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On Full k-ideals of a Ternary Semiring

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Abstract

In this paper, we generalize the concept of the full k-ideals of a semiring to ternary semiring and prove that the set of zeroids$Z(S)$, annihilator $A_S(M)$ of a right ternary S-semimodule M, and the Jacobson radical $J(S)$ of a ternary semiring S are all full k-ideals of S. Also we prove that the set $\mathfrak{I}(S)$ of all full k-ideals of a ternary semiring$S$ is a complete lattice which is also modular.

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1. Introduction

The notion of a ternary semiring was first introduced by T. K. Dutta and S. Kar in [1]. They also define the notion of a right ternary semimodule over a ternary semiring and Jacobson radical of a ternary semiring in [2]. In [3], M. K. Sen and M. R. Adhikari, consider full k-ideals of a semiring and they prove that the set of all full k-ideals of a semiring is a complete lattice which is also modular and also they discussed several characterizations of k-ideals of a semiring. The present paper extends some results of [3] to a ternary semirings.

2. Preliminary Definitions

Definition 2.1. [1] A non-empty set S together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if S is an additive commutative semigroup satisfying the following conditions:

i. $(abc)de = a(bcd)e = ab(cde)$(Associative Law)

ii. $(a+b)cd = acd+bcd$ (Right Distributive Law)

iii. $a(b+c)d = abd+acd$ (Lateral Distributive Law)

iv. $ab(c+d) = abc+abd$ (Left Distributive Law) for all $a,b,c,d,e \in T$.

Example 2.2. [1] Let $\mathbb{Z}_0^-$ be the set of all negative integers with zero. Then with the usual binary addition and ternary multiplication, $\mathbb{Z}_0^-$ forms a ternary semiring.

Definition 2.3[1] An additive subsemigroup$I$ of a ternary semiring S is called ideal of S if $SSI \subseteq I$, $SIS \subseteq I$ and $ISS \subseteq I$. An ideal I of a ternary semiring S is called k-ideal (subtractive) if for $a \in I, a+b \in I, b \in S$ imply $b \in I$. We denote $I \trianglerighteq S$, a ternary semiring ideal in S.

Definition 2.4 [1] A ternary semiring$S$ is said to be regular if for each element $a$in Sthere exists an element x in Ssuch that $a = axa$. If the element x is unique and satisfies $x = xax$, then S is called an inverse ternary semiring. x is called the inverse of a.

Definition 2.5 A ternary semiring$S$ is called additive inversive if S is an inverse ternary semiring under addition. In an additively inverse ternary semiring the unique inverse x of an element a is usually denoted by a'.

Definition 2.6 [1] An element a in a ternary semiring S is called an idempotent if $aaa = a$ that is$a^3 = a$. And if each element of S is idempotent, then S is called an idempotent ternary semiring.

Definition 2.7[1] An element a in a ternary semiring S is called an additively idempotent if $a+a = a$. And if each element of S is additively idempotent, then S is called an additively idempotent ternary semiring. The set of all additively idempotent elements of ternary semiring$S$ will be denoted by $I^+(S)$.

Definition 2.8[2] An additive commutative semigroup M with a zero element 0_M is called a right ternary semimodule over a ternary semiring S or simply a right ternary S-semimodule if there exists a mapping $M \times S \times S \to M$ (images to be denoted by ms_1s_2 for all $m \in M$ and $s_1, s_2 \in S$) satisfying the following conditions:

(i) $(m_1 + m_2)s_1s_2 = m_1s_1s_2 + m_2s_1s_2$

(ii)$m_1s_1(s_2 + s_3) = m_1s_1s_2 + m_1s_1s_3$

(iii) $m_1((s_1 + s_2)s_3 = m_1s_1s_3 + m_1s_2s_3$

(iv) $(m_1s_1s_2)s_3s_4 = m_1(s_1s_2s_3)s_4 = m_1s_1(s_2s_3s_4)$

(v)$0_Ms_1s_2 = 0_M = m_1s_10_S = m_10_Ss_1$; for all $m_1, m_2 \in M$ and for all $s_1, s_2, s_3, s_4 \in S$.

