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SOLVABILITY OF BOUNDARY VALUE PROBLEMS WITH TRANSMISSION CONDITIONS FOR DISCONTINUOUS ELLIPTIC DIFFERENTIAL OPERATOR EQUATIONS

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ABSTRACT.

We consider nonlocal boundary value problems which includes discontinuous coefficients elliptic differential operator equations of the second order and nonlocal boundary conditions together with boundary-transmission conditions. We prove coerciveness and Fredholmness for these nonlocal boundary value problems.

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1. Introduction

We investigated nonlocal boundary value problem with discontinuous operator coefficients for the second order elliptic differential-operator equation and boundary conditions which have transmission conditions in a Hilbert space H :

$$L(D)u := -u''(x) + Au(x) + \varphi(x)u(x) = g(x), \quad x \in [-1,0) \cup (0,1] \tag{1.1}$$

$$L_\upsilon u = \alpha_\upsilon u^{(m_k)}(-1) + \beta_\upsilon u^{(m_k)}(-0) + \eta_\upsilon u^{(m_k)}(+0) + \gamma_\upsilon u^{(m_k)}(1) = f_\upsilon, \tag{1.2}$$

$$\upsilon = 1,2,3,4, k = 1,2$$

where $m_k \in \{0,1\}$; α_υ, β_υ, η_υ, γ_υ, f_υ are complex numbers; $g(x) := g_1(x)$, $A := A_1$, $\varphi := \varphi_1$ for $x \in [-1,0)$ and $g(x) := g_2(x)$, $A := A_2$, $\varphi := \varphi_2$ for $x \in (0,1]$; A_k are unbounded operators in H and φ is a measurable function on $[-1,0) \cup (0,1]$. Problems of such type arise in heat and mass transfer in various physical transfer problems and in diffraction problems (see, for example, A. V. Likov [10]; A. A. Shkalikov [17] and references cited therein). Elliptic functional differential equations are closely associated with differential equations with nonlocal boundary conditions, which arises plasma theory, and boundary value problems with elliptic differential equations have some important applications such as to elasticity theory, control theory and diffusion processes (see, for example, A. L. Skubachevskii [16]). There are many papers that the spektral properties of such problem are investigated (see, [1], [2], [4], [14], [17]). Some boundary value problems with discontinuous coefficient and eigenvalue parameter in both the differential equation and boundary conditions have been studied by O. Sh. Mukhtarov, M. Kandemir and others (see, [5]-[8], [8], [11]-[13]). In this study, we investigated coerciveness and Fredholmness of nonlocal boundary value problem with discontinuous operator coefficients and transmission conditions at point zero in [-1,1] for elliptic differential-operator equations on which S. Yakubov, G. Dore and S. Yakubov have suggestion results for nonlocal boundary value problems with elliptic differential equation in [0,1] (see, [3],[19]). Besides, we have considered methods of solution of boundary value problems for elliptic differential-operator equations, which are suggested by S. G. Krein (see, [9]).

2. Preliminaries

In this section, we give some definitions and auxiliary results which are used through the paper.

Lemma 2.1. (18, section 1.2) Let $\{A_0, A_1\}$ be an interpolation couple. Hence, $A_0 \cap A_1$ and $A_0 + A_1$ are Banach spaces. It holds that

$$A_0 \cap A_1 \subset A_j \subset A_0 + A_1, \ j = 0,1,$$

where A_0 and A_1 are Banach spaces continuously embedded into the Banach space A : $A_0 \subset \mathrm{A}$, $A_1 \subset \mathrm{A}$ and the space $A_0 + A_1$ is defined as

$$A_0 + A_1 := \{ a \,|\, a \in \mathrm{A} \ , \exists a_j \in A_j, j = 0,1, \text{ where } a = a_0 + a_1 \},$$

$$\|a\|_{A_0+A_1} := \inf_{\substack{a=a_0+a_1 \\ a_j \in A_j}} (\|a_0\|_{A_0} + \|a_1\|_{A_1}) .$$

Theorem 2.2. (18, Theorem 1.8.2) Let $\{A_0, A_1\}$ be an interpolation couple. Further, let j and m be integer, $0 \le j \le m-1$, η_0 and η_1 are real numbers with $\eta_0 + j > 0$, $\eta_1 + j < m$. Also, let $1 \le p_0, p_1 \le \infty$ or $p_0 = p_1 = \infty$. Then (in the sense of equivalent norms)

$$T_j^m(p_0, \eta_0, A_0; p_1, \eta_1, A_1) = (A_0, A_1)_{\theta,p}$$

holds, where $\theta := \dfrac{\eta_0 + j}{m + \eta_0 - \eta_1}$, $\dfrac{1}{p} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$.

Theorem 2.3. (9, Theorem 14.1) Each operator $A = U^{-1} + T$, where U is a bounded operator from F into E and T is a compact operator from E into F, is Fredholm. Each Fredholm operator A can be represented in the form $A = U^{-1} + T$, T is a finite dimensional operator.

The K-Functional and K-Method are expressed by H. Triebel ([18], section 1.3) in the following forms.

Definition 2.1. The K-Functional: If $\{A_0, A_1\}$ is an interpolation couple, the functional

$K(t,a) := K(t,a; A_0, A_1) := \inf_{\substack{a=a_0+a_1 \\ a_j \in A_j}} (\|a_0\|_{A_0} + t\|a_1\|_{A_1})$, $u \in A_0 + A_1$, $(0 < t < \infty)$, is an equivalent norm in the space $A_0 + A_1$ and continuous on $(0, \infty)$. Further, if $(0 < t < \infty)$ it can be obtained in the form

$$\min(1,t)\|a\|_{A_0+A_1} \le K(t,a) \le \max(1,t)\|a\|_{A_0+A_1} .$$

The K-Method:

Let $\{A_0, A_1\}$ be an interpolation couple and $0 < \theta < 1$. If $1 \le q < \infty$, then

$$(A_0, A_1)_{\theta,q} := \left\{ a \,\middle|\, a \in A_0 + A_1;\ \|a\|_{(A_0,A_1)_{\theta,q}} := \left(\int_0^\infty [t^{-\theta} K(t,a)]^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty \right\}.$$

Definition 2.2. (Direct sum) The Banach space $W_q^k(-1,0) + W_q^k(0,1)$, an integer $k \ge 0$, a real $q > 1$, is defined as

$$W_q^k(-1,0) + W_q^k(0,1) = \left\{ u = \begin{cases} u_1(x), & for \quad x \in (-1,0) \\ u_2(x), & for \quad x \in (0,1) \end{cases} \,\middle|\, u_1 \in W_q^k(-1,0),\ u_2 \in W_q^k(0,1), \right.$$

$$\left. \|u\| = \|u_1\|_{W_q^k(-1,0)} + \|u_2\|_{W_q^k(0,1)} \right\}.$$

For convenience below, the direct sum $W_q^k(-1,0) + W_q^k(0,1)$ is denoted by $W_q^k(-1,0,1)$.

Obviously, $W_q^0(-1,0,1) := L_q(-1,0,1) = L_q(-1,1)$.

Definition 2.3. Let H_1 and H_2 be Hilbert spaces. The set

$$H_1 \oplus H_2 := \left\{ (u,v) \,\middle|\, u \in H_1, v \in H_2;\ \|(u,v)\|_{H_1 \oplus H_2} := (\|u\|_{H_1}^2 + \|v\|_{H_2}^2)^{\frac{1}{2}} \right\},$$

with coordinatewise linear operations is a Hilbert space which is called the ortogonal sum of H_1 and H_2 Hilbert spaces. Let A be a closed operator in Hilbert space H. The domain of definition $D(A)$ of the operator A is turned into a Hilbert space with respect to the norm

$$\|u\|_{H(A)} := (\|u\|^2 + \|Au\|^2)^{\frac{1}{2}}.$$

$W_p^l((-1,0,1);H)$, $p\in[1,\infty)$, l-integer, denotes a Banach space of functions $u(x)$ with values from H which have generalized derivatives up to the l-th order inclusive on $(-1,0)\cup(0,1)$ and the norm $\|u\|_{W_p^l((-1,0,1);H)} := \sum_{k=0}^{l}[(\int_{-1}^{0}\|u^{(k)}(x)\|^p dx)^{\frac{1}{p}} + (\int_{0}^{1}\|u^{(k)}(x)\|^p dx)^{\frac{1}{p}}]$ is finite.

