ISSN 2347-1921
Volume 13 Number 2
Journal of Advance in Mathematics

Bayesian One- Way Repeated Measurements Model Based on Bayes Quadratic Unbiased Estimator

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ABSTRACT

In this paper, bayesian approach based on Bayes quadratic unbiased estimator is employed to the linear one- way repeated measurements model which has only one within units factor and one between units factor incorporating univariate random effects as well as the experimental error term.

The prior information obtained by using variance analysis technique to represent prior estimates of the parameters of the model. Then, the prior distribution is considered as a uniform distribution.

Keywords

Repeated Measurements Model; ANOVA; Mixed Model, Bayes Quadratic Unbiased Estimator; prior distribution; loss function,risk function .

1.INTRODUCTION

The repeated measures model had its origins in analysis of variance , so the term repeated measures ANOVA is often used to refer to the generic design . In fact , the independent variables in repeated measures can be any combination of categorical and continuous variables . Repeated measurements occur frequently in observational studied which are longitudinal in nature , and in experimental studies incorporating repeated measures design . For longitudinal studies , the underlying metameter for the occasions at which measurements are taken is usually time. Here , interest often centers around modeling the response as a linear or nonlinear function of time . Repeated measures designs , on the other hand, entail one or more response variables being measured repeatedly on each individuals over arrange of conditions. Here the metameter may be time or it may be set of experimental conditions (e.g., dose levels of a drug) . Repeated measurements model has been investigated by many researchers as Vonesh and Chinchilli in (1997) discussed the univariate repeated measurements model , analysis of variance model (RM ANOVA),[10]. Al-Mouel in (2004) studied the multivariate repeated measures models and comparison of estimators,[1]. Al-Mouel and Wang in (2004) presented the sphericity test for the one-way multivariate repeated measurements analysis of variance model . They studied the asymptotic expansion of the sphericity test for the one –way multivariate repeated measurements analysis of variance model, [2].Mohaisen and Swadi in (2014) studied Bayesian approach based on Markov Chain Monte Carlo, which is employed to making inferences on the one-way repeated measurements model, bayesian approach is employed to making inferences on the one- way repeated measurements model as mixed model and they investigate the consistency property of Bayes factor for testing the fixed effects linear one- way repeated measurements model against the mixed one- way repeated measurements alternative model. Under some conditions on the prior and design matrix, they identified the analytic form of the Bayes factor and showed that the Bayes factor is consistent,[4],[5],[6],[7] .

Variance covariance components play an important role in many fields of sciences. There are several procedures to estimate variance components.Rao in (1971) has introduced the minimum norm quadratic estimation (MINQUE) method. In this approach a quadratic estimator is sought that satisfies the minimum norm criterion,[8]. Stuchly in (1989) presentedthe necessary and sufficient conditions forthe existence and an explicit expression for the Bayes invariant quadratic unbiased estimate of the linear function of the variance components for the mixed linear model with three unknown variance components in the normal case, [9]. Witkovsky in (1996) has considered (MINQUE) approach with first order autoregressive disturbances,[11]. Fathy in (2006) presented Bayes invariant quadratic unbiased estimator (BAIQUE) . he used Bayesian approach to estimate the covariance functions of the regionalized variables which appear in the spatial covariance structure in mixed linear model,[3].

This paper deals with Bayesian estimation of one-way repeated measurements model. Bayesian approach based on Bayes quadratic unbiased estimator of the linear one- way repeated measurements model which has only one within units factor and one between units factor incorporating univariate random effects as well as the experimental error term. The prior information obtained by using variance analysis technique to represent prior estimates of the parameters of the model. Then, the prior distribution is considered as a uniform distribution.