Definition 2.9[2] An element s of a ternary semiring S is called a zeroid, if $s + a = a + s = a$ for some a in S. The set of all zeroids of S denoted by $Z(S)$. A ternary semiring S is called zeroic if $Z(S) = S$. If $Z(S) \neq \phi$ and $Z(S)$ is proper subset of S then S is called non-zeroic.

Definition 2.10 [2] Let M be a right ternary S-semimodule. We put$(0 : M) = \{x \in S : msx = 0 \text{ and } mxs = 0 \text{ for all } s \in S\}$. Then we call $(0 : M)$ the annihilator of M in S, denoted by$A_S(M)$. The set of zeroids $Z(S)$ of a ternary semiring S is contained in $A_S(M)$.

Definition 2.11 [2] A right ternary S-semimodule $M \neq \{0\}$ is said to be irreducible if for every arbitrary fixed pair $u_1, u_2 \in M$ with $u_1 \neq u_2$ and for any $x \in M$, there exist$a_i, b_i, c_j, d_j \in S$ $(1 \leq i \leq p, 1 \leq j \leq q;\ p, q$ are positive integers) such that

$$x + \sum_{i=1}^{p} u_1 a_i b_i + \sum_{j=1}^{q} u_2 c_j d_j = \sum_{j=1}^{q} u_1 c_j d_j + \sum_{i=1}^{p} u_2 a_i b_i.$$

Definition 2.12 [2] Let S be a ternary semiring and Δ be the set of all irreducible right ternary S-semimodules. Then $J(S) = \cup_{M \in \Delta} A_S(M)$ is called the Jacobson radical of S. If Δ is empty then S itself is considered as $J(S)$ i.e. $J(S) = S$ and in this case, we say that S is a radical ternary semiring. A ternary semiring$S$ is said to be Jacobson semisimple (or simply, J-semisimple) if $J(S) = \{0\}$.

3. Full k-ideal

Lemma 3.1 Let S be a ternary semiring, then set of additive idempotent elements $I^+(S)$ of$S$ is an ideal of S.

Remark 3.2 Clearly,$I^+(S) \subseteq Z(S)$.

Proposition 3.3 [2] Let S be a ternary semiring, then set of zeroids $Z(S)$ of$S$ is a k-ideal of$S$.

Lemma 3.4 Every k-ideal of a ternary semiring S is an additively inverse ternary subsemiring of$S$.

Lemma 3.5 [1] Let A be an ideal of a ternary semiring$S$. Then $\bar{A} = \{a \in S: a + b = c \text{ for some } b, c \in A\}$ is a k-ideal of$S$.

Corollary 3.6 [1] Let A be an ideal of a ternary semiring$S$. Then$\bar{A} = A$ iff A is a k-ideal.

Definition 3.7 A k-ideal I of a ternary semiring S is called full k-ideal if the set of all additive idempotent elements $I^+(S)$ of S is contained in I.

Example 3.8 In a ternary semiring S, every k-ideal I is a full k-ideal. Since 0 is the only additive idempotent element in S which belongs to any k-ideal I of S. So I is full k-ideal of S.

Proposition 3.9 Let S be a ternary semiring, then set of zeroids $Z(S)$ of S is a full k-ideal of S.

Proof: By proposition 3.3 $Z(S)$ is a k-ideal of S and also $I^+(S) \subseteq Z(S)$. Hence $Z(S)$ is a full k-ideal of S.

Lemma 3.10 [1] Let A, B be any two k-ideals of a ternary semiring S, then $A \cap B$ is also a k-ideal of S.

Proposition 3.11 Let A, B be any two full k-ideal of a ternary semiring S, then $A \cap B$ is also a full k-ideal of S.

Proposition 3.12 Every k-ideal of an additively inversive ternary semiring S is an inversive ternary sub-semiring of S.

Lemma 3.13 [2] Let M be a right ternary S-semimodule. Then $A_S(M)$ is an k-ideal of S.

Proposition 3.14 Let M be a right ternary S-semimodule. Then $A_S(M)$ is a full k-ideal of S.