$W_p^2((-1,0,1);H(A),H) := L_p((-1,1);H(A)) \cap W_p^2((-1,0,1);H)$ is a Banach space with the norm $\|u\|_{W_p^2((-1,0,1);H(A),H)} := \|Au\|_{L_p((-1,1);H)} + \|u''\|_{L_p((-1,1);H)}$.

Theorem 2.4. [3] Let E be a complex Banach space, A be a closed operator in E of type φ with bound L and m be a positive integer, $p\geq 1$, $\alpha\in(\frac{1}{2p}, m+\frac{1}{2p})$. Then there exists $C>0$ (depending only on L, φ, m, α and p) such that for every $u\in(E,E(A^m))_{\frac{\alpha}{m}-\frac{1}{2mp},p}$ and $\lambda\in\sum_\varphi := \{\lambda\in\mathbb{C} \mid |\arg\lambda|\leq\varphi,\ \varphi\in(0,\pi)\}\cup\{0\}$,

$$\int_0^\infty \left\|(A+\lambda I)^\alpha e^{-x(A+\lambda I)^{1/2}}u\right\|^p dx \leq C(\|u\|^p_{(E,E(A^m))_{\frac{\alpha}{m}-\frac{1}{2mp},p}} + |\lambda|^{p\alpha-\frac{1}{2}}\|u\|^p).$$

This theorem has been proved in ([20], Theorem 5.4.2/1).

3. Homogeneous problem of transmission-boundary value problem

We will consider in this section a boundary value problem

$$L_0(\lambda)u := -u''(x) + (A+\lambda I)u(x) = 0,\ x\in[-1,0)\cup(0,1] \quad (3.1)$$

$$L_{\upsilon 0}u := \alpha_\upsilon u^{(m_k)}(-1) + \beta_\upsilon u^{(m_k)}(-0) + \eta_\upsilon u^{(m_k)}(+0) + \gamma_\upsilon u^{(m_k)}(1),\quad \upsilon=1,2,3,4, \quad (3.2)$$

where λ is a complex parameter.

Denote:

1) $$\theta := \begin{vmatrix} (-1)^{m_1}\beta_1 & \alpha_1 & (-1)^{m_1}\eta_1 & \gamma_1 \\ (-1)^{m_2}\beta_2 & \alpha_2 & (-1)^{m_2}\eta_2 & \gamma_2 \\ (-1)^{m_1}\beta_3 & \alpha_3 & (-1)^{m_1}\eta_3 & \gamma_3 \\ (-1)^{m_2}\beta_4 & \alpha_4 & (-1)^{m_2}\eta_4 & \gamma_4 \end{vmatrix}$$

2) $u(x)$ is a solution of equation (3.1) where $u(x) := u_1(x) + u_2(x)$; $u_1(x) := u_{11}(x) + u_{21}(x)$ for $x\in[-1,0)$, $u_2(x) := u_{12}(x) + u_{22}(x)$ for $x\in(0,1]$.

3) $R(\lambda, A_k) := (\lambda I - A_k)^{-1}$ is the resolvent of the operator A_k, $k=1,2$.

We will assume that A_k is a closed densely defined operator in a Hilbert space H.

Theorem 3.1. Let the following conditions be satisfied:

$\theta\neq 0$, $\|R(\lambda, A_k)\| \leq C(1+|\lambda|)^{-1}$, $|\arg\lambda|\geq\pi-\varphi$, $k=1,2$, $0\leq\varphi<\pi$.

Then problem (3.1)-(3.2) for $f_k \in (H(A_k),H)_{\theta_k,p}$, where $\theta_\upsilon := \frac{m_\upsilon}{2} + \frac{1}{2p}$, $\upsilon = 1,2,3,4$. $p \in (1,\infty)$, $|\arg\lambda| \le \varphi$ and $\lambda \to \infty$, has a unique solution that belongs to the space $W_p^2((-1,0,1);H(A_k),H)$ and for these λ the following coercive estimate holds for the solution of problem (3.1)-(3.2) :

$$\sum_{k=1}^{2}\left(|\lambda|\|u_k\|_{L_p((-1,1);H)} + \|u_k''\|_{L_p((-1,1);H)} + \|A_k u_k\|_{L_p((-1,1);H)}\right)$$

$$\le C\sum_{j=1}^{2}\sum_{k=1}^{2}\left(\|f_{jk}\|_{(H(A_k),H)_{\theta_{jk},p}} + |\lambda|^{1-\theta_{jk}}\|f_{jk}\|\right), \tag{3.3}$$

where, $\theta_{11} := \theta_1, \theta_{21} := \theta_2, \theta_{12} := \theta_3, \theta_{22} := \theta_4$, $f_{11} := f_1$, $f_{21} := f_2$, $f_{12} := f_3$, $f_{22} := f_4$.

Proof. In view of the condition (1) and Theorem 2.4 for $|\arg\lambda| \le \varphi$, there exist semigroups $e^{-x(A_1+\lambda I)^{\frac{1}{2}}}$, $e^{-x(A_2+\lambda I)^{\frac{1}{2}}}$ which are the holomorphic and strongly continuous for $x<0$ and $x>0$, respectively. By virtue of S. G. Krein ([9], p 252) and ([20], section 5.4.3), an arbitrary solution of equation (3.1) in the space $W_p^2((-1,0,1);H(A_k),H)$, $k=1,2$, has the form

$$u(x) := u_1(x) + u_2(x)$$

$$= e^{-x(A_1+\lambda I)^{\frac{1}{2}}}h_1 + e^{(1+x)(A_1+\lambda I)^{\frac{1}{2}}}h_2 + e^{-x(A_2+\lambda I)^{\frac{1}{2}}}h_3 + e^{-(1-x)(A_2+\lambda I)^{\frac{1}{2}}}h_4, \tag{3.4}$$

where

$$u_1(x) := e^{-x(A_1+\lambda I)^{\frac{1}{2}}}h_1 + e^{(1+x)(A_1+\lambda I)^{\frac{1}{2}}}h_2, \quad u_2(x) := e^{-x(A_2+\lambda I)^{\frac{1}{2}}}h_3 + e^{-(1-x)(A_2+\lambda I)^{\frac{1}{2}}}h_4;$$

$h_1, h_2 \in (H(A_1),H)_{\frac{1}{2p},p}$ and $h_3, h_4 \in (H(A_2),H)_{\frac{1}{2p},p}$. Let $u \in W_p^2((-1,0,1);H(A_k),H)$ be a solution of equation (3.1). So, from (3.1) we have

$$[D-(A_k+\lambda I)^{\frac{1}{2}}][D+(A_k+\lambda I)^{\frac{1}{2}}]u_k(x) = 0, \; k=1,2.$$

Denote:

$$r_k(x) := [D+(A_k+\lambda I)^{\frac{1}{2}}]u_k(x), \; k=1,2. \tag{3.5}$$

Therefore, $r_k \in W_p^2((-1,0,1);H(A_k^{\frac{1}{2}}),H)$ and

$$[D-(A_k+\lambda I)^{\frac{1}{2}}]r_k(x) = 0.$$

So, we have

$$r_1(x) = e^{(1+x)(A_1+\lambda I)^{\frac{1}{2}}}r_1(-1) \text{ and } r_2(x) = e^{-(1-x)(A_2+\lambda I)^{\frac{1}{2}}}r_2(1), \tag{3.6}$$

where, in view of the Theorem 2.2. $r_1(-1) \in (H(A_1^{\frac{1}{2}}),H)_{\frac{1}{p},p}$ and $r_2(1) \in (H(A_2^{\frac{1}{2}}),H)_{\frac{1}{p},p}$.

