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# BIFURCATION IN SINGULARITY PARAMETERIZED ODE$_S$

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## ABSTRACT

In this paper , we will study differential algebraic equations (DAEs) through studying singularly perturbed ODEs. That's the ODEs will be transformed to an DAEs when the perturbed parameter approach to 0. This will permit us to apply the classical bifurcation theory of ODEs for the new system (DAEs). So we will show by giving theorems, sufficient conditions for fold, pitchfork and transcritical bifurcation to be occurred in (DAEs). An illustrative example is given.

## Keywords

Singularly perturbed ODE$_S$, Differential Algebraic Equations, Bifurcation theory.

## 1.INTRODUCTION

The perturbation theory in mathematics refers to the methods that studying how to find an approximate solution to a problem, by starting from the exact solution of related problem [3]. Perturbation theory problem are classified as regular or singular. A regular problem is one for which a simple asymptotic expansion can be found with the property that the expansion is uniform in the independent variable [6]. A Singular problem of a dynamical system refers to the derivatives of some of the states are multiplied by a small positive parameter $\epsilon$ [5]. In this paper we will introduce the singularity perturbed ODE which has the standard form:

$$\dot{x} = f(x, y, \epsilon), \qquad (1.1)$$

$$\epsilon\dot{y} = g(x, y, \epsilon). \qquad (1.2)$$

The theory of singular perturbations has been studied for a little over a century (from the 1940s). In singular perturbation theory, a similar reduction can be seen as the limiting system as some dynamics become arbitrarily fast which studied by[Kokotovic et al., 1986]. It studied the mathematical problems that make extensive use of a small parameter were probably first described by J. H. Poincare (1854 − 1912) and Stieltjes (1886). The aim of Singular Perturbation problem is to use the limiting behavior of the system, when $\epsilon$ approach to 0 [3] [6], to get an idea of what the system looks like when perturbation parameter $\epsilon$ is small.This work is devoted to investigate the sufficient conditions for the bifurcation to beoccurred in singularity perturbed ODEs when a $\epsilon$ pproach to zero. Then this impliesto study the bifurcation of solution for differential algebraic equations(DAEs). Thebifurcation theory will study two cases that's when DAEs is of index one and indextwo.

This paper is organized as follows: In Section 2 we introduce some definitions and concepts on singularity perturbed ODEs and explain the method that we used in the singularity perturbed ODEs. Section 3 will illustrate the relationship between the singularity perturbed ODEs and the differential algebraic equations also bifurcation theory on the singularity perturbed ODEs when the perturbed parameter $\epsilon$approach to 0 will be studied.

## 2.DEFINITIONS AND CONCEPTS

### Definition 2.1. [2][5](A Fast-Slow Vector Field (or (m, n)-Fast-Slow System))

Is a system of ordinary differential equations taking the form:

$$\frac{dx}{dt} = x' = \epsilon f(x, y, \epsilon), \qquad (2.1)$$

$$\frac{dy}{dt} = y' = g(x, y, \epsilon). \qquad (2.2)$$

Where $f: R^m \times R^n \times R \to R^m, g: R^m \times R^n \times R \to R^n$, (A prime $'$ denotes differentiation with respect to t). Furthermore, the $x$ variables are called fast variables, and the $y$ variables are called slow variables.

Here we will state a simple example that illustrate the perturbation method on the system of algebraic equation.

Setting $t = \frac{\tau}{\epsilon} \implies \tau = \epsilon t$

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} \Rightarrow \frac{dx}{d\tau} = \frac{dx}{dt}\frac{1}{\epsilon} = f(x, y, \epsilon),$$

and

$$\frac{dy}{d\tau}=\frac{dy}{dt}\frac{dt}{d\tau}\Longrightarrow\frac{dy}{d\tau}=\frac{1}{\epsilon}\frac{dy}{dt}=\frac{1}{\epsilon}g(x,y,\epsilon)\Longrightarrow\epsilon\frac{dy}{d\tau}=g(x,y,\epsilon),$$

that gives the equivalent form:

$$\frac{dx}{d\tau}=\dot{x}=f(x,y,\epsilon), \qquad (2.3)$$

$$\frac{dy}{d\tau}=\epsilon\dot{y}=g(x,y,\epsilon), \qquad (2.4)$$

where (A dot $\dot{}$ denotes differentiation with respect to $\tau$). It refers to t as the fast time scale or fast time and to $\tau$as the slow time scale or slow time.