2.One- Way Repeated Measurements Model

Consider the model

$$y_{ijk} = \mu + \tau_j + \delta_{i(j)} + \gamma_k + (\tau\gamma)_{jk} + e_{ijk} \tag{1}$$

Where

i=1,…., n is an index for experimental unit within group j ,

j=1,…,q is an index for levels of the between-units factor (Group) , k=1,…,p is an index for levels of the within-units factor (Time) ,

y_{ijk} is the response measurement at time k for unit i within group j,

μ is the overall mean ,

τ_j is the added effect for treatment group j ,

$\delta_{i(j)}$ is the random effect for due to experimental unit i within treatment group j ,

γ_k is the added effect for time k ,

$(\tau\gamma)_{jk}$ is the added effect for the group j × time k interaction ,

e_{ijk} is the random error on time k for unit i within group j ,

For the parameterization to be of full rank, we imposed the following set of conditions

$\sum_{j=1}^{q} \tau_{j=0}$, $\sum_{k=1}^{p} \gamma_{k=0}$, $\sum_{j=1}^{q} (\tau\gamma)_{jk=0}$ for each k=1,…,p

$\sum_{k=1}^{p} (\tau\gamma)_{jk=0}$ for each j=1,…,q

And we assumed that the e_{ijk} and $\delta_{i(j)}$ are indepndent with

$e_{ijk} \sim \text{i.i.d } N(0,\sigma_\varepsilon^2)$, $\delta_{i(j)} \overset{iid}{\sim} N(0,\sigma_\delta^2)$ (2)

Sum of squares due to groups, subjects(group), time, group*time and residuals are then defined respectively as follows:

$SS_G = np\sum_{j=1}^{q}\left(\bar{y}_{.j.} - \bar{y}_{...}\right)^2$, $SS_{U(G)} = p\sum_{i=1}^{n}\sum_{j=1}^{q}(\bar{y}_{ij.} - \bar{y}_{.j.})^2$

$SS_{time} = nq\sum_{k=1}^{p}(\bar{y}_{..k} - \bar{y}_{...})^2$, $SS_{G\times time} = n\sum_{j=1}^{q}\sum_{k=1}^{p}(\bar{y}_{.jk} - \bar{y}_{.j.} - \bar{y}_{..k} + \bar{y}_{...})^2$

$$SS_E = \sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk} - \bar{y}_{.jk} - \bar{y}_{ij.} + \bar{y}_{.j.})^2$$

Where

$\bar{y}_{...} = \dfrac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p} y_{ijk}}{nqp}$ is the overall mean.

$\bar{y}_{.j.} = \dfrac{\sum_{i=1}^{n}\sum_{k=1}^{p} y_{ijk}}{np}$ is the mean for group j.

$\bar{y}_{ij.} = \dfrac{\sum_{k=1}^{p} y_{ijk}}{p}$ is the mean for the i^{th} subject in group j.

$\bar{y}_{..k} = \dfrac{\sum_{i=1}^{n}\sum_{j=1}^{q} y_{ijk}}{nq}$ is the mean for time k.

$\bar{y}_{.jk} = \dfrac{\sum_{i=1}^{n} y_{ijk}}{n}$ is the mean for group j at time k.

Table 1 : ANOVA table for one-way Repeated measures model

Source of variation	d.f	SS	MS	E(MS)
Group	$q-1$	SS_G	$\frac{SS_G}{q-1}$	$\frac{np}{(q-1)}\sum_{j=1}^{q}\tau_j^2 + p\sigma_\delta^2 + \sigma_\varepsilon^2$
unit (Group)	$q(n-1)$	$SS_{U(G)}$	$\frac{SS_{U(G)}}{q(n-1)}$	$p\sigma_\delta^2 + \sigma_\varepsilon^2$
Time	$p-1$	SS_{time}	$\frac{SS_{time}}{p-1}$	$\frac{nq}{(p-1)}\sum_{k=1}^{p}\gamma_k^2 + \sigma_\varepsilon^2$

Group*Time	$(q-1)(p-1)$	$SS_{G\times time}$	$\frac{SS_{GT}}{(q-1)(p-1)}$	$\frac{n}{(p-1)(q-1)}\sum_{j=1}^{q}\sum_{k=1}^{p}(\tau\gamma)^2_{jk}+\sigma^2_{\varepsilon}$
Residual	$q(p-1)(n-1)$	SS_E	$\frac{SS_E}{q(p-1)(n-1)}$	σ^2_e