From (3.5) and (3.6) we have

$$u_1(x)=e^{-x(A_1+\lambda I)^{\frac{1}{2}}}u_1(-0)+\frac{1}{2}(A_1+\lambda I)^{-\frac{1}{2}}[e^{(1+x)(A_1+\lambda I)^{\frac{1}{2}}}-e^{(1-x)(A_1+\lambda I)^{\frac{1}{2}}}]r_1(-1),$$

$$u_2(x)=e^{-x(A_2+\lambda I)^{\frac{1}{2}}}u_2(+0)+\frac{1}{2}(A_2+\lambda I)^{-\frac{1}{2}}[e^{-(1-x)(A_2+\lambda I)^{\frac{1}{2}}}-e^{-(1+x)(A_2+\lambda I)^{\frac{1}{2}}}]r_2(1).$$

Hence, we can write

$$\begin{aligned}u(x)&=e^{-x(A_1+\lambda I)^{\frac{1}{2}}}u_1(-0)+e^{-x(A_2+\lambda I)^{\frac{1}{2}}}u_2(+0)]\\&+\frac{1}{2}e^{(1+x)}[(A_1+\lambda I)^{-\frac{1}{2}}e^{(A_1+\lambda I)^{\frac{1}{2}}}r_1(-1)-(A_2+\lambda I)^{-\frac{1}{2}}e^{-(A_2+\lambda I)^{\frac{1}{2}}}r_2(1)]\\&+\frac{1}{2}e^{-(1-x)}[(A_2+\lambda I)^{-\frac{1}{2}}e^{(A_2+\lambda I)^{\frac{1}{2}}}r_2(1)]-(A_1+\lambda I)^{-\frac{1}{2}}e^{-(A_1+\lambda I)^{\frac{1}{2}}}r_1(-1)],\end{aligned} \tag{3.7}$$

where, by Theorem 2.2. (see also, [20], Theorem 1.7.7/1) $u_1(-0)\in(H(A_1),H)_{\frac{1}{2p},p}$ and $u_2(+0)\in(H(A_2),H)_{\frac{1}{2p},p}$.

By virtue of ([18], Theorem 1.15.2), the operator $A_k^{\frac{1}{2}}$ is an isomorphism from $(H(A_k),H)_{\frac{1}{2p},p}$ onto $(H(A_k^{\frac{1}{2}}),H)_{\frac{1}{p},p}$.

Hence, (3.7) has form (3.4). Let us prove that the function (3.6) belongs to the space $W_p^2((-1,0,1);H(A_k),H)$. In view of Theorem 2.4, from (3.4) we have

$$\begin{aligned}\|u\|_{W_p^2((-1,0,1);H(A),H)}&\le(A_1(A_1+\lambda I)^{-1}+1)[(\int_{-1}^{0}\left\|(A_1+\lambda I)e^{-x(A_1+\lambda I)^{1/2}}h_1\right\|^p dx)^{\frac{1}{p}}\\&+(\int_{-1}^{0}\left\|(A_1+\lambda I)e^{(1+x)(A_1+\lambda I)^{1/2}}h_2\right\|^p dx)^{\frac{1}{p}}]\\&+(A_2(A_2+\lambda I)^{-1}+1)[(\int_{0}^{1}\left\|(A_2+\lambda I)e^{-x(A_2+\lambda I)^{1/2}}h_3\right\|^p dx)^{\frac{1}{p}}\\&+(\int_{0}^{1}\left\|(A_2+\lambda I)e^{-(1-x)(A_2+\lambda I)^{1/2}}h_4\right\|^p dx)^{\frac{1}{p}}]\\&\le C[\sum_{k=1}^{2}(\|h_k\|_{(H(A_1),H)_{\frac{1}{2p},p}}+|\lambda|^{1-\frac{1}{2p}}\|h_k\|)\\&+\sum_{k=3}^{4}(\|h_k\|_{(H(A_2),H)_{\frac{1}{2p},p}}+|\lambda|^{1-\frac{1}{2p}}\|h_k\|)].\end{aligned} \tag{3.8}$$

Let us now write (3.4) the function $u(x)$ in (3.2) boundary conditions.

$$\begin{aligned}&(-1)^{m_k}(\alpha_\upsilon e^{(A_1+\lambda I)^{\frac{1}{2}}}+\beta_\upsilon)(A_1+\lambda I)^{\frac{m_k}{2}}h_1\\&+(\alpha_\upsilon+\beta_\upsilon e^{(A_1+\lambda I)^{\frac{1}{2}}})(A_1+\lambda I)^{\frac{m_k}{2}}h_2\\&+(-1)^{m_k}(\eta_\upsilon+\gamma_\upsilon e^{-(A_2+\lambda I)^{\frac{1}{2}}})(A_2+\lambda I)^{\frac{m_k}{2}}h_3\end{aligned}$$

$$+(\eta_\upsilon e^{-(A_2+\lambda I)^{\frac{1}{2}}}+\gamma_\upsilon)(A_2+\lambda I)^{\frac{m_k}{2}} h_4 = f_\upsilon , \qquad (3.9)$$

$$\upsilon = 1,2,3,4 , k = 1,2 , \ m_k \in \{0,1\} .$$

This system can be written as a matrix in the form

$$\begin{pmatrix} (-1)^{m_1}(\alpha_1 e^{(A_1+\lambda I)^{1/2}}+\beta_1)e^{(A_1+\lambda I)^{\frac{m_1}{2}}} & (\alpha_1+\beta_1 e^{(A_1+\lambda I)^{1/2}})e^{(A_1+\lambda I)^{\frac{m_1}{2}}} & (-1)^{m_1}(\eta_1+\gamma_1 e^{-(A_2+\lambda I)^{1/2}})e^{(A_2+\lambda I)^{\frac{m_1}{2}}} & (\eta_1 e^{-(A_2+\lambda I)^{1/2}}+\gamma_1)e^{(A_2+\lambda I)^{\frac{m_1}{2}}} \\ (-1)^{m_2}(\alpha_2 e^{(A_1+\lambda I)^{1/2}}+\beta_2)e^{(A_1+\lambda I)^{\frac{m_2}{2}}} & (\alpha_2+\beta_2 e^{(A_1+\lambda I)^{1/2}})e^{(A_1+\lambda I)^{\frac{m_2}{2}}} & (-1)^{m_2}(\eta_2+\gamma_2 e^{-(A_2+\lambda I)^{1/2}})e^{(A_2+\lambda I)^{\frac{m_2}{2}}} & (\eta_2 e^{-(A_2+\lambda I)^{1/2}}+\gamma_2)e^{(A_2+\lambda I)^{\frac{m_2}{2}}} \\ (-1)^{m_1}(\alpha_3 e^{(A_1+\lambda I)^{1/2}}+\beta_3)e^{(A_1+\lambda I)^{\frac{m_1}{2}}} & (\alpha_3+\beta_3 e^{(A_1+\lambda I)^{1/2}})e^{(A_1+\lambda I)^{\frac{m_1}{2}}} & (-1)^{m_1}(\eta_3+\gamma_3 e^{-(A_2+\lambda I)^{1/2}})e^{(A_2+\lambda I)^{\frac{m_1}{2}}} & (\eta_3 e^{-(A_2+\lambda I)^{1/2}}+\gamma_3)e^{(A_2+\lambda I)^{\frac{m_1}{2}}} \\ (-1)^{m_2}(\alpha_4 e^{(A_1+\lambda I)^{1/2}}+\beta_4)e^{(A_1+\lambda I)^{\frac{m_2}{2}}} & (\alpha_4+\beta_4 e^{(A_1+\lambda I)^{1/2}})e^{(A_1+\lambda I)^{\frac{m_2}{2}}} & (-1)^{m_2}(\eta_4+\gamma_4 e^{-(A_2+\lambda I)^{1/2}})e^{(A_2+\lambda I)^{\frac{m_2}{2}}} & (\eta_4 e^{-(A_2+\lambda I)^{1/2}}+\gamma_4)e^{(A_2+\lambda I)^{\frac{m_2}{2}}} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} .$$