## 2.1 REDUCED SYSTEM WITH INITIAL CONDITION

Now we will apply the following (reduced system with initial condition) method on the system (2.3),(2.4) in order to get an DAEs which is the reduced model when ϵ approach to 0. We know that we cannot directly integration on the DAEs form, so we will use the implicit function theorem to eliminate fast variable y to get a reduced model in terms of slow dynamics x. Consider the standard form of the singularity perturbed ODEs as follows:

$$\dot{x}=f(x,y,\epsilon)(2.5)$$

$$\dot{y}=\epsilon g(x,y,\epsilon)(2.6)$$

$$x(0)=x_0, y(0)=y_0$$

where $(x,y)\in R^m\times R^n$ and $0<\epsilon\ll 1$ is a small parameter .

If $f(.,.,\epsilon)$ and $g(.,.,\epsilon)$ are globally Lipschitz and uniformly bounded in $\epsilon$ then $\dot{y}$ will be of order $\frac{1}{\epsilon}$ faster than $\dot{x}$. Accordingly, we call x the slow variables and y the fast variable

of the system. Now, we may choose $(x_0,y_0)$ such that $g(x_0,y_0,0)=0$. Moreover, the corresponding reduced problem when $\epsilon$approach to 0consists of the nonlinear differential-algebraic system which is:

$$\dot{x}=f(x,y,0), \qquad (2.7)$$

$$0=g(x,y,0), \qquad (2.8)$$

$$x(0)=x_0, y(0)=y_0.$$

By construction, the dynamics remain on the invariant manifold which define as follows:

$$S=\{(x,y)\in R^m\times R^n: g(x,y,0)=0\} \qquad (2.9)$$

The reduced problem could provide the limiting solution on $0<t<1$ if the corresponding limiting the problem

$$\frac{d\delta}{d\tau}=g(x(0),\delta,0), \qquad \delta(0)=y_0,$$

had a bounded solution $\delta(\tau)$ for all $\tau>0$ which matched $y$ in the sense that $\delta(\infty)=y_0$.

If $\frac{\partial g}{\partial y}$is nonsingular along the solution $(x,y)$ of the reduced problem, we can differentiate the algebraic constraint $g(x,y,0)=0$ we get:

$$\frac{\partial g}{\partial x}f+\frac{\partial g}{\partial y}\dot{y}=0.$$

Thus, $y$ must satisfy the initial value problem

$$\dot{y}=-(\frac{\partial y}{\partial x})^{-1}(x,y,0)\left[\frac{\partial g}{\partial x}(x,y,0)f(x,y,0)\right], \qquad \delta(0)=y_0,$$

which is coupled to the remaining initial value problem

$$\dot{x}=f(x,y,0), \qquad x(0)=x_0.$$

According to implicit function theorem, where the algebraic variables $y$ have been eliminated. This yields an equation $y=h(x)$, where $h\in C^I(R^n)$is a function defined on x that can plug into the first equation (2.8), by which we find

$$\dot{x}=f(x,h(x),0), \qquad x(0)=x_0. \qquad (2.10)$$

The condition on the initial values may be relaxed if the fast subsystem is strictly hyperbolic. Let $f(x,h(x),0)=F(x)$, then our system will become:

$$\dot{x}=F(x,\alpha), \qquad x(0)=x_0, \qquad (2.11)$$

where $F$is a locally unique smooth function [7], $x \in X \subset R^m$, parameter $\alpha$refers to the bifurcation parameter, $x$ denoted to the dynamic state (slow modes). The following example will explain applying of the reduce system with initial condition method.

**Example 2.1.**Consider the slow system:

$$\dot{x} = xy, \tag{2.12}$$

$$\epsilon\dot{y} = -y - x^2, \tag{2.13}$$

with invariant manifold:

$$S_0 = \{(x, y) \in R^m \times R^n : -y - x^2 = 0\} \tag{2.14}$$

When $\epsilon$ approach to 0 we have:

$$\dot{x} = xy,$$

$$0 = -y - x^2,$$

and $\frac{\partial g}{\partial y} = -1 \neq 0$, which is nonsingular. Then by implicit function theorem we get:

$$y = x^2,$$

then

$$\dot{x} = x^3.$$

## 3. Singularity Parameterized ODEs and Differential Algebraic Equations

Consider the singularity parameterized ODEs which has the following form:

$$\dot{x} = f(x, y, \alpha, \epsilon),$$

$$\epsilon\dot{y} = g(x, y, \alpha, \epsilon), \tag{3.1}$$

where $x \in R^m$ and $y \in R^n$, $x$ is a slow variable and $y$ is a fast variable, $0 < \epsilon \ll 1$. When $\epsilon$approach to $0$ we have DAEs as follows:

$$\dot{x} = f(x, y, \alpha, 0)(3.2)$$

$$0 = g(x, y, \alpha, 0) \tag{3.3}$$

Where $f: R^m \times R^n \times R \longrightarrow R^m, g: R^m \times R^n \times R \longrightarrow R^n$ and $\alpha$is the bifurcation parameter. Note that the critical point of (3.2),(3.3) should satisfy the constraint condition.

