. A mapping $T: K \to K$ is said to be nonexpansive mapping if

$$\|Tx - Ty\| \le \|x - y\| \tag{1}$$

for all $x, y \in K$.

In 1965, Browder [1] showed that if K is bounded, then every nonexpansive mapping on K has a fixed point. In 1980, Ray [2] showed that the converse of Browder's theorem is true, i.e. every nonexpansive self mapping on K has a fixed point, then K is bounded. There are many versions of Ray's theorem for nonexpansive mapping. For examples, in 1987, Sine [3], proved Ray's theorem by applying a version of the uniform boundedness principle (see, for instance, [6]) and the convex combination of a sequence of a metric projections onto closed and convex sets. In 2010, Aoyama et al. [4], obtained a strong version of Ray's theorem for the class of λ −hybrid mappings in Hilbert spaces.

Recently, Kohsaka [5] given another proof of a strong version of Ray's theorem [4] ensuring that every unbounded closed convex subset of a Hilbert space admits a fixed point free firmly nonexpansive mapping. He used in his proof a version of uniform boundedness principle and single metric projection onto a closed and convex set.

In this paper, we define two new class of weaker firmly nonexpansive called G-firmly and F-firmly nonexpansive. We present new two versions of Ray's theorem for mappings satisfying the conditions of weaker firmly nonexpansive.

We begin with some notations and preliminaries.

Definition 1.1. [5] A mapping $T: K \to K$ is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \le \langle Tx - Ty, x - y \rangle \tag{2}$$

for all $x, y \in K$.

Definition 1.2. [7] A linear subspace M of a normed space X is called proximinal (resp. Chebyshev) if for each $x \in X$, the set of best approximations to x from M,

$$P_M := \{ y \in M : \|x - y\| = \inf_{m \in M} \|x - m\| \},$$

is nonempty (resp. a singleton). It well know that for each element of the Hilbert space there exist Chebyshev convex subset.

Definition 1.3. [5] The mapping $P_K : H \to K$ which is defined by $P_K x = z_x$ for $x \in H$ such that $\|P_K x - x\| \leq \|y - x\|$ for all $y \in K$ is called the metric projection of H onto K, therefore, $z = P_K x$ if and only if $\sup_{y\in K}\langle y - z, x - z\rangle \leq 0$ for all $(x, y) \in H \times K.$

Theorem 1.1. (A strong version of Rays theorem [4]) Let K be a nonempty closed convex subset of a Hilbert space H. If every firmly nonexpansive self-mapping on K has a fixed point, then K is bounded.

2. MAIN RESULTS

We now present our new conditions of weak nonexpansive.

Definition 2.1. A self mapping T on K is said to be G-firmly nonexpansive if

$$\frac{1}{3}\|x - Tx\|^2 \leq \langle Tx - Ty, x - y\rangle \Rightarrow \|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle,\ \forall\, x, y \in K. \tag{3}$$

Proposition 2.1. Every firmly nonexpansive is G-firmly nonexpansive.

Remark 2.1. The converse of proposition 2.1 is not true as we will see in the following example.

Example 2.1. Define a mapping T on [0, 4] such that $Tx = 0$ as $x \neq 4$ and $Tx = 0.5$ as $x = 4$. Then T is G-firmly nonexpansive but T is not firmly nonexpansive. Where the inner product $\langle x, y\rangle = x.y$ for all real numbers x and y.

Proof. It is clear that T is not continuous, therefore it is not firmly nonexpansive . If $x < y$ and $x \in [0,2] \cup \{4\}$ and $y \in [0,4)$, then $\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$ holds. If $x \in (2,4)$ and $y = 4,$ then

$$\frac{1}{3}\|x - Tx\|^2 = \frac{x^2}{3} > 1, \langle Tx - Ty, x - y\rangle < 1 \text{ and } \frac{1}{3}\|y - Ty\|^2 > 1.$$

Thus T is generalized firmly nonexpansive mapping.■

Definition 2.2. A self mapping T on K is said to be F-firmly nonexpansive if

$$\|T^2 x - Ty\|^2 \leq \langle T^2 x - Ty, Tx - y\rangle, \forall\, x, y \in K. \tag{4}$$

Proposition 2.2. Every firmly nonexpansive is F-firmly nonexpansive.

Remark 2.2. The converse of proposition 2.2 is not true as we will see in the following example.

Example 2.2. Define the mapping T on [0, 2] by

$$Tx = \begin{cases} 0 & \text{if} \quad x \neq 2, \\ 1 & \text{if} \quad x = 2. \end{cases}$$

And the inner product $\langle x, y\rangle = x.y$ for all real numbers x and $y.$

Then T is F-firmly nonexpansive but T is not firmly nonexpansive.

Proof. Let $x = 2$ and $y = 1.5$. Then $\|Tx - Ty\|^2 = 1$, but $\langle Tx - Ty, x - y\rangle = 0.5$. Thus T is not firmly nonexpansive mapping.

If $x, y \in [0,2)$, then $\|T^2 x - Ty\| = 0$ and $\langle T^2 x - Ty, Tx - y\rangle = 0.$ If $x = 2$ and $y \in [0,2),$ then we have that:

$$\|T^2 x - Ty\| = 1 \text{ and } \langle T^2 x - Ty, Tx - y\rangle = 1.$$

Last case, if $x \in [0,2)$ and $y = 2$, we get that:

$$\|T^2x - Ty\| = 1 \text{ and } \langle T^2x - Ty, Tx - y\rangle = 2.$$

Therefore T is F-firmly nonexpansive. ■

Lemma 2.1.