Denote: $w_1 := (A_1+\lambda I)^{\frac{m}{2}} h_1$, $w_2 := (A_1+\lambda I)^{\frac{m}{2}} h_2$,

$$w_3 := (A_2+\lambda I)^{\frac{m}{2}} h_3 , \ w_4 := (A_2+\lambda I)^{\frac{m}{2}} h_4 ,$$

where $m = \max\{m_1, m_2\}$. Therefore, from system (3.9) we have

$$(-1)^{m_k}(\alpha_\upsilon e^{(A_1+\lambda I)^{\frac{1}{2}}}+\beta_\upsilon)(A_1+\lambda I)^{\frac{m_k-m}{2}} w_1$$

$$+(\alpha_\upsilon+\beta_\upsilon e^{(A_1+\lambda I)^{\frac{1}{2}}})(A_1+\lambda I)^{\frac{m_k-m}{2}} w_2$$

$$+(-1)^{m_k}(\eta_\upsilon+\gamma_\upsilon e^{-(A_2+\lambda I)^{\frac{1}{2}}})(A_2+\lambda I)^{\frac{m_k-m}{2}} w_3$$

$$+(\eta_\upsilon e^{-(A_2+\lambda I)^{\frac{1}{2}}}+\gamma_\upsilon)(A_2+\lambda I)^{\frac{m_k-m}{2}} w_4 = f_\upsilon , \qquad (3.10)$$

System (3.10) has the form

$$\left[\begin{pmatrix} (-1)^{m_1}\beta_1 & \alpha_1 & (-1)^{m_1}\eta_1 & \gamma_{11} \\ (-1)^{m_2}\beta_2 & \alpha_{21} & (-1)^{m_2}\eta_2 & \gamma_2 \\ (-1)^{m_1}\beta_3 & \alpha_3 & (-1)^{m_1}\eta_3 & \gamma_3 \\ (-1)^{m_2}\beta_4 & \alpha_4 & (-1)^{m_2}\eta_4 & \gamma_4 \end{pmatrix} + R(\lambda)\right] \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$$

$$=\begin{pmatrix} (A_1+\lambda I)^{-\frac{m_1-m}{2}} & 0 & 0 & 0 \\ 0 & (A_1+\lambda I)^{-\frac{m_2-m}{2}} & 0 & 0 \\ 0 & 0 & (A_2+\lambda I)^{-\frac{m_1-m}{2}} & 0 \\ 0 & 0 & 0 & (A_2+\lambda I)^{-\frac{m_2-m}{2}} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}, \qquad (3.11)$$

where, $R(\lambda)$ can be written as a matrix in the form

$$R(\lambda):=\begin{pmatrix} (-1)^{m_1}\alpha_1 e^{(A_1+\lambda I)^{1/2}} & \beta_1 e^{(A_1+\lambda I)^{1/2}} & (-1)^{m_1}\gamma_1 e^{-(A_2+\lambda I)^{1/2}} & \eta_1 e^{-(A_2+\lambda I)^{1/2}} \\ (-1)^{m_2}\alpha_2 e^{(A_1+\lambda I)^{1/2}} & \beta_2 e^{(A_1+\lambda I)^{1/2}} & (-1)^{m_2}\gamma_2 e^{-(A_2+\lambda I)^{1/2}} & \eta_2 e^{-(A_2+\lambda I)^{1/2}} \\ (-1)^{m_1}\alpha_3 e^{(A_1+\lambda I)^{1/2}} & \beta_3 e^{(A_1+\lambda I)^{1/2}} & (-1)^{m_1}\gamma_3 e^{-(A_2+\lambda I)^{1/2}} & \eta_3 e^{-(A_2+\lambda I)^{1/2}} \\ (-1)^{m_2}\alpha_4 e^{(A_1+\lambda I)^{1/2}} & \beta_4 e^{(A_1+\lambda I)^{1/2}} & (-1)^{m_2}\gamma_4 e^{-(A_2+\lambda I)^{1/2}} & \eta_4 e^{-(A_2+\lambda I)^{1/2}} \end{pmatrix}.$$

Hence, from (3.10), we obtain the system

$$(R(\lambda)w)_1:=(-1)^{m_1}\alpha_1 e^{(A_1+\lambda I)^{1/2}}w_1+\beta_1 e^{(A_1+\lambda I)^{1/2}}w_2 +(-1)^{m_1}\gamma_1 e^{-(A_2+\lambda I)^{1/2}}w_3+\eta_1 e^{-(A_2+\lambda I)^{1/2}}w_4,$$

$$(R(\lambda)w)_2:=(-1)^{m_2}\alpha_2 e^{(A_1+\lambda I)^{1/2}}w_1+\beta_2 e^{(A_1+\lambda I)^{1/2}}w_2 +(-1)^{m_2}\gamma_2 e^{-(A_2+\lambda I)^{1/2}}w_3+\eta_2 e^{-(A_2+\lambda I)^{1/2}}w_4,$$

$$(R(\lambda)w)_3:=(-1)^{m_1}\alpha_3 e^{(A_1+\lambda I)^{1/2}}w_1+\beta_3 e^{(A_1+\lambda I)^{1/2}}w_2 +(-1)^{m_1}\gamma_3 e^{-(A_2+\lambda I)^{1/2}}w_3+\eta_3 e^{-(A_2+\lambda I)^{1/2}}w_4,$$

$$(R(\lambda)w)_4:=(-1)^{m_2}\alpha_4 e^{(A_1+\lambda I)^{1/2}}w_1+\beta_4 e^{(A_1+\lambda I)^{1/2}}w_2 +(-1)^{m_2}\gamma_4 e^{-(A_2+\lambda I)^{1/2}}w_3+\eta_4 e^{-(A_2+\lambda I)^{1/2}}w_4.$$

By virtue of (20, Lemma 5.4.2/6) for $|\arg\lambda|\le\varphi$ and $|\lambda|\to\infty$

$$\|R(\lambda)\|_{B(H^2)}\to 0,\ \|R(\lambda)\|_{B(H[(A_k)]^2)}\to 0. \qquad (3.12)$$

Therefore, by $\theta\neq 0$ and Neumann identity for $|\arg\lambda|\le\varphi$ and $|\lambda|\to\infty$,

$$\left[\begin{pmatrix} (-1)^{m_1}\beta_1 & \alpha_1 & (-1)^{m_1}\eta_1 & \gamma_1 \\ (-1)^{m_2}\beta_2 & \alpha_2 & (-1)^{m_2}\eta_2 & \gamma_2 \\ (-1)^{m_1}\beta_3 & \alpha_3 & (-1)^{m_1}\eta_3 & \gamma_3 \\ (-1)^{m_2}\beta_4 & \alpha_4 & (-1)^{m_2}\eta_4 & \gamma_4 \end{pmatrix}+R(\lambda)\right]^{-1}=\begin{pmatrix} (-1)^{m_1}\beta_1 & \alpha_1 & (-1)^{m_1}\eta_1 & \gamma_1 \\ (-1)^{m_2}\beta_2 & \alpha_2 & (-1)^{m_2}\eta_2 & \gamma_2 \\ (-1)^{m_1}\beta_3 & \alpha_3 & (-1)^{m_1}\eta_3 & \gamma_3 \\ (-1)^{m_2}\beta_4 & \alpha_4 & (-1)^{m_2}\eta_4 & \gamma_4 \end{pmatrix}^{-1}$$

$$\times\sum_{k=0}^{\infty}\left[-R(\lambda)\begin{pmatrix}(-1)^{m_1}\beta_1 & \alpha_1 & (-1)^{m_1}\eta_1 & \gamma_1\\ (-1)^{m_2}\beta_2 & \alpha_2 & (-1)^{m_2}\eta_2 & \gamma_2\\ (-1)^{m_1}\beta_3 & \alpha_3 & (-1)^{m_1}\eta_3 & \gamma_3\\ (-1)^{m_2}\beta_4 & \alpha_4 & (-1)^{m_2}\eta_4 & \gamma_4\end{pmatrix}^{-1}\right]^k. \quad (3.13)$$