Suppose that $G$ be a set of all equilibrium points of (3.2),(3.3) defines as follows:

$$G = \{(x, y, \alpha, 0) \in R^m \times R^n \times R \times R: f(x, y, \alpha, 0) = 0 = g(x, y, \alpha, 0)\},$$

and we define the function $F$ as follows:

$$F(x, y, \alpha, 0) = \begin{pmatrix} f(x, y, \alpha, 0) \\ g(x, y, \alpha, 0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.4}$$

where $F$ is equivalent to $G$.

Now we will study the bifurcation kinds on the DAEs that we obtain from the singularity parameterized ODEs when $\epsilon$approach to 0.

### 3.1 Bifurcation Theory

In this section we will study the bifurcation theory in the singularity parameterized ODEs. First we will state the bifurcation definition as follows:

**Definition 3.1. [8]**The appearance of a topologically nonequivalent phase portrait under variation of parameters is called a bifurcation. Thus, bifurcation is a complex phenomena occurs in nonlinear systems, it is refers to the branching of solutions at some critical value parameters, which results in a loss of the structural stability and it is one of routes to chaos [1]. Here we will state the bifurcation kinds such as fold, pitchfork and transcritical bifurcation.

### 3.2 Fold Bifurcation in Singularity Parameterized ODEs

A fold bifurcation point is a pair of equilibria, meets and disappears with a zero eigenvalue [4]. One of the equilibria (saddle) is unstable while the other (node) is stable [7]. Now, consider the DAEs (3.2), (3.3). If we differentiate the constraint equation $0 = g(x, y)$ one time w.r.t $x$ and $y$ we get:

$$\frac{\partial g}{\partial x}\dot{x} + \frac{\partial g}{\partial y}\dot{y} = 0 \qquad (3.5)$$

since $\frac{\partial g}{\partial x}(0) = 0$ and $\frac{\partial g}{\partial y}(0) \neq 0$ (i.e.) has an inverse, then from (3.6) we get that $\frac{dy}{dt}(0) = 0$, that implies $\frac{dy}{dx}(0) = 0$, which is resulting from ( chain rule).

Now we will study fold bifurcation of the singularity parameterized ODEs system by the following theorem:

**Theorem 3.1.** Consider the singularity parameterized ODEs (3.1) defined on the set of critical points $G$ with an equilibrium point $(0,0,0,0)$. If the following conditions are holds:

**(1)** $\boldsymbol{\frac{\partial F}{\partial \alpha}(0,0,0,0) \neq 0}$,

**(2)** $\boldsymbol{\frac{\partial^2 F}{\partial x^2} \neq 0}$,

then $(0,0,0,0)$ is a fold bifurcation point for the singularity parameterized ODEs (3.1) when $\epsilon$approach to $0$.

**Proof.** By implicit function theorem we get a function $\alpha(x,y)$. So we rewrite the equation $F(x,y,\alpha,0) = 0$ as follows:

$$F(x,y,\alpha(x,y),0) = 0 \qquad (3.6)$$

Here we will differentiate the equation (3.6) w.r.t $x$ we get:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial \alpha}\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial y}\frac{dy}{dx}\right) = 0.$$

Evaluate the above equation at $(0,0,0,0)$ and substitute that $\frac{dy}{dx}(0) = 0$we get that:

$$\frac{\partial f}{\partial x}(0,0,0,0) + \frac{\partial f}{\partial \alpha}(0,0,0,0)\frac{\partial \alpha}{\partial x}(0,0,0,0) = 0,$$

Or

$$\frac{\partial g}{\partial x}(0,0,0,0) + \frac{\partial g}{\partial \alpha}(0,0,0,0)\frac{\partial \alpha}{\partial x}(0,0,0,0) = 0.$$

By substituting the condition (1) with the non-hyperbolic condition $\frac{\partial f}{\partial x}(0,0,0,0) = 0$ we have:

$$\frac{\partial \alpha}{\partial x}(0,0,0,0) = \frac{\partial f}{\partial x}(0,0,0,0)\left(\frac{\partial f}{\partial \alpha}(0,0,0,0)\right)^{-1} = 0.$$

Applying the same way w.r.t $g$we can prove the first property $\frac{\partial \alpha}{\partial x}(0) = 0$.

