Near Rings Admitting Certain Decomposition Theorems

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Abstract: In the present paper we shall investigate some decomposition theorems for near rings satisfying any one of the conditions; (1) $xy = y^m (xy)^p y^n$

(2) $xy = y^m (yx)^p y^n$, where $m = m(x, y) \geq 0$, $n = n(x, y) \geq 0$ and $p = p(x, y) > 1$ are integers.

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1 Introduction

The property $x^n = x$ has been a great favorite with many ring theorists over the last few decades since Jacobson [1] rst studied the commutativity of rings sat-isfying this condition in order to generalize the classical Wedderburn Theorem [2]. Further, Searcoid and MacHale [3] studied commutativity of rings satisfying the condition $(xy)^{n(x,y)} = xy$. Also, Ligh and Luh [4] established that a ring R satisfying the above condition is the direct sum of a J-ring and a nil ring. Later, Bell and Ligh [5] obtained a direct sum decomposition of the rings satisfying the related properties like $(xy) = (xy)^2 f(x, y)$ for $f(X, Y) \in Z(X, Y)$ the ring of polynomials in two non-commuting indeterminate over the ring Z of integers and remarked that in the case of near rings the analogous hypothesis does not quite yield a direct sum decomposition. The authors have defined a weaker notion of orthogonal sum. We say that a near ring R is an orthogonal sum of sub-near rings A and B denoted by $R = A \oplus B$ if $AB = BA = \{0\}$ and each element of R has a unique representation in the form $a + b$ with $a \in A$ and $b \in B$. The present paper is the continue study to investigate the structure of certain near rings satisfying the following conditions:

(P$_1$) For every $x, y \in R$ there exists integers $m = m(x, y) \geq 0$, $n = n(x, y) \geq 0$, $p = n(x, y) > 1$ such that $xy = y^m (xy)^p y^n$.

(P$_2$) For every $x, y \in R$ there exists integers $m = m(x, y) \geq 0$, $n = n(x, y) \geq 0$, $p = n(x, y) > 1$ such that $xy = y^m (yx)^p y^n$.

2 Preliminaries

Throughout, R is a left near ring with multiplicative center Z. We shall denote by N the set of nilpotent elements and by P the set of potent elements of R that is $\{x \in R / x^{n(x)} = x$, for some positive integer n(x) > 1\}. The set of commutators is denoted by C. An element $x \in R$ is said to be distributive if $(y + z) x = yx + zx$ for all y, z $\in$ R. A near ring R is said to be distributively generated (d-g) if it contains a multiplicative sub-semigroup of distributive elements which generates the additive group (R, +). A near ring R is called periodic if for every x $\in$ R there exist distinct positive integers m = m(x), n = n(x) such that $x^m = x^n$.

An ideal of a near ring R is defined to be a normal subgroup I of (R, +) such that

(1) RI $\subseteq$ I;

(2) (x + i) y - xy $\in$ I for all x, y $\in$ R and i $\in$ I.

A near ring R is called zero- commutative if xy = 0 implies yx = 0 for all x, y $\in$ R and if for all x $\in$ R, 0x = 0 then, R is called zero-symmetric (recall that left distributivity in R yields x0 = 0).

3 Main Results

Lemma 3.1. Let R be a zero-symmetric near ring satisfying the following properties:

(1)   For each $x \in$ R, there exists a positive integer n(x) > 1 such that x^n = x.

(2)   Every non-trivial homomorphic image of R contains a non-zero central idempotent. Then (R, +) is commutative.

Lemma 3.2. Let R be a zero-symmetric near ring having no non-zero nilpotent ele-ments. Then every distributive idempotent is multiplicatively central.

Lemma 3.3. Let R be a zero-commutative periodic near ring, then R = P + N.

Lemma 3.4. Let R be a near ring in which idempotent elements are multiplicatively central. If e and f are any idempotent elements, then there exists an idempotent g such that ge = e and gf = f.

Lemma 3.5. If R is a zero-commutative near ring, then N is an ideal of R.

Lemma 3.l to 3.5 are essentially proved in references [5- 7].

Lemma 3.6. Let R be a zero-symmetric near ring satisfying either of the conditions (P_1) and (P_2). Then RN = NR = {0}

Proof. Let R satisfy (P_1). Notice that R is zero-commutative. Indeed, if xy = 0, then there exist integers m_1 = m (y, x) $\geq$ 0, n_1 = n (y; x) $\geq$ 0 and p_1 = p (y; x) > 1, such that yx = x^{m1} $(yx)^{p1}$ x^{n1} = 0. Replacing, y by x in condition (P_1), we get

$$x^2 = x^{r+2s+t}; \text{ for all } r + 2s + t \geq 4. \tag{3.1}$$

If u $\in$ N then making repeated use of (3.1), we get u^2 = 0. Now for any x$\in$ R, by condition (P_1), we have xu = u^{r1} $(xu)^{s1}$ u^{t1} = 0. But, the zero-commutativity of R implies that ux = 0 for u $\in$ N and x $\in$ R. Hence

$$RN = NR = \{0\}. \tag{3:2}$$

The proof of the lemma runs on the similar lines if R satisfies condition (P_2).

Theorem 3.1. Let R be a zero-symmetric near ring satisfying either of the conditions (P_1) or (P_2). Suppose further that the idempotent elements of R are multiplicatively central. Then, P is a sub-near ring with (P, +) abelian. N is a sub-near ring with trivial multiplication and R = P $\oplus$ N.