We can rewrite the model (1) as a mixed model as following

$$Y = X\beta + Zb + \epsilon \tag{3}$$

Where

$$Y=\begin{bmatrix} y_{111}\\ y_{112}\\ \vdots \\ y_{nqp}\end{bmatrix}_{nqp\times 1} \quad , \quad \beta=\begin{bmatrix}\mu\\ \tau\\ \gamma\\ (\tau\gamma)\end{bmatrix}_{(qp+q+p+1)\times 1} \quad , \quad Z=\begin{pmatrix}1_{P\times 1} & \cdots & 0_{P\times 1}\\ \vdots & \ddots & \vdots\\ 0_{P\times 1} & \cdots & 1_{P\times 1}\end{pmatrix}_{nqp\times nq} ,$$

$$b=\begin{bmatrix}\delta_{1(1)}\\ \delta_{1(2)}\\ \vdots\\ \delta_{n(q)}\end{bmatrix}_{nq\times 1} \quad , \quad \epsilon = \begin{bmatrix} e_{111}\\ e_{112}\\ \vdots\\ e_{nqp}\end{bmatrix}_{nqp\times 1} ,$$

and design matrix X is an $\big(nqp * (qp+q+p+1)\big)$matrix.. And we assumed that

$e\sim N\big(0,\sigma^2_\epsilon I_{nqp}\big)$, $b\sim N\big(0,\sigma^2_\delta I_{nq}\big)$, $Cov(b,\epsilon)=0$ (4)

3.Bayes Quadratic Unbiased Estimator

From (3) we have

$Y \sim N\big(X\beta, \sigma^2_\epsilon I_{nqp} + \sigma^2_\delta I_{nq}\big)$,(5)

we can write the model (3) as

$$MY = MX\beta + MZb + M\epsilon , \tag{6}$$

where M are known $(nqp \times nqp)$ projection matrix and

$$M = I - X(X'X)^{-1}X' . \tag{7}$$

Then

$$MX = (I - X(X'X)^{-1}X')X = X - (X'X)^{-1}X'X$$

$= X - X.I = X - X = O$.

According to property projection matrix M we have

$$MY = MZb + +M\epsilon . \tag{8}$$

Let $MY = H$, then

$$H = MZb + M\epsilon . \tag{9}$$

Since$E(H)=0,\ E(\epsilon)=0 \to E(M\epsilon)=ME(\epsilon)=0$and

$$Var(H) = MZVar(b)(MZ)' + MVar(\epsilon)M', \tag{10}$$

let $MZ = G$

$$\therefore Var(H) = GVar(b)G' + MVar(\epsilon)M'$$

$$= G(\sigma^2_\delta I_{nq})G' + M(\sigma^2_\epsilon I_{nqp})M' = \sigma^2_\delta GG' + \sigma^2_\epsilon MM'. \tag{11}$$

Since the matrix M symmetric and idempotent i.e

$$M = M' = M^2$$

$\therefore Var(H) = \sigma^2_\delta GG' + \sigma^2_\epsilon M$.(12)

Let $V_1 = GG'$ and $V_2 = M$

$\rightarrow Var(H) = \sigma_\delta^2 V_1 + \sigma_\epsilon^2 V_2$, (13)

we can write (13) as

$Var(H) = \theta_1 V_1 + \theta_2 V_2$, where $\theta_1 = \sigma_\delta^2$ and $\theta_2 = \sigma_\varepsilon^2$. Let $\sum(\theta) = \theta_1 V_1 + \theta_2 V_2$,

$\rightarrow Var(H) = \sum(\theta)$(14)

where $\sum(\theta)$ is an $(nqp \times nqp)$ symmetric matrix. The parameters θ_1 and θ_2 are estimate by linear function estimation which proposed by Rao (1974) as follows:

$\alpha(H) = \ell_1\theta_1 + \ell_2\theta_2 = \acute{\ell}\theta$, (15)

where $\ell = (\ell_1, \ell_2)\acute{}$ and $\theta = (\theta_1, \theta_2)\acute{}$. The linear function $\alpha(H)$estimated by the following quadratic form:

$\hat{\alpha} = \acute{H}AH$,

where *A* is an $(nqp \times nqp)$ symmetric matrix which must be found from the data. If we have the prior distribution function $g(\theta)$ of the parameter θ , then the quadratic loss function is:

$L(\hat{\alpha}, \alpha) = (\hat{\alpha} - \alpha)^2$. (16)

And the risk function is:

$R(\hat{\alpha}, \alpha) = E\big(L(\hat{\alpha}, \alpha)\big) = E[(\hat{\alpha} - \alpha)^2]$, (17)

then the Bayes risk function $B(\hat{\alpha})$ takes the following form:

$$B(\hat{\alpha}) = E_\theta[R(\hat{\alpha}, \alpha)] = E_\theta[E(\hat{\alpha} - \alpha)^2]$$

$B(\hat{\alpha}) = \int_{\theta\in\Omega} R(\hat{\alpha}, \alpha) g(\theta) d\theta$

$= \int_{\theta\in\Omega} E_\theta(\hat{\alpha} - \alpha)^2 g(\theta) d\theta$, $\Omega = \{\theta: \theta_1, \theta_2 > 0\}$ (18)

THEOREM (1):

$\hat{\alpha}$ satisfies the unbiasedness condition i.e $E(\hat{\alpha}) = \alpha$.

Proof:

$E(\hat{\alpha}) = E\big(\acute{H}AH\big) = E(tr(\acute{H}AH) = E\big(trAH\acute{H}\big) = trAE(H\acute{H})$

$$= trA(Var(H)) = trA\sum(\theta)$$

$$= trA(\theta_1 V_1 + \theta_2 V_2) = trA\sum_{i=1}^{2}\theta_i V_i = \sum_{i=1}^{2}\theta_i trAV_i$$

$\hat{\alpha}$ be unbiased if and only if

$trAV_i = \ell_i \qquad i = 1,2$ (19)

$$\therefore E(\hat{\alpha}) = \sum_{i=1}^{2}\ell_i\theta_i = \acute{\ell}\theta = \alpha \ \blacksquare$$

THEOREM (2):

$\hat{\alpha}$ satisfies theminimizing property of the Bayes risk function.

Proof:

From (18)

$$B(\hat{\alpha}) = \int_{\theta\in\Omega} E_\theta(\hat{\alpha} - \alpha)^2 g(\theta) d\theta$$

$= E_\theta[E(\hat{\alpha} - \alpha)^2] = E_\theta[E(\hat{\alpha} - E(\hat{\alpha}))^2] = E_\theta[Var(\hat{\alpha})] = E_\theta\big[Var(\acute{H}AH)\big]$

$$= E_\theta\big[2trAVar(\acute{H})AVar(H)\big] = E_\theta\left[2trA\sum(\theta)A\sum(\theta)\right]$$

$$= E_\theta\left[2trA\sum_{i=1}^{2}V_i\theta_i A\sum_{j=1}^{2}V_j\theta_j\right] = E_\theta\left[2\sum_{i=1}^{2}\sum\nolimits_{j=1}^{2}\theta_i\theta_j\,trAV_iAV_j\right]$$

$\rightarrow B(\hat{\alpha}) = 2\sum_{i=1}^{2}\sum_{j=1}^{2}E(\theta_i\theta_j)trAV_iAV_j$ (20)

Where $E(\theta_i\theta_j)$ is the second moment for θ_i, and the matrix of the second moment is

$E(\theta\acute{\theta}) = C = E(\theta_i\theta_j) = Var(\theta) + E(\theta)E(\acute{\theta})$, $i,j = 1,2$