So, system (3.11) has a unique solution for $|\arg\lambda|\le\varphi$ and $|\lambda|\to\infty$, and the solution can be expressed in the form of

$$w_j=\sum_{k=1}^{2}[C_{jk}+R_{jk}(\lambda)](A_1+\lambda I)^{\frac{m-m_k}{2}}f_j,\quad j=1,2,$$

$$w_j=\sum_{k=1}^{2}[C_{jk}+R_{jk}(\lambda)](A_2+\lambda I)^{\frac{m-m_k}{2}}f_j,\quad j=3,4,$$

where C_{jk} are complex numbers and in view of (3.12) for $|\arg\lambda|\le\varphi$ and $|\lambda|\to\infty$

$$\|R_{jk}(\lambda)\|_{B(H)}\to 0,\ \|R_{jk}(\lambda)\|_{B(H(A_k))}\to 0,\ k=1,2 \text{ for every } j,\ j=1,2,3,4. \quad (3.14)$$

Therefore, we have

$$h_j=\sum_{k=1}^{2}[C_{jk}+R_{jk}(\lambda)](A_1+\lambda I)^{\frac{-m_k}{2}}f_j,\quad j=1,2,$$

$$h_j=\sum_{k=1}^{2}[C_{jk}+R_{jk}(\lambda)](A_2+\lambda I)^{\frac{-m_k}{2}}f_j,\quad j=3,4. \quad (3.15)$$

Substituting (3.15) into (3.4) we have

$$u(x)=\sum_{k=1}^{2}[(C_{1k}+R_{1k}(\lambda))e^{-x(A_1+\lambda I)^{1/2}}+(C_{2k}+R_{2k}(\lambda))e^{(1+x)(A_1+\lambda I)^{1/2}}](A_1+\lambda I)^{-\frac{m_k}{2}}f_k$$

$$+\sum_{k=1}^{2}[(C_{3k}+R_{3k}(\lambda))e^{-x(A_2+\lambda I)^{1/2}}+(C_{4k}+R_{4k}(\lambda))e^{-(1-x)(A_2+\lambda I)^{1/2}}](A_2+\lambda I)^{-\frac{m_k}{2}}f_{k+1}. \quad (3.16)$$

Hence, for $|\arg\lambda|\le\varphi$ and for $|\lambda|\to\infty$, we get

$$\sum_{k=1}^{2}(|\lambda|\|u_k\|_{L_p((-1,1);H)}+\|u_k''\|_{L_p((-1,1);H)}+\|A_k u_k\|_{L_p((-1,1);H)})$$

$$\le C\sum_{k=1}^{2}\{|\lambda|[(\int_{-1}^{0}\|(A_1+\lambda I)^{-m_k/2}e^{-x(A_1+\lambda I)^{1/2}}f_k\|^p dx)^{\frac{1}{p}}$$

$$+\|R_{1k}(\lambda)\|(\int_{-1}^{0}\|(A_1+\lambda I)^{-m_k/2}e^{-x(A_1+\lambda I)^{1/2}}f_k\|^p dx)^{\frac{1}{p}}$$

$$+(\int_{-1}^{0}\|(A_1+\lambda I)^{-m_k/2}e^{(1+x)(A_1+\lambda I)^{1/2}}f_k\|^p dx)^{\frac{1}{p}}$$

$$+\|R_{2k}(\lambda)\|(\int_{-1}^{0}\|(A_1+\lambda I)^{-m_k/2}e^{(1+x)(A_1+\lambda I)^{1/2}}f_k\|^p dx)^{\frac{1}{p}}]$$

$$+(1+\|A_1(A_1+\lambda I)^{-1}\|)[(\int_{-1}^{0}\|(A_1+\lambda I)^{1-m_k/2}e^{-x(A_1+\lambda I)^{1/2}}f_k\|^p dx)^{\frac{1}{p}}$$

$$+\|R_{1k}(\lambda)\|(\int_{-1}^{0}\|(A_1+\lambda I)^{1-m_k/2}e^{-x(A_1+\lambda I)^{1/2}}f_k\|^p dx)^{\frac{1}{p}}$$

$$+(\int_{-1}^{0}\|(A_1+\lambda I)^{1-m_k/2}e^{(1+x)(A_1+\lambda I)^{1/2}}f_k\|^p dx)^{\frac{1}{p}}$$

$$+\|R_{2k}(\lambda)\|(\int_{-1}^{0}\|(A_1+\lambda I)^{1-m_k/2}e^{(1+x)(A_1+\lambda I)^{1/2}}f_k\|^p dx)^{\frac{1}{p}}]\}$$

$$+\sum_{k=1}^{2}\{|\lambda|[(\int_{0}^{1}\|(A_2+\lambda I)^{-m_k/2}e^{-x(A_2+\lambda I)^{1/2}}f_{k+1}\|^p dx)^{\frac{1}{p}}$$

$$+\|R_{3k}(\lambda)\|(\int_{0}^{1}\|(A_2+\lambda I)^{-m_k/2}e^{-x(A_2+\lambda I)^{1/2}}f_{k+1}\|^p dx)^{\frac{1}{p}}$$

$$+(\int_{0}^{1}\|(A_2+\lambda I)^{-m_k/2}e^{-(1-x)(A_2+\lambda I)^{1/2}}f_{k+1}\|^p dx)^{\frac{1}{p}}$$

$$+\|R_{4k}(\lambda)\|(\int_{0}^{1}\|(A_2+\lambda I)^{-m_k/2}e^{-(1-x)(A_2+\lambda I)^{1/2}}f_{k+1}\|^p dx)^{\frac{1}{p}}]$$

$$+(1+\|A_2(A_2+\lambda I)^{-1}\|)[(\int_{0}^{1}\|(A_2+\lambda I)^{1-m_k/2}e^{-x(A_2+\lambda I)^{1/2}}f_{k+1}\|^p dx)^{\frac{1}{p}}$$

$$+\|R_{3k}(\lambda)\|(\int_{0}^{1}\|(A_2+\lambda I)^{1-m_k/2}e^{-x(A_2+\lambda I)^{1/2}}f_{k+1}\|^p dx)^{\frac{1}{p}}$$

$$+(\int_{0}^{1}\|(A_2+\lambda I)^{1-m_k/2}e^{-(1-x)(A_2+\lambda I)^{1/2}}f_{k+1}\|^p dx)^{\frac{1}{p}}$$

$$+\|R_{4k}(\lambda)\|(\int_{0}^{1}\|(A_2+\lambda I)^{1-m_k/2}e^{-(1-x)(A_2+\lambda I)^{1/2}}f_{k+1}\|^p dx)^{\frac{1}{p}}]\}. \quad (3.17)$$

In view of the inequality (3.8) or Theorem 2.4, for the terms of the right-hand side of inequality (3.17) we get the following inequalities. We have for the first term

$$|\lambda|(\int_{-1}^{0}\|(A_1+\lambda I)^{-m_k/2}e^{-x(A_1+\lambda I)^{1/2}}f_k\|^p dx)^{\frac{1}{p}}$$

$$\le C|\lambda|\|(A_1+\lambda I)^{-1}\|(\int_{-1}^{0}\|(A_1+\lambda I)^{1-m_k/2}e^{-x(A_1+\lambda I)^{1/2}}f_k\|^p dx)^{\frac{1}{p}}$$

$$\le C\sum_{k=1}^{2}(\|f_{k1}\|_{(H(A_1),H)_{\theta_k,p}}+|\lambda|^{1-\theta_k}\|f_k\|), \quad (3.18)$$

for the second term

$$|\lambda|\|R_{1k}(\lambda)\|(\int_{-1}^{0}\|(A_1+\lambda I)^{-m_k/2}e^{-x(A_1+\lambda I)^{1/2}}f_k\|^p dx)^{\frac{1}{p}}$$

$$\le C|\lambda|\left\|(A_1+\lambda I)^{-1}\right\|\left\|R_{1k}(\lambda)\right\|(\int_{-1}^{0}\left\|(A_1+\lambda I)^{1-m_k/2}e^{-x(A_1+\lambda I)^{1/2}}f_k\right\|^p dx)^{\frac{1}{p}}$$