Proof: By Lemma 3.13, $A_S(M)$ is a k-ideal of S and also $I^+(S) \subseteq Z(S) \subseteq A_S(M)$. Hence $A_S(M)$ is a full k-ideal of S.

Remark 3.15 [2] The set of zeroids $Z(S)$ of S is contained in the Jacobson radical $J(S)$, since $Z(S) \subseteq A_S(M)$ for all right ternary S-semimodule M. And hence $I^+(S) \subseteq Z(S) \subseteq J(S)$.

Lemma 3.16[2]$J(S)$ is an k-ideal of S.

Proposition 3.17$J(S)$ is a full k-ideal of S.

Proof: By Lemma 3.16, $J(S)$ is a k-ideal of S and also $I^+(S) \subseteq J(S)$. Hence $J(S)$ is a full k-ideal of S.

Lemma 3.18 Let A and B be two full k-ideals of a ternary semiring S. Then $\overline{A+B}$ is a full k-ideal of S such that $A \subseteq \overline{A+B}$ and $B \subseteq \overline{A+B}$.

Proof: Clearly $A+B$ is an ideal of S. From Lemma 3.5, we prove $\overline{A+B}$ is a k-idealand $A+B \subseteq \overline{A+B}$. Since A, B are full k-ideals, $I^+(S) \subseteq A, B$.Hence$I^+(S) \subseteq A+B \subseteq \overline{A+B}$. This implies that $\overline{A+B}$ is a full k-ideal. Now $a = a + a' + a = a + (a' + a) \in A + B$ as$a' + a \in I^+(S) \subseteq B$. Hence $A \subseteq \overline{A+B}$ and similarly $B \subseteq \overline{A+B}$.

Theorem 3.19 Let $\mathfrak{T}(S)$ denotes the set of all full k-ideals of ternary semiring S. Then $\mathfrak{T}(S)$ is a complete lattice which is also modular.

Proof: Clearly$\mathfrak{T}(S)$is a partially ordered set with respect to usual set inclusion. Let $A, B \subseteq \mathfrak{T}(S)$. Then $A \cap B \subseteq \mathfrak{T}(S)$ and from Lemma 3.18$\overline{A+B} \subseteq \mathfrak{T}(S)$. Now we define $A \wedge B = A \cap B$ and $A \vee B = \overline{A+B}$. Let $C \subseteq \mathfrak{T}(S)$ such that $A, B \subseteq C$. Then $A + B \subseteq C$ and$\overline{A+B} \subseteq \bar{C}$. But$\bar{C} = C$. Hence$\overline{A+B} \subseteq C$. Now $\overline{A+B}$ is the l.u.b. of A, B. Thus $\mathfrak{T}(S)$ is a lattice. Now $I^+(S)$ is an ideal of S. Hence $I^+(S) \in \mathfrak{T}(S)$ and also $S \in \mathfrak{T}(S)$. Therefore $\mathfrak{T}(S)$is a complete lattice.

Now consider $A, B, C \in \mathfrak{T}(S)$ such that $A \wedge B = A \wedge C$ and $A \vee B = A \vee C$ and $B \subseteq C$. Let $c \in C$. Then $c \in A \vee C = A \vee B = \overline{A+B}$. Hence there exists $a + b \in A + B$ such that $c + a + b = a_1 + b_1$ for some$a_1 \in A, b_1 \in B$. Then $c + a + a' + b = a_1 + b_1 + a'$. Now $c \in C, a + a' \in C$ and $b \in B \subseteq C$. Hence $a_1 + b_1 + a' \in C$. But$b_1 \in C$. Consequently, $a_1 + a' \in A \cap C = B \cap C$. Hence$a_1 + a' \in B$. So from $c + a + b = a_1 + b_1$ we prove that $c + a + a' + b = a_1 + a' + b \in B$. But$(a + a') + b \in B$ and B is a k-ideal. Hence $c \in B$and $B = C$. This proves that $\mathfrak{T}(S)$ is a modular lattice.

5. References

5.1. Journal Article

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