Now we will prove the second property of the fold bifurcation to be occurred. By differentiate (3.6) w.r.t $x$ twice we get:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial x \partial y}\frac{dy}{dx} + \frac{\partial^2 F}{\partial x \partial \alpha}\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial y}\frac{dy}{dt}\right) + \frac{\partial F}{\partial y}\frac{d^2 y}{dx^2} + \left[\frac{\partial^2 F}{\partial y \partial x} + \frac{\partial^2 F}{\partial y^2}\frac{dy}{dx} + \frac{\partial^2 F}{\partial y \partial x}\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial y}\frac{dy}{dx}\right)\right]\frac{dy}{dx}$$
$$+ \frac{\partial F}{\partial \alpha}\left[\frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial x \partial y}\frac{dy}{dx} + \frac{\partial \alpha}{\partial y}\frac{d^2 y}{dx^2} + \left(\frac{\partial^2 \alpha}{\partial y \partial x} + \frac{\partial^2 \alpha}{\partial y^2}\frac{dy}{dx}\right)\frac{dy}{dx}\right] + \left[\frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial y}\frac{dy}{dx}\right]\left[\frac{\partial^2 F}{\partial \alpha \partial x} + \frac{\partial^2 F}{\partial \alpha \partial y}\frac{dy}{dx} + \frac{\partial^2 F}{\partial \alpha^2}\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial y}\frac{dy}{dx}\right)\right]$$
$$= 0.$$

Evaluate at $(0,0,0,0)$ and substitute the non-hyperbolic condition and condition (1) and (2) with application of the property that we proved $\frac{\partial \alpha}{\partial x}(0,0,0,0) = 0$ we get:

$$\frac{\partial^2 \alpha}{\partial x^2}(0,0,0,0) = -\left[\frac{\partial^2 f}{\partial x^2} + \left(\frac{\partial f}{\partial y}(0,0,0,0) + \frac{\partial f}{\partial \alpha}(0,0,0,0)\frac{\partial \alpha}{\partial y}(0,0,0,0)\right)\frac{d^2 y}{dx^2}(0)\right]\left(\frac{\partial f}{\partial \alpha}(0,0,0,0)\right)^{-1} \neq 0,$$

then

$$\frac{\partial^2 \alpha}{\partial x^2}(0,0,0,0) \neq 0.$$

Applying the same way w.r.t $g$ we can prove the second property of the fold bifurcation which is $\frac{\partial^2 \alpha}{\partial x^2}(0,0,0,0) \neq 0$, then $(0,0,0,0)$ is a fold bifurcation point for the singularity parameterized ODE system (3.1) when $\epsilon$approach to 0.

The above theorem apply the fold bifurcation conditions for index one DAEs which is resulting from the singularity parameterized ODEs when $\epsilon$approach to 0. If we differentiate the constraint $0 = g(x,y)$ twice we have:

$$\frac{\partial g}{\partial x}\ddot{x}+\left(\frac{\partial^2 g}{\partial x^2}\dot{x}+\frac{\partial^2 g}{\partial x\,\partial y}\dot{y}\right)\dot{x}+\frac{\partial g}{\partial y}\ddot{y}+\left(\frac{\partial^2 g}{\partial y\,\partial x}\dot{x}+\frac{\partial^2 g}{\partial y^2}\dot{y}\right)\dot{y}=0, \qquad (3.7)$$

If we assume that $\frac{\partial^2 g}{\partial x^2}(0)=0$ and $\frac{\partial^2 g}{\partial x\,\partial y}(0)=0$, are satisfied for the constraint of index 2 then we have:

$$\frac{\partial g}{\partial y}\ddot{y}+\left(\frac{\partial^2 g}{\partial y^2}\dot{y}\right)\dot{y}=0,$$

since $\frac{\partial g}{\partial y}(0)=0$ because the constraint of index 2, then we have:

$$\left(\frac{\partial^2 g}{\partial y^2}(0)\dot{y}\right)\dot{y}=0,$$

if $\frac{\partial^2 g}{\partial y^2}(0)\neq 0$, then

$$\frac{d^2 y}{dt^2}(0)=0,$$

which it give that:

$$\frac{dy}{dx}(0)=0.$$

Now the following theorem talking about fold bifurcation conditions for index two DAEs:

**Theorem 3.2.** Consider the singularity parameterized ODEs (3.1) defined on the set of critical points $G$ with an equilibrium point $(0,0,0,0)$. If the following conditions are holds:

**(1)** $\frac{\partial F}{\partial \alpha}(0,0,0,0)\neq 0$,

**(2)** $\frac{\partial^2 f}{\partial x^2}(0,0,0,0)\neq 0$,

then $(0,0,0,0)$ is a fold bifurcation point for the singularity parameterized ODEs (3.1) when $\epsilon$ approach to $0$.