The matrix C is

$$C = (C_{ij}) = (\sum_{k=1}^{2} r_{ik}r_{kj}) \quad , \quad i,j = 1,2 \tag{21}$$

i.e

$$C = \sqrt{C}\sqrt{C} = RR$$

$$\rightarrow B(\hat{\alpha}) = 2\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} r_{ik}r_{kj}\,trAV_iAV_j = 2\sum_{k=1}^{2} trA\left[\sum_{i=1}^{2} r_{ik}V_i\right]A\left[\sum_{j=1}^{2} r_{kj}V_j\right]$$

$\therefore B(\hat{\alpha}) = 2\sum_{k=1}^{2} trAT_kAT_k$(22)

where$T_k = \sum_{i=1}^{2} r_{ik}V_i$, $k = 1,2.$

To solve (22) use Lagrange method according to unbiasedness condition to get minimum value assume that

$$S = 2\sum_{k=1}^{2} trAT_kAT_k + 4\sum_{i=1}^{2}\lambda_i(trAV_i - \ell_i) \tag{23}$$

Where λ_i is Lagrange Multipliers.

Derive (23) for the A, and by equality to zero get:

$$\frac{dS}{dA} = 4\sum_{k=1}^{2} T_kAT_k + 4\sum_{i=1}^{2}\lambda_iV_i = 0$$

$$= \sum_{k=1}^{2} T_kAT_k + \sum_{i=1}^{2}\lambda_iV_i = 0 \tag{24}$$

Then derive (23) for λ_i, and by equality to zero get:

$$\frac{dS}{d\lambda_i} = 4(trAV_i - \ell_i) = 0$$

$\therefore trAV_i = \ell_i$, $i = 1,2$ (25)

Now by using system of linear equations get

$[\sum_{k=1}^{2} T_k \otimes T_k]VecA + \sum_{i=1}^{2}\lambda_i VecV_i = 0$(26)

$(VecV_i)\acute{}VecA = \ell_i$, $i = 1,2$. (27)

We can write (26) as

$WVecA + \sum_{i=1}^{2}\lambda_i VecV_i = 0$, (28)

where $W = \sum_{k=1}^{2} T_k \otimes T_k$.

Then we can write the system (27) and (28) as

$$\begin{bmatrix} VecV_1 & VecV_2 & W \\ O & O & Vec\acute{V_1} \\ O & O & Vec\acute{V_2} \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ VecA \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \ell_1 \\ \ell_2 \end{bmatrix}, (29)$$

and by using matrices form

$GO = B$, (30)

where

G is an $(nqp^2+2)\times(nqp^2+2)$ system matrix,

O is an $(nqp^2+2)\times 1$ unknown values vector in the system,

B is an $(nqp^2+2)\times 1$ constants vector in the system,

$\rightarrow O = G^{-1}B$ (if G reversible) (31)

4.Prior distribution

Assuming that little is known about variance components for one-way repeated measurement model, it makes sense to put an improper uniform prior on this components, then we will assume that the prior distribution on θ_1 and θ_2 are

$\boldsymbol{\theta_1 \sim u(a_1, b_1)}$, $\boldsymbol{\theta_2 \sim u(a_2, b_2)}$and $\theta_1$andθ_2 are independent, then *Cov*$(\theta_1,\theta_2)=0$,

$$\therefore C = \begin{bmatrix} \frac{b_1+a_1}{2} \\ \frac{b_2+a_2}{2} \end{bmatrix} \begin{bmatrix} \frac{b_1+a_1}{2} & \frac{b_2+a_2}{2} \end{bmatrix} + \begin{bmatrix} \frac{(b_1-a_1)^2}{12} & 0 \\ 0 & \frac{(b_2-a_2)^2}{12} \end{bmatrix} \quad (32)$$

5.Example (The storage experiment)