(1) the metric projection mapping P_K (as in definition 1.3) of a Hilbert space H onto a nonempty closed subset K of H is F-firmly nonexpansive,

(2) if K be a nonempty closed convex subset of H, $a \in H$, and $T : K \to K$ such that $Tx = P_K(x+a)$ for all $x \in K$. Then T is a F-firmly nonexpansive self -mapping on K,

(3) $u \in K$ is fixed point of T if and only if $\langle u, a\rangle = \sup_{y \in K} \langle y, a\rangle$.

Proof. (1) Let $x, y \in H$, thus we have that:

$\sup_{w\in K}\langle w - P_K^2 x, P_K x - P_K^2 x\rangle \le 0$ and $\sup_{k\in K}\langle k - P_K y, y - P_K y\rangle \le 0$ and hence

$$\begin{aligned}
\|P_K^2 x - P_K y\|^2 - \langle P_K^2 x - P_K y, P_K x - y\rangle &= \langle P_K^2 x - P_K y, P_K^2 x - P_K y\rangle - \langle P_K^2 x - P_K y, P_K x - y\rangle \\
&= \langle P_K^2 x - P_K y, P_K^2 x - P_K y - P_K x + y\rangle \\
&= \langle P_K^2 x - P_K y, y - P_K y\rangle + \langle P_K^2 x - P_K y, P_K^2 x - P_K x \rangle \\
&= \langle P_K^2 x - P_K y, y - P_K y \rangle + \langle P_K y - P_K^2 x, P_K x - P_K^2 x\rangle \\
&= \sup_{w\in K}\langle w - P_K y, y - P_K y \rangle + \sup_{k\in K}\langle k - P_K^2 x, P_K x - P_K^2 x\rangle \\
&\le 0.
\end{aligned}$$

Which implies that: $\|P_K^2 x - P_K y\|^2 \le \langle P_K^2 x - P_K y, P_K x - y\rangle.$ Thus P_K is F-firmly nonexpansive.

(2) $\|Tx - Ty\|^2 = \|P_K(x+a) - P_K(y+a)\|^2 \le \langle P_K(x+a) - P_K(y+a), x + a - y - a\rangle = \langle Tx - Ty, x - y\rangle$..

Put, $x = Tu$ and $v = y$, hence T is a F-firmly nonexpansive self-mapping on K.

(3) $u \in F(T) \Leftrightarrow P_K(u+a) = u \Leftrightarrow \sup_{y\in K}\langle y - u, u + a - u\rangle \le 0 \Leftrightarrow \langle u, a\rangle = \sup_{y\in K}\langle y, a\rangle$. ■

Lemma 2.2. The metric projection mapping of a Hilbert space H onto a nonempty closed subset K of H is G-firmly nonexpansive. Furthermore, if K be a nonempty closed convex subset of H, and $a \in H$, and $T : K \to K$ such that $Tx = P_K(x+a)$ for all $x \in K$. Then T is a G-firmly nonexpansive self-mapping on K such that : $u \in K$ is fixed point of T if and only if $\langle u, a\rangle = \sup_{y\in K}\langle y, a\rangle$.

Proof. Let $x, y \in K$, we have that:

$$\|Tx - Ty\|^2 = \|P_K(x+a) - P_K(y+a)\|^2 \le \langle P_K(x+a) - P_K(y+a), x + a - y - a\rangle = \langle Tx - Ty, x - y\rangle.$$

Hence T is a firmly self mapping on K. Then the same argument as in the proof of lemma 2.1 leads to $u \in F(T)$ if and only if $\langle u, a\rangle = \sup_{y\in K}\langle y, a\rangle$. ■

We are now ready to introduce our new versions of Ray's theorem for weak firmly nonexpansive self-mappings.

Theorem 2.1 . (F-firmly version of Ray's theorem) Let K be a nonempty closed convex of a Hilbert space H. If the following fixed point property (F) hold then K is bounded.

(F) If every F-firmly nonexpansive mapping $T : K \to K$ has a fixed point.

Proof. Suppose that K is unbounded. Thus there exist $x^* \in H$ such that $x^*(K)$ is unbounded (see, for instance, [6]). Then we have $a \in H$ such that : $\sup_{y \in K} \langle y, a \rangle = \infty$. Define $Tx = P_K(x+a)$ and by (3) in Lemma 2.1, then T is a fixed point free F-firmly nonexpansive self mapping on K. ■

Theorem 2.2. (G-firmly version of Ray's theorem) Let K be a nonempty closed convex of a Hilbert space H. If the following fixed point property (E) hold then K is bounded.

(E) If every G-firmly nonexpansive mapping $T : K \to K$ has a fixed point.

ACKNOWLEDGMENTS

The authors wish to thank the referees for the valuable remarks and commonts in the revision of this manuscript.

REFERENCES

[1] F. E. Browder, Non-expansive nonlinear operators in Banach spaces, Proc. Natl. Acad. Sci. USA 54 (1965) 1041-1044.

[2] W. O. Ray, The fixed point property and unbounded sets in Hilbert space, Trans. Am. Math. Soc. 258 (1980) 531- 537.

[3] R. Sine, On the converse of the nonexpansive map fixed point theorem for Hilbert space, Proc. Am. Math. Soc. 100 (1987) 489- 490.

[4] K. Aoyama, S. Lemoto, F. Kohsaka and W. Takahashi, Fixed point and ergodic theorems for λ-hybrid mapping in Hilber Spaces, J. Nonlinear Convex Anal. 11 (2010) 335-343.

[5] F. Kohsaka , Ray's theorem revisited: a fixed point free firmly nonexpansive mapping in Hilbert spaces, Journal of Inequalities and Applications (2015) 2015:86.

[6] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, Berlin (2011).

[7] F. Deutsch, Linear selections for the metric projection, Journal of Functional Analysis 49 (1982), 269-292.