**Proof.** By implicit function theorem we get a function $\alpha(x,y)$. So we rewrite the equation $F(x,y,\alpha,0)=0$ as follows:

$$F(x,y,\alpha(x,y),0)=0. \qquad (3.8)$$

Differentiate the equation (3.8) w.r.t $x$ and evaluate the equation at $(0,0,0,0)$ we get that:

$$\frac{\partial f}{\partial x}(0,0,0,0)+\frac{\partial f}{\partial \alpha}(0,0,0,0)\frac{\partial \alpha}{\partial x}(0,0,0,0)=0,$$

by substituting the condition (1) with the non-hyperbolic condition $\frac{\partial f}{\partial x}(0,0,0,0)=0$ we have:

$$\frac{\partial \alpha}{\partial x}(0,0,0,0)=\frac{\partial f}{\partial x}(0,0,0,0)(\frac{\partial f}{\partial \alpha}(0,0,0,0))^{-1}=0.$$

Now we will prove the second property of the fold bifurcation to be occurred.

We differentiate (3.8) w.r.t $x$ twice and evaluate at $(0,0,0,0)$ with applying of the property that we proved $\frac{\partial \alpha}{\partial x}(0,0,0,0)=0$ with condition (2) we get:

$$\frac{\partial^2 \alpha}{\partial x^2}(0,0,0,0)=-[\frac{\partial^2 f}{\partial x^2}+\left(\frac{\partial f}{\partial y}(0,0,0,0)+\frac{\partial f}{\partial \alpha}(0,0,0,0)\frac{\partial \alpha}{\partial y}(0,0,0,0)\right)\frac{d^2 y}{dx^2}(0)](\frac{\partial f}{\partial \alpha}(0,0,0,0))^{-1}\neq 0,$$

then

$$\frac{\partial^2 \alpha}{\partial x^2}(0,0,0,0)\neq 0.$$

Applying the same way w.r.t $g$ with applying the giving conditions $\frac{\partial^2 g}{\partial x^2}(0)=0$, $\frac{\partial^2 g}{\partial y^2}(0)\neq 0$ and $\frac{\partial^2 g}{\partial x\,\partial y}(0)=0$, we can prove the second property of the fold bifurcation which is $\frac{\partial^2 \alpha}{\partial x^2}(0,0,0,0)\neq 0$, Then $(0,0,0,0)$ is a fold bifurcation point for the singularity parameterized ODE system (3.1) when $\epsilon$ approach to 0.

### 3.3 Pitchfork Bifurcation in Singularity Parameterized ODEs

In the pitchfork bifurcation, an equilibrium point reverses its stability, and two new equilibrium points are born [4].

Define:

$$f(x,y,\alpha,\epsilon) = xV(x,y,\alpha,\epsilon),$$

$$g(x,y,\alpha,\epsilon) = xW(x,y,\alpha,\epsilon),$$

Where $V: R^m \times R^n \times R \longrightarrow R^m, W: R^m \times R^n \times R \longrightarrow R^n$ and $\alpha$ is the bifurcation parameter, $x \in R^m$ and $y \in R^n, 0 < \epsilon \ll 1$.

$$F(x,y,\alpha,0) = \begin{pmatrix} xV(x,y,\alpha,0) \\ xW(x,y,\alpha,0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

And $V(0,y,\alpha,0) = \frac{\partial f}{\partial x}$ at $x = 0$, $W(0,y,\alpha,0) = \frac{\partial g}{\partial x}$ at $x = 0$.

Now we will state the pitchfork bifurcation theorem for the singularity parameterized ODEs as follows:

**Theorem 3.3.** Consider the singularity parameterized ODEs (3.1) defined on the set of critical points $G$ with an equilibrium point $(0,0,0,0)$. If the following conditions are holds:

**(1)** $\frac{\partial \mathbf{F}}{\partial \boldsymbol{\alpha}}(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) = \mathbf{0}$,

**(2)** $\frac{\partial^2 \mathbf{F}}{\partial \mathbf{x}^2}(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) = \mathbf{0}$, $\frac{\partial^2 \mathbf{F}}{\partial \mathbf{x}\partial \boldsymbol{\alpha}}(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) \neq \mathbf{0}$,

**(3)** $\frac{\partial^3 \mathbf{F}}{\partial \mathbf{x}^3}(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) \neq \mathbf{0}$.

then $(0,0,0,0)$ is a pitchfork bifurcation point for the singularity parameterized ODEs (3.1) when $\epsilon$ approach to $0$.

**Proof.** By implicit function theorem we get a function $\alpha(x,y)$ s.t. $V(x,y,\alpha(x,y),0) = 0$, since $\frac{\partial^2 F}{\partial x\, \partial \alpha}(0,0,0,0) \neq 0$, then $\frac{\partial V}{\partial \alpha}(0,0,0,0) \neq 0$.