$$\le C\sum_{k=1}^{2}(\|f_{k1}\|_{(H(A_1),H)_{\theta_k,p}}+|\lambda|^{1-\theta_k}\|f_k\|) \qquad (3.19)$$

for the ninth term

$$|\lambda|[(\int_0^1\left\|(A_2+\lambda I)^{-m_k/2}e^{-x(A_2+\lambda I)^{1/2}}f_{k+1}\right\|^p dx)^{\frac{1}{p}}$$

$$\le C|\lambda|\left\|(A_2+\lambda I)^{-1}\right\|(\int_0^1\left\|(A_2+\lambda I)^{1-m_k/2}e^{-x(A_2+\lambda I)^{1/2}}f_{k+1}\right\|^p dx)^{\frac{1}{p}}$$

$$\le C\sum_{k=1}^{2}(\|f_{k2}\|_{(H(A_2),H)_{\theta_{k+1},p}}+|\lambda|^{1-\theta_{k+1}}\|f_{k+1}\|), \qquad (3.20)$$

for the tenth term

$$|\lambda|\|R_{3k}(\lambda)\|(\int_0^1\left\|(A_2+\lambda I)^{-m_k/2}e^{-x(A_2+\lambda I)^{1/2}}f_{k+1}\right\|^p dx)^{\frac{1}{p}}$$

$$\le C|\lambda|\left\|(A_2+\lambda I)^{-1}\right\|\left\|R_{3k}(\lambda)\right\|(\int_0^1\left\|(A_2+\lambda I)^{1-m_k/2}e^{-x(A_2+\lambda I)^{1/2}}f_{k+1}\right\|^p dx)^{\frac{1}{p}}$$

$$\le C\sum_{k=1}^{2}(\|f_{k2}\|_{(H(A_2),H)_{\theta_{k+1},p}}+|\lambda|^{1-\theta_{k+1}}\|f_{k+1}\|). \qquad (3.21)$$

Therefore, we can obtain the same result for all the other terms in the right-hand side of inequality (3.17), and the inequality (3.3) has been proved too.

4. Nonhomogeneous transmission-boundary value problem

In this section, we will consider the boundary value problem which has the nonhomogeneous equation with a parameter and transmission conditions

$$L_0(\lambda)u := -u''(x)+(A+\lambda I)u(x)=g(x),\ x\in[-1,0)\cup(0,1], \qquad (4.1)$$

$$L_{\upsilon 0}u := \alpha_\upsilon u^{(m_k)}(-1)+\beta_\upsilon u^{(m_k)}(-0)+\eta_\upsilon u^{(m_k)}(+0)+\gamma_\upsilon u^{(m_k)}(1)=f_\upsilon, \qquad (4.2)$$

$$\upsilon=1,2,3,4,\ k=1,2$$

Theorem 4.1. Let the following conditions be satisfied:

1) $R(\lambda,A_kI)\le C(1+|\lambda|)^{-1}$, $|\arg\lambda|\ge\pi-\varphi$, $k=1,2$, $0\le\varphi<\pi$

where $R(\lambda,A_kI):=(1+|\lambda|)^{-1}$ is the resolvent of the operator A_k,

2) $\theta\ne 0$,

Then the operator $\mathrm{L}_0(\lambda):u\to\mathrm{L}_0(\lambda)u:=(L_0(\lambda)u,L_1,L_2,L_3,L_4)$, for $|\arg\lambda|\le\varphi$ and sufficiently large $|\lambda|$, is an isomorphism from $W_p^2((-1,0,1);H(A_k),H)$ onto

$$(L_p(-1,1);H)+(H(A_1),H)_{\theta_1,p}+(H(A_1),H)_{\theta_2,p}+(H(A_2),H)_{\theta_3,p}+(H(A_2),H)_{\theta_4,p}),\ \theta_\upsilon:=\frac{m_k}{2}+\frac{1}{2p},$$

$\upsilon=1,2,3,4$, $k=1,2$, $p\in(1,\infty)$, and for these λ the following coercive estimation holds for the solution of the problem (1.1)-(1.2):

$$\sum_{k=1}^{2}(|\lambda|\|u_k\|_{L_p((-1,1);H)}+\|u_k''\|_{L_p((-1,1);H)}+\|A_k u_k\|_{L_p((-1,1);H)})$$

$$\leq C[\sum_{k=1}^{2}\|f_k\|_{L_p((-1,1);H)}+\sum_{j=1}^{2}\sum_{k=1}^{2}(\|f_{jk}\|_{(H(A_k),H)_{\theta_{jk},p}}+|\lambda|^{1-\theta_{jk}}\|f_{jk}\|)]. \qquad (4.3)$$

where, $\theta_{11}:=\theta_1, \theta_{21}:=\theta_2, \theta_{12}:=\theta_3, \theta_{22}:=\theta_4$, $f_{11}:=f_1$, $f_{21}:=f_2$, $f_{12}:=f_3$, $f_{22}:=f_4$.

Proof. We have proved the uniqueness in Theorem 3.1. Let us define $f(x):=f_1(x):=\tilde{f}_1(x)$, at $x\in[-1,0)$; $f(x):=f_2(x):=\tilde{f}_2(x)$, at $x\in(0,1]$,

$$f(x):=f_1(x):=0, \quad \text{at } x\notin[-1,0); \quad f(x):=f_2(x):=0, \quad \text{at } x\notin(0,1].$$

We will show that a solution of the problem (4.1)-(4.2) which belongs to $W_p^2((-1,0,1);H(A_k),H)$ as a sum in the following form

$$u(x)=u_1(x)+u_2(x)$$

$$=u_{11}(x)+u_{21}(x)+u_{12}(x)+u_{22}(x),$$

where

$$u_1(x)=u_{11}(x)+u_{21}(x) \quad \text{for} \quad x\in[-1,0),$$

$$u_2(x)=u_{12}(x)+u_{22}(x) \quad \text{for} \quad x\in(0,1],$$

besides,

$$u_{10}(x):=u_{11}(x)+u_{12}(x),$$

$$\tilde{u}_{10}(x):=\tilde{u}_{11}(x)+\tilde{u}_{12}(x),$$

$$u_{20}(x):=u_{21}(x)+u_{22}(x),$$

where $u_{11}(x)$ and $u_{12}(x)$ are the restriction on $[-1,0)$ and $(0,1]$, respectively, of the solutions $\tilde{u}_{11}(x)$ and $\tilde{u}_{12}(x)$ of the equation

$$L_0(\lambda)\tilde{u}_{10}=\tilde{f}(x), \quad x\in\mathbb{R}, \qquad (4.4)$$

and $u_{20}(x)$ is a solution of the problem

$$L_0(\lambda)u_{20}=0,$$

$$L_{k\upsilon 0}(\lambda)u_{20}=f_k-L_{k\upsilon 0}(\lambda)u_{10}, \ \upsilon=1 \text{ for } k=1,2 \text{ and } \upsilon=2 \text{ for } k=1,2. \qquad (4.5)$$

A solution of equation (4.4) is given by the formula

$$\tilde{u}_{10}(x)=\frac{1}{2\pi}\sum_{k=1}^{2}\int_{\mathbb{R}}e^{i\mu_k x}L_0(\lambda,i\mu_k)^{-1}F_k\tilde{f}_k(\mu_k)d\mu_k, \qquad (4.6)$$

where $F_k\tilde{f}_k$ is the Fourier transform of the function $\tilde{f}_k(x)$, and $L_0(\lambda,t_k)$ is a characteristic pencil of equation (4.4), that is, $L_0(\lambda,t_k)=-t_k^2 I+A_k+\lambda I$, $k=1,2$. It follows from (4.6) that

$$\sum_{k=1}^{2}(|\lambda|\|\tilde{u}_{1k}\|_{L_p(\mathbb{R};H)}+\|\tilde{u}_{1k}\|_{W_p^2(\mathbb{R};H(A_k),H)})$$

$$=\sum_{k=1}^{2}(|\lambda|\|\tilde{u}_{1k}\|_{L_p(\mathbb{R};H)}+\|A_k\tilde{u}_{1k}\|_{L_p(\mathbb{R};H)}+\|\tilde{u}''_{1k}\|_{L_p(\mathbb{R};H)})$$

$$\leq\sum_{k=1}^{2}(\|F_k^{-1}\lambda L_0(\lambda,i\mu_k)^{-1}F_k f_k(\mu_k)\|_{L_p(\mathbb{R};H)}$$

$$+\|F_k^{-1}A_k L_0(\lambda,i\mu_k)^{-1}F_k f_k(\mu_k)\|_{L_p(\mathbb{R};H)}$$

$$+\|F_k^{-1}(i\mu_k)^2 L_0(\lambda,i\mu_k)^{-1}F_k f_k(\mu_k)\|_{L_p(\mathbb{R};H)}). \qquad (4.7)$$