In this section, we illustrate the effectiveness of the our methodology. We have choosing the data set which an experiment was conducted during winter season 2008-2009 in one of unhted plastic house which belong to tomato development project in Basrah / Agriculture directorate of Basrah (Khor Al – zubiar) in order to investigate the effect of calcium on growth and yield of cultivars of cucumber (Sayff) and the storage temperature on storage capability and quality. The field experiment included 84 variable treatments which were the interaction of three factors of three storage temperatures, which were room temperature, 5$C^{\circ}$and 12 C° and two concentration of calcium chloride 0, 1, 2, 3 % and three storage period 0 , 10 , 15 , 20 , 25 , 30 day. After harvest, the fruit was treated with $CaCl_2$. $2H_2O$ for five minute, then it was stored in three temperatures. Then the chemical changes characteristics of the fruit was reviewed during the storage at 5 days periods. the design of the experiment was done according to the model (1). Table (2) below show the results for the analysis of variance for model, from this table we can see that the calculated F-values is greater than the tabulated F-values at 0.05 level significant that is means there is significant effectfor calcium chloride on storage capability for cucumber fruits under different temperatures. The values of variance components for the model (1) based on ANOVA are $\theta_1=\sigma_1^2=1.6431$ and $\theta_2=\sigma_2^2=0.2515$. Then the prior distribution of θ_1 and θ_2 are

$\pi_0(\theta_1)=\frac{1}{1.6431}$ =0.6086 $\quad 0<\theta_1<1.6431$

$\pi_0(\theta_2)=\frac{1}{0.2515}$ =3.9761 $\quad 0<\theta_2<0.2515$

Table 2 : ANOVA table for one-way Repeated measures model

Source of variation	d.f	SS	MS	E(MS)	F-Test
Group	2	5.9874	2.9937	4.6368	$F_c=\frac{MS_G}{MS_{U(G)}}=8.9498$*
Unit (Group)	9	3.0102	0.3345	1.6431	F_t(2,9,0.05)=4.26
Time	6	13.8254	2.3042	2.5557	$F_c=\frac{MS_T}{MS_E}=9.1618$* F_t(6,34,0.05)=2.36
Group*Time	12	9.2196	0.7683	1.0198	$F_c=\frac{MS_{G\times T}}{MS_E}=3.0549$* F_t(12,34,0.05)=2.03
Residual	34	8.5518	0.2515	0.2515	
Total	63	40.5944			

The matrix C by using (24) is

$$C=\begin{bmatrix} 0.9602 & 0.1067 \\ 0.1067 & 0.0211 \end{bmatrix}.$$

We next applied our methodology Bayes quadratic unbiased estimator to the storage experiment data. The values of variance components for the one-way repeated measurement model based on this method are $\theta_1 = \sigma_1^2 = 1.601$ and $\theta_2 = \sigma_2^2 = 0.2312$. Then we can see that the values of variance components for the one-way repeated measurement model obtained in both ANOVA and Bayes quadratic unbiased estimator are nearly alike and encouraging.

6.CONCLUSIONS

The conclusions obtained throughout this paper are as follows:

(1) The linear function $\alpha(H)$estimated by the quadratic form $\hat{\alpha} = \acute{H}AH$ where $\hat{\alpha}$ satisfies the unbiasedness condition i.e. $E(\hat{\alpha}) = \alpha$ and $\hat{\alpha}$ satisfies theminimizing property of the Bayes risk function.

(2) There is significant effectfor calcium chloride on storage capability for cucumber fruits under different temperatures.

(3) The values of variance components for the model (1) based on ANOVA to the storage experiment data are $\theta_1 = \sigma_1^2 = 1.6431$ and $\theta_2 = \sigma_2^2 = 0.2515$.

(4) The values of variance components for the one-way repeated measurement model based on the Bayes quadratic unbiased estimator method to the storage experiment data are $\theta_1 = \sigma_1^2 = 1.601$ and $\theta_2 = \sigma_2^2 = 0.2312$.

(5) The values of variance components for the one-way repeated measurement model obtained in both ANOVA and Bayes quadratic unbiased estimator are nearly alike and encouraging.

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