We want to prove that $\frac{\partial \alpha}{\partial x} = 0$, differentiate the equation $V(x,y,\alpha(x,y),0) = 0$ w.r.t $x$ we get:

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}\frac{dy}{dx} + \frac{\partial V}{\partial \alpha}\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial y}\frac{dy}{dx}\right) = 0.$$

Evaluate the above equation at $(0,0,0,0)$ and substitute that $\frac{dy}{dx}(0) = 0$ with condition (2) and $\frac{\partial V}{\partial \alpha}(0,0,0,0) \neq 0$, then we have:

$$\frac{\partial \alpha}{\partial x}(0) = -\left(\frac{\partial V}{\partial x}(0)\right)\left(\frac{\partial V}{\partial \alpha}(0)\right)^{-1} = 0.$$

which is the first property of the pitchfork bifurcation.

We can apply the same way w.r.t $W(x,y,\alpha,0) = 0$ to prove the first property $\frac{\partial \alpha}{\partial x} = 0$ of the pitchfork bifurcation.

Now we will prove the second property of pitchfork bifurcation to be occurred as follows:

Differentiate $V(x,y,\alpha(x,y),0) = 0$ twice w.r.t $x$ we get:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x\, \partial y}\frac{dy}{dx} + \frac{\partial^2 V}{\partial x\, \partial \alpha}\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial y}\frac{dy}{dt}\right) + \frac{\partial V}{\partial y}\frac{d^2 y}{dx^2} + \left[\frac{\partial^2 V}{\partial y\, \partial x} + \frac{\partial^2 V}{\partial y^2}\frac{dy}{dx} + \frac{\partial^2 V}{\partial y\, \partial x}\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial y}\frac{dy}{dx}\right)\right]\frac{dy}{dx}$$
$$+ \frac{\partial V}{\partial \alpha}\left[\frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial x\, \partial y}\frac{dy}{dx} + \frac{\partial \alpha}{\partial y}\frac{d^2 y}{dx^2} + \left(\frac{\partial^2 \alpha}{\partial y\, \partial x} + \frac{\partial^2 \alpha}{\partial y^2}\frac{dy}{dx}\right)\frac{dy}{dx}\right] + \left[\frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial y}\frac{dy}{dx}\right]\left[\frac{\partial^2 V}{\partial \alpha\, \partial x} + \frac{\partial^2 V}{\partial \alpha\, \partial y}\frac{dy}{dx} + \frac{\partial^2 V}{\partial \alpha^2}\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial y}\frac{dy}{dx}\right)\right]$$
$$= 0.$$

Evaluate at $(0,0,0,0)$ and substitute $\frac{dy}{dx}(0) = 0$, conditions (2) and (3) with application of the property that we proved $\frac{\partial \alpha}{\partial x}(0,0,0,0) = 0$ we get:

$$\frac{\partial^2 \alpha}{\partial x^2}(0,0,0,0) = -\left[\frac{\partial^2 V}{\partial x^2} + \left(\frac{\partial V}{\partial y}(0,0,0,0) + \frac{\partial V}{\partial \alpha}(0,0,0,0)\frac{\partial \alpha}{\partial y}(0,0,0,0)\right)\frac{d^2 y}{dx^2}(0)\right]\left(\frac{\partial V}{\partial \alpha}(0,0,0,0)\right)^{-1} \neq 0,$$

then

$$\frac{\partial^2 \alpha}{\partial x^2}(0,0,0,0) \neq 0.$$

Applying the same way w.r.t $W(x,y,\alpha,0) = 0$, we can prove the second property of the pitchfork bifurcation which is $\frac{\partial^2 \alpha}{\partial x^2}(0,0,0,0) \neq 0$. Then (0, 0, 0, 0) is a pitchfork bifurcation point for the singularity parameterized ODEs system (3.1) when $\epsilon$ approach to 0.

The above theorem apply the pitchfork bifurcation conditions for index one DAEs which is resulting from the singularity parameterized ODEs when $\epsilon$approach to 0.Now the following theorem apply the pitchfork bifurcation conditions for index twoDAEs:

**Theorem 3.4.**Consider the singularity parameterized ODEs (3.1) defined on the set of critical points $G$ with an equilibrium point $(0,0,0,0)$. If the following conditions are holds:

**(1)** $\frac{\partial F}{\partial \alpha}(0,0,0,0)=0$,

**(2)** $\frac{\partial^2 F}{\partial x^2}(0,0,0,0)=0$, $\frac{\partial^2 F}{\partial x \partial \alpha}(0,0,0,0)\neq 0$,

(3) $\frac{\partial^3 F}{\partial x^3}(0,0,0,0)\neq 0$.

then $(0,0,0,0)$ is a pitchfork bifurcation point for the singularity parameterized ODEs (3.1) when $\epsilon$approach to $0$.