Then, by using (4.7) we obtain

$$\sum_{k=1}^{2}(|\lambda|\|\tilde{u}_{1k}\|_{L_p(\mathbb{R};H)}+\|\tilde{u}_{1k}\|_{W_p^2(\mathbb{R};H(A_k),H)})\leq C\sum_{k=1}^{2}\|\tilde{f}_{1k}\|_{L_p(\mathbb{R};H)},\quad |\arg\lambda|\leq\varphi \qquad (4.8)$$

and, therefore, $u_{10}(x)\in W_p^2((-1,0,1);H(A_k),H)$. In view of ([18], Theorem 1.8.2) and inequality (4.8), we have $u_{10}^{(m_k)}(-0)\in W_p^2((-1,0);H(A_1),H)_{\frac{m_k}{2}+\frac{1}{2p},p}$, $u_{10}^{(m_k)}(+0)\in W_p^2((0,1);H(A_2),H)_{\frac{m_k}{2}+\frac{1}{2p},p}$. Therefore, $L_{k\upsilon 0}u_{10}\in(H(A_k),H)_{\theta_{k\upsilon},p}$. Hence, by virtue of Theorem 3.1, problem (4.5) has a unique solution $u_{20}(x)$ that belongs to $W_p^2((-1,0,1);H(A_k),H)$ as $|\arg\lambda|\leq\varphi$ and $|\lambda|\to\infty$. Also, for a solution of problem (4.5), for $|\arg\lambda|\leq\varphi$ and $|\lambda|\to\infty$ we have

$$\sum_{k=1}^{2}(|\lambda|\|u_{2k}\|_{L_p((-1,1);H)}+\|A_k u_{2k}\|_{L_p((-1,1);H)}+\|u''_{2k}\|_{L_p((-1,1);H)})$$

$$\leq C\sum_{j=1}^{2}\sum_{k=1}^{2}(\|f_{jk}-L_{jk0}u_{1k}\|_{(H(A_k),H)_{\theta_{jk},p}}+|\lambda|^{1-\theta_{jk}}\|f_{jk}-L_{jk0}u_{1k}\|)$$

$$\leq C\sum_{j=1}^{2}\sum_{k=1}^{2}[(\|f_{jk}\|_{(H(A_k),H)_{\theta_{jk},p}}+|\lambda|^{1-\theta_{jk}}\|f_{jk}\|)$$

$$+(\|u_{1k}^{(m_k)}\|_{C([-1,0)\cup(0,1];(H(A_k),H)_{\theta_{jk},p})}+|\lambda|^{1-\theta_{jk}}\|u_{1k}^{(m_k)}\|_{C([-1,0)\cup(0,1];H)})]. \qquad (4.9)$$

From (4.8) for $|\arg\lambda|\leq\varphi$ it follows that

$$\sum_{k=1}^{2}(|\lambda|\|u_{1k}\|_{L_p((-1,1);H)}+\|u_{1k}\|_{W_2^p((-1,0,1);H(A_k),H)})\leq C\sum_{k=1}^{2}\|g_k\|_{L_p((-1,1);H)}. \qquad (4.10)$$

Therefore, from Theorem 2.2 and inequality (4.10) we have

$$\|u_{11}^{(m_k)}(-0)\|_{(H(A_1),H)_{\theta_{k1}}}\leq C\|u_{11}^{(m_k)}\|_{W_p^2((-1,0,1);H(A),H)}\leq C\|g_1\|_{L_p((-1,1);H)}$$

$$\|u_{12}^{(m_k)}(+0)\|_{(H(A_2),H)_{\theta_{k2}}}\leq C\|u_{12}^{(m_k)}\|_{W_p^2((-1,0,1);H(A),H)}\leq C\|g_2\|_{L_p((-1,1);H)}. \qquad (4.11)$$

In view of ([20], Theorem 1.7.7/2), for $\mu_k \in \mathbb{C}$, $u \in W_2^2((-1,0,1);H(A_k),H)$ it holds that

$$|\mu_1|^{2-m_k}\left\|u^{(m_k)}(-0)\right\| \le C(|\mu_1|^{\frac{1}{p}}\|u\|_{W_p^2((-1,0,1);H)} + |\mu_1|^{2+\frac{1}{p}}\|u\|_{L_p((-1,1);H)}),$$

$$|\mu_2|^{2-m_k}\left\|u^{(m_k)}(+0)\right\| \le C(|\mu_2|^{\frac{1}{p}}\|u\|_{W_p^2((-1,0,1);H)} + |\mu_2|^{2+\frac{1}{p}}\|u\|_{L_p((-1,1);H)}). \qquad (4.12)$$

If the inequalities in (4.12) divided by $|\mu_1|^{\frac{1}{p}}$ and $|\mu_2|^{\frac{1}{p}}$, respectively, and substitute $\lambda = \mu_k^2$ for $\lambda \in \mathbb{C}$, $u \in W_p^2((-1,0,1);H)$, we have

$$|\lambda|^{1-\theta_{k1}}\left\|u^{(m_k)}(-0)\right\| \le C(\|u\|_{W_p^2((-1,0,1);H)} + |\lambda|\|u\|_{L_p((-1,0,1);H)}),$$

$$|\lambda|^{1-\theta_{k2}}\left\|u^{(m_k)}(+0)\right\| \le C(\|u\|_{W_p^2((-1,0,1);H)} + |\lambda|\|u\|_{L_p((-1,0,1);H)}). \qquad (4.13)$$

Therefore, from (4.10) and (4.13) we get

$$|\lambda|^{1-\theta_{k1}}\left\|u_{11}^{(m_k)}(-0)\right\| \le C(\|u_{11}\|_{W_2^p((-1,0,1);H(A_1),H)} + |\lambda|\|u_{11}\|_{L_p((-1,1);H)})$$
$$\le C\|g_1\|_{L_p((-1,1);H)},$$

$$|\lambda|^{1-\theta_{k2}}\left\|u_{12}^{(m_k)}(+0)\right\| \le C(\|u_{12}\|_{W_2^p((-1,0,1);H(A_2),H)} + |\lambda|\|u_{12}\|_{L_p((-1,1);H)})$$
$$\le C\|g_2\|_{L_p((-1,1);H)}, \quad |\arg\lambda| \le \varphi. \qquad (4.14)$$

From (4.9), (4.11), and (4.14) for $|\arg\lambda| \le \varphi$ and $|\lambda| \to \infty$ we have

$$\sum_{k=1}^{2}(|\lambda|\|u_{2k}\|_{L_p((-1,1);H)} + \|A_k u_{2k}\|_{L_p((-1,1);H)} + \|u_{2k}''\|_{L_p((-1,1);H)})$$

$$\le C[\sum_{k=1}^{2}\|f_k\|_{L_p((-1,1);H)} + \sum_{j=1}^{2}\sum_{k=1}^{2}(\|f_{jk}\|_{(H(A_k),H)_{\theta_{jk},p}} + |\lambda|^{1-\theta_{jk}}\|f_{jk}\|)]. \qquad (4.15)$$

Hence, from (4.14) and (4.15) we can obtain (4.3). This completed the proof.