Proof. In view of Lemma 3.6, the nilpotent elements of R annihilate R on both sides and hence, in particular, N^2 = {0} and N $\subseteq$ Z. Replacing y by x in condition (P_1), we see that R is periodic. Hence, every clement x $\in$ R can be expressed in the form x = a + b where a $\in$ P and b $\in$ N by Lemma 3.3. Next, we will show that P is a sub-near ring. Let a, b $\in$ P and choose integers p = p(a) > 1 and q = q(b) > 1 such that a^p = a and b^q = b. Let

k = (p -1) q - (p- 2) = (q- 1) p- (q- 2).

Then, it is clear that a^k = a and b^k = b. Note that e = q^{k-1} and f = b^{k-1} are idempotents in R with ea = a and fb = b. Using (P_1), we get

ab = eafb = (ab)(ef) = $(ef)^m (abcf)^p (ef)^n$;

for some positive integers m = m (ab, ef) $\geq$ 0, n=n (ab, ef) $\geq$ 0 and p = p (ab, ef) $\geq$ 1. Thus, we find that ab = $(ab)^p$, hence ab $\in$ P. Moreover, since R/N has x^p = x property, we have an integer j > 1, such that

$$(a- b)^j = a- b- u, u \in N(R) \tag{3:3}$$

Using Lemma 3.4, we can choose an idempotent g for which ge = e and gf = f. Therefore, ga = a and gb = b. Now multiplying (3.3) by g, we have $(a-b)^j$ = a-b, that is a-b $\in$ P. Also by the application of Lemma 3.1, (P, +) is abelian.

Trivially, P $\cap$ N = {0}. Let a + u = b + v, where a, b $\in$ P and u, v $\in$ N. Then

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a- b = v- u $\in$ P $\cap$ N = {0}

which yields that a = b and u = v. Hence R = P $\oplus$ N.

The proof runs on the same lines if R satisfies condition (P$_2$).

The following example shows that in the hypotheses of Theorem 3.1, the centrality of idempotents is not superfluous.

Example 3.1. Let R = {0, a, b, c} with addition and multiplication tables defined as follows.

+	0	a	b	c	*			0	a	b	c
0	0	a	b	c	0			0	0	0	0
a	a	0	c	b	a			0	a	b	c
b	b	c	0	a	b			0	a	b	c
c	c	b	a	0	c			0	a	b	c

It can be easily verified that (R, +,*) is a near ring satisfying both the conditions (P$_1$) and (P$_2$). However, the set P = {0, a, b} is not a sub-near ring.

4 Decomposition Theorem for D-Near Rings

Despite the existence of such adverse examples, we need not give up the possibility of extending our Theorem 3.1 for larger classes of near rings known as D-near rings.

A near ring R is called a D-near ring if every non-zero homomorphic image T of R satisfies the following conditions:

(1) T has a non-zero right distributive element;

(2) That the additive group (T, +) of T is abelian implies that T is a ring.

All distributive and (d-g)-near rings are the examples of D-near rings. However, example 2.5 of [8] illustrates that D-near rings are the generalizations of (d-g)-near rings.

Lemma 4.1. Let R be a zero-symmetric D-near ring. If for each x $\in$ R there exists a positive integer n(x) > 1 such that $x^n = x$, then R is a commutative ring.

Proof. Since R is a D near ring, then every non-zero homomorphic image of R con-tains a non-zero distributive element, say, a. Choose m = m(a) > 1 such that $a^m = a$, so that a^{m-1} is a distributive idempotent. Also, since R has no non-zero nilpotent elements, then application of Lemma 3.2 yields that a^{m-1} is a distributive central idempotent. Hence, by Lemma 3.1. (R, +) is abelian. This together with the fact that R is a D-near ring implies that R is a ring.

Theorem 4.1. Let R be a zero-symmetric D-near ring satisfying either of the conditions (P$_1$) or (P$_2$). Then P is a sub-ring, N is a sub-near ring with trivial multiplication and R = P $\oplus$ N.

Proof. Let R satisfy condition (P$_1$). In view of Lemma 3.5. R/N is a D-near ring being the homomorphic image of R. Choose positive integers m$_1$, n$_1$, and p$_1$ such that $\bar{x}^{-m_1+n_1+p_1}$ for $\bar{x} \in R/N$, so that $\bar{x}\,(\bar{x}^{-m_1+n_1+p_1-1} - \bar{x}) = 0$.

Since R/N has no non-zero divisors, $\bar{x}^{-m_1+n_1+p_1-1} = \bar{x}$, for $\bar{x} \in R/N$. Hence R/N is a commutative ring by Lemma 4.1 and consequently

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$$C \subseteq N \subseteq Z$$

Let e be an idempotent in R. Then e(xe-ex) =0 by using (3.2), i.e. exe=ex for all x in R. Also by condition (P_1) there exist integers $r = r(x,e) \geq 0$, $s = s(x,e) \geq 0$ and $t = t(x,e) > 1$ such that $xe = e^r(xe)^t e^s$. Multiplying by e on the left, we get exe=xe. Hence ex=xe for all x in R. For the rest of the proof, it is argued in the similar manner as we have done in proof of Theorem 3.1.

The proof of the theorem can be developed on the same lines if R satisfies property (P_2).

The following example due lo Clay (2.5) of reference [8] shows that one cannot get a direct sum decomposition under the hypotheses of Theorem 3.1 and Theorem 4.1.

Example 4.1. Consider non-abelian group (R, +) isomorphic to symmetric group S_3 and de ne the multiplication in R as follows.

*	0	a_1	a_2	a_3	a_4	a_5
0	0	0	0	0	0	0
a_1	0	a_1	a_1	a_1		0	0
a_2	0	a_1	a_1	a_1		0	0
a_3	0		a_1	a_1	a_1	0	0
a_4	0	0	0	0	0	0
a_5	0	0	0	0	0	0

Then (R, +, *) is a commutative near ring satisfying both conditions (P_1) and (P_2). However, P = {0, a_1} is not an ideal of R.

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