**Proof.**From implicit function theorem there is a function $\alpha(x,y)\ s.t.$

$V(x,y,\alpha(x,y),0)=0.$

Differentiate the equation$V(x,y,\alpha(x,y),0)=0\ w.r.t\ x$andevaluate at $(0,0,0,0)$ and substitute that $\frac{dy}{dx}(0)=0$ with condition (2), then we have:

$$\frac{\partial \alpha}{\partial x}(0)=-(\frac{\partial V}{\partial x}(0))(\frac{\partial V}{\partial \alpha}(0))^{-1}=0.$$

which is the first property of the pitchfork bifurcation. Now we will prove the second property of the pitchfork bifurcation to be occurred which is as follows:

Differentiate $V(x,y,\alpha(x,y),0)=0$ twice w.r.t $x$ and evaluate at $(0,0,0,0)$ with substituting condition (3) and apply the property that we proved $\frac{\partial \alpha}{\partial x}(0,0,0,0)=0$ we get:

$$\frac{\partial^2 \alpha}{\partial x^2}(0,0,0,0)=-[\frac{\partial^2 V}{\partial x^2}+\left(\frac{\partial V}{\partial y}(0,0,0,0)+\frac{\partial V}{\partial \alpha}(0,0,0,0)\frac{\partial \alpha}{\partial y}(0,0,0,0)\right)\frac{d^2 y}{dx^2}(0)](\frac{\partial V}{\partial \alpha}(0,0,0,0))^{-1}\neq 0,$$

Applying the same way w.r.t $W(x,y,\alpha,0)=0$, with applying the giving conditions $\frac{\partial^2 g}{\partial x^2}(0)=0\ and\ \frac{\partial^2 g}{\partial x \partial y}(0)=0$, to prove the first property $\frac{\partial \alpha}{\partial x}(0)\neq 0$ of the pitchfork bifurcation.

we can prove the second property of the pitchfork bifurcation which is $\frac{\partial^2 \alpha}{\partial x^2}(0,0,0,0)\neq 0$. Then (0, 0, 0, 0) is a pitchfork bifurcation point for the singularity parameterized ODEs system (3.1) when $\epsilon$approach to 0.

## 3.4 Transcritical Bifurcation in Singularity Parameterized ODEs

A transcritical bifurcation is one in which an equilibrium point exists for all values of a parameter and is never destroyed [4]. In transcritical bifurcation there is an exchange of stability between two equilibrium points, there is one unstable and the other is stable equilibrium point.

Now we will introduce the transcritical bifurcation theorem for the singularity parameterized ODEs as follows:

**Theorem 3.5**. Consider the singularity parameterized ODEs (3.1) defined on the set of critical points $G$ with an equilibrium point $(0,0,0,0)$. If the following conditions are holds:

**(1)** $\frac{\partial F}{\partial \alpha}(0,0,0,0)=0$,

**(2)** $\frac{\partial^2 F}{\partial x^2}(0,0,0,0)\neq 0$, $\frac{\partial^2 F}{\partial x \partial \alpha}(0,0,0,0)\neq 0$,

then $(0,0,0,0)$ is a transcritical bifurcation point for the singularity parameterized ODEs (3.1) when $\epsilon$approach to $0$.

**Proof**. By implicit function theorem we get a function $\alpha(x,y)\ s.t. V(x,y,\alpha(x,y),0)=0$.

We want to prove that $\frac{\partial \alpha}{\partial x}\neq 0$, for this purpose, differentiate the equation$V(x,y,\alpha(x,y),0)=0\ w.r.t\ x$and evaluate at $(0,0,0,0)$ and substitute that $\frac{dy}{dx}(0)=0$ with condition (2) and $\frac{\partial V}{\partial \alpha}(0,0,0,0)\neq 0$, then we have:

$$\frac{\partial \alpha}{\partial x}(0)=-(\frac{\partial V}{\partial x}(0))(\frac{\partial V}{\partial \alpha}(0))^{-1}\neq 0.$$

which is the first property of the transcritical bifurcation.

To prove that $\frac{\partial^2\alpha}{\partial x^2} \neq 0$, we follow the same procedure in Theorem3.3, but with different conditions given above.

Then (0, 0, 0, 0) is a transcritical bifurcation point for the singularity parameterized ODEs system (3.1) when $\epsilon$approach to 0.

Now for index 2 DAEs we have the following theorem related to transcritical bifurcation:

**Theorem 3.6.** Consider the singularity parameterized ODEs (3.1) defined on the set of critical points $G$ with an equilibrium point $(0,0,0,0)$. If the following conditions are holds:

**(1)** $\frac{\partial F}{\partial \alpha}(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) = \mathbf{0}$,

**(2)** $\frac{\partial^2 f}{\partial x^2}(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) \neq \mathbf{0}$, $\frac{\partial^2 F}{\partial x \partial \alpha}(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) \neq \mathbf{0}$,

then $(0,0,0,0)$ is a transcritical bifurcation point for the singularity parameterized ODEs (3.1) when $\epsilon$approach to $0$.