5. Coerciveness and Fredholmness for transmission-boundary value problem

In this section, we will consider the following boundary value problem

$$Lu := -u''(x) + Au(x) + \varphi(x)u(x) = g(x), \quad x \in [-1,0) \cup (0,1] \qquad (5.1)$$

$$L_\upsilon u = \alpha_\upsilon u^{(m_k)}(-1) + \beta_\upsilon u^{(m_k)}(-0) + \eta_\upsilon u^{(m_k)}(+0) + \gamma_\upsilon u^{(m_k)}(1) = f_\upsilon \qquad (5.2)$$
$$\upsilon = 1,2,3,4, \; k = 1,2$$

where, $u_1(x) = u_{11}(x) + u_{21}(x)$ for $x \in [-1,0)$, $u_2(x) = u_{12}(x) + u_{22}(x)$ for $x \in (0,1]$,
$u(x) = u_1(x) + u_2(x) = u_{11}(x) + u_{21}(x) + u_{12}(x) + u_{22}(x)$.

Theorem 5.1. We assume that the following conditions be satisfied:

1) $R(\lambda, A_k I) \leq C(1+|\lambda|)^{-1}$, $\arg\lambda = \pi$, $|\lambda| \to \infty$, $k=1,2$,

2) the embedding $H(A_k) \subset H$ is compact,

3) $\theta \neq 0$,

4) for any $\varepsilon > 0$ and for almost all $x \in [-1,0) \cup (0,1]$,

$$\|\varphi_k(x)u_k\| \leq \varepsilon \|A_k u_k\| + C(\varepsilon)\|u_k\|,\ u_k \in D(A_k),\ k=1,2,$$

for $u_k \in (H(A_k), H)_{\frac{1}{2},1}$ and $\varphi_k(x)u_k$ are measurable on $[-1,0) \cup (0,1]$ in H.

Then

a) for $u_k \in W_p^2((-1,0,1); H(A_k), H)$, the coercive estimate

$$\sum_{k=1}^{2}(\|u_k''\|_{L_p((-1,1);H)} + \|A_k u_k\|_{L_p((-1,1);H)})$$

$$\leq C[\sum_{k=1}^{2}(\|Lu_k\|_{L_p((-1,1);H)} + \|u_k\|_{L_p((-1,1);H)}) + \sum_{j=1}^{2}\sum_{k=1}^{2}\|L_{jk}u_k\|_{(H(A_k);H)_{\theta_{jk},p}}], \quad (5.3)$$

where $\theta_\upsilon := \dfrac{m_k}{2} + \dfrac{1}{2p}$, $\upsilon = 1,2,3,4$, $k=1,2$, $p \in (1,\infty)$, holds;

b) the operator $\mathrm{L}_0(\lambda): u_k \to \mathrm{L}_0(\lambda)u_k := (L(D)u_k, L_1u_k, L_2u_k, L_3u_k, L_4u_k)$ from $W_p^2((-1,0,1); H(A_k), H)$ into

$$L_p((-1,1);H) \dot{+} (H(A_1),H)_{\theta_1,p} \dot{+} (H(A_1),H)_{\theta_2,p} \dot{+} (H(A_2),H)_{\theta_3,p} \dot{+} (H(A_2),H)_{\theta_4,p}$$

is bounded and Fredholm.

Proof. Assume that condition (1) is satisfied for $\arg\lambda = \pi$. The general case is a reduced to the latter if the operator $A_k + \lambda_0 I$ for some $\lambda_0 > 0$ sufficiently large, is considered instead of the operator A_k, and the operator $\varphi_k(x) - \lambda_0 I$ is considered instead of the operator $\varphi_k(x)$, $k=1,2$.

a) Let $u_k(x) \in W_p^2((-1,0,1); H(A_k), H)$ be a solution of problem (5.1)-(5.2). Then $u_k(x)$ is a solution of the problem

$$-u_k''(x) + (A_k + \lambda I)u_k(x) = f_k(x) + \lambda u_k(x) - \varphi_k(x)u_k(x),\ k=1,2$$

$$L_{\upsilon 0}u_k = f_\upsilon,\ \upsilon = 1,2,3,4,\ k=1,2$$

where $L_{k\upsilon 0}$ are defined by equalities (3.2). By Theorem 4.1 for $\lambda \to \infty$ we have the estimate

$$\sum_{k=1}^{2}(\|u_k''\|_{L_p((-1,1);H)} + \|A_k u_k\|_{L_p((-1,1);H)})$$

$$\leq C(\sum_{k=1}^{2}\|g_k(x) + \lambda u_k - \varphi_k(x)u_k(x)\|_{L_p((-1,1);H)} + \sum_{\upsilon=1}^{2}\sum_{k=1}^{2}\|f_{\upsilon k}\|_{(H(A_k),H)_{\theta_{\upsilon k},p}})$$

$$\leq C[\sum_{k=1}^{2}(\|g_k\|_{L_p((-1,1);H)} + \|u_k\|_{L_p((-1,1);H)} + \|\varphi_k u_k\|_{L_p((-1,1);H)})$$

$$+\sum_{\upsilon=1}^{2}\sum_{k=1}^{2}\| f_{\upsilon k}\|_{(H(A_k),H)_{\theta_{\upsilon k},p}}], \tag{5.4}$$

where, $\theta_{11} := \theta_1, \theta_{21} := \theta_2, \theta_{12} := \theta_3, \theta_{22} := \theta_4$, $f_{11} := f_1$, $f_{21} := f_2$, $f_{12} := f_3$, $f_{22} := f_4$.

In view of condition (4) and ([13], Theorem 4), for $u_k(x) \in W_p^2((-1,0,1); H(A_k), H)$,

$$\|\varphi_k u_k\|_{L_p((-1,1);H)} \le \varepsilon \|u_k\|_{W_p^1((-1,0,1);H(A_k),H)} + C(\varepsilon)\|u_k\|_{L_p((-1,1);H)}, \; k=1,2. \tag{5.5}$$

By S. and Y. Yakubov ([20], Lemma 1.7.7/2), for $u_k(x) \in W_p^2((-1,0,1); H)$,

$$\|\varphi_k u_k\|_{L_p((-1,1);H)} \le \varepsilon \|u_k\|_{W_p^2((-1,0,1);H(A_k),H)} + C(\varepsilon)\|u_k\|_{L_p((-1,1);H)}, \; k=1,2. \tag{5.6}$$

Substituting (5.6) into (5.4) we have (5.3).

b) The operator L_k can be written in the form

$$\mathrm{L}_k = \mathrm{L}_0(\lambda_0) + \mathrm{L}_k, \; k=1,2 \tag{5.7}$$

where $\mathrm{L}_0(\lambda)u_1 := \mathrm{L}_{01}(\lambda)u_1 := (L_0(\lambda)u_1, L_1u_1, L_2u_1L_3, u_1L_4u_1)$,

$L_0(\lambda)$, L_1, L_2, L_3, L_4 are defined by equalities (3.1)-(3.2), and

$$\mathrm{L}_1u_1 = (-\lambda u_1(x) + \varphi_1(x)u_1(x), 0, 0)$$

$$\mathrm{L}_2u_2 = (-\lambda u_2(x) + \varphi_2(x)u_2(x), 0, 0).$$

We can conclude from Theorem 4.1 that $\mathrm{L}_0(\lambda_0)$ from $W_p^2((-1,0,1); H(A_k), H)$ onto $L_p((-1,1);H) + (H(A_1),H)_{\theta_{11},p} + (H(A_1),H)_{\theta_{21},p} + (H(A_2),H)_{\theta_{12},p} + (H(A_2),H)_{\theta_{22},p})$ has an inverse. From (5.7), (5.9) it follows that the operator L_{1k} from $W_p^2((-1,0,1); H(A_k), H)$ into $L_p((-1,1);H) \dot{+} (H(A_1),H)_{\theta_{11},p} \dot{+} (H(A_1),H)_{\theta_{21},p} \dot{+} (H(A_2),H)_{\theta_{12},p} \dot{+} (H(A_2),H)_{\theta_{22},p}$ is compact. The proof of the theorem is completed in view of application by Theorem 2.3 to operator (5.7).

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