**Proof.** By implicit function theorem we get a function $\alpha(x,y)\ s.t.$

$$V(x,y,\alpha(x,y),0) = 0,$$

differentiate the equation$V(x,y,\alpha(x,y),0) = 0$ w.r.t xand evaluate at $(0,0,0,0)$ with substituting $\frac{dy}{dx}(0) = 0$, condition (2) and $\frac{\partial V}{\partial \alpha}(0,0,0,0) \neq 0$, then we have:

$$\frac{\partial \alpha}{\partial x}(0) = -(\frac{\partial V}{\partial x}(0))(\frac{\partial V}{\partial \alpha}(0))^{-1} \neq 0.$$

which is the first property of the transcritical bifurcation.

We can apply the same way w.r.t $W(x,y,\alpha(x,y),0) = 0$, with applying the giving conditions $\frac{\partial^2 g}{\partial x^2}(0) = 0$ and $\frac{\partial^2 g}{\partial x\, \partial y}(0) = 0$, to prove the first property $\frac{\partial \alpha}{\partial x}(0) \neq 0$ of the transcritical bifurcation.

Then (0, 0, 0, 0) is a transcritical bifurcation point for the singularity parameterized ODEs system (3.1) when $\epsilon$approach to 0.

Now we will show how we apply the fold bifurcation theory on the singularity parameterized ODEs by the following example:

**Example 3.4.** Consider the singularity parameterized ODEs:

$$\dot{x} = \alpha - x^2,$$

$$\epsilon\dot{y} = -y,$$

when $\epsilon$approach to 0 we get::

$$\dot{x} = \alpha - x^2,$$

$$0 = -y.$$

Now we have DAEs. There is a fold bifurcation at the non-hyperbolic critical point $(0,0,0,0)$ at the bifurcation value $\alpha = 0$. If $\alpha = 0$ we get only one equilibrium point which is $(0,0,0,0)$. If $\alpha > 0$ then there are two equilibrium points which are $(\mp\sqrt{\alpha}, 0)$. If $\alpha < 0$ then there are no equilibrium points at all . since the condition of the set of critical points satisfy (i.e.)$f(0,0,0,0) = 0 = g(0,0,0,0)$, since $\frac{\partial g}{\partial y}(0) = -1$, then $\frac{\partial g}{\partial y}$ has an inverse.

Hence the DAEs satisfy the non-hyperbolic condition $\frac{\partial f}{\partial x}(0) = 0, \frac{\partial g}{\partial x}(0) = 0$, our DAEs satisfied the fold bifurcation conditions as follows:

$$\frac{\partial f}{\partial \alpha}(0) = -1 \neq 0,$$

$$\frac{\partial^2 f}{\partial x^2}(0) = -2 \neq 0.$$

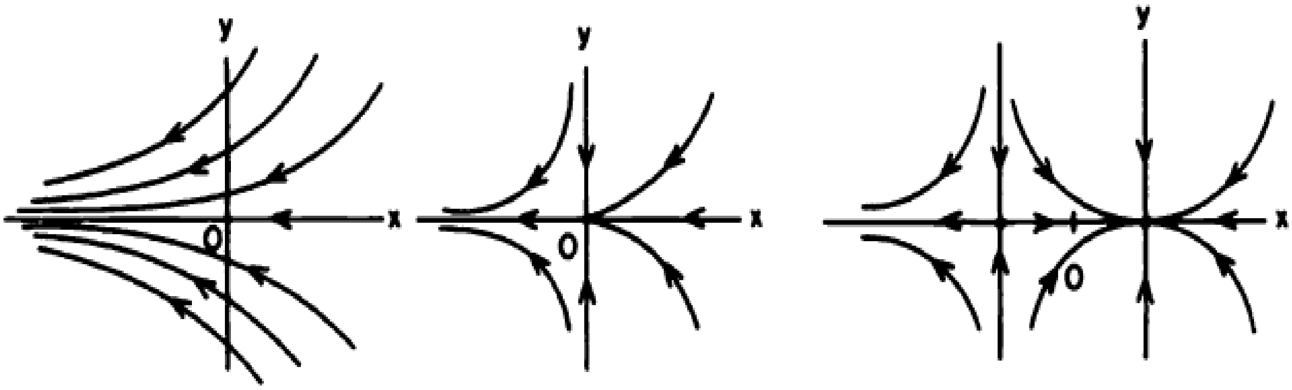


Figure 1: Bifurcation diagrams for the Fold Bifurcation. From left to right, $\alpha < 0, \alpha = 0$ and $\alpha > 0$.

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