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Characterized Fuzzy $R_{2\frac{1}{2}}$ and Characterized Fuzzy $T_{3\frac{1}{2}}$ Spaces

A. S. ABD-ALLAH and A. AL-KHEDHAIRI
Department of Mathematics, College of Science, El-Mansoura University, El-Mansoura, Egypt
Current Address: Mathematics Department, College of Science and Humanity Studies, Prince Sattam Bin Abdul-Aziz University, P. O. Box 132012, Hotat Bani Tamim 11941, Saudi Arabia.
asabdallah@hotmail.com
Department of Statistics and Operations Research, College of Science, King Saud University, P. O. Box 2455, Riyadh 1145, Saudi Arabia
akhediri@ksu.edu.sa

ABSTRACT: This paper, deals with. introduce and study the notions of characterized fuzzy $R_{2\frac{1}{2}}$ spaces and of characterized fuzzy $T_{3\frac{1}{2}}$ spaces by using the notion of fuzzy function family presented in [21] and the notion of $\varphi_{1,2}\psi_{1,2}$ - fuzzy continuous mappings presented in [5] as a generalization of all the weaker and stronger forms of the fuzzy completely regular spaces introduced in [11,24,26,29]. We denote by characterized fuzzy $T_{3\frac{1}{2}}$ space or characterized fuzzy Tychonoff space to the characterized fuzzy space that is characterized fuzzy T_1 and characterized fuzzy $R_{2\frac{1}{2}}$ space in this sense. The relations between the characterized fuzzy $T_{3\frac{1}{2}}$ spaces, the characterized fuzzy T_4 spaces and the characterized fuzzy T_3 spaces are introduced. When the given fuzzy topological space is normal, then the related characterized fuzzy space is finer than the associated characterized fuzzy proximity space that is presented in [1]. Moreover, the associated characterized fuzzy proximity spaces and the characterized fuzzy T_4 spaces are identical with help of the complementarily symmetric fuzzy topogenous structure, that is, identified with the fuzzy proximity δ . More generally, the fuzzy function family of all $\varphi_{1,2}\psi_{1,2}$ - fuzzy continuous mappings are applied to show that the characterized fuzzy $R_{2\frac{1}{2}}$ spaces and the associated characterized fuzzy proximity spaces are identical.

KEYWORDS: fuzzy filter. Fuzzy; topological space; fuzzy proximity; fuzzy topogenous structure; operations; characterized fuzzy space; $\varphi_{1,2}\delta$ - fuzzy neighborhood; fuzzy topogenous order, fuzzy function family; characterized fuzzy proximity space; characterized fuzzy $R_{2\frac{1}{2}}$ space, characterized fuzzy R_k spaces for $k \in \{0,1,2,2\frac{1}{2},3\}$, characterized fuzzy T_s spaces for $s \in \{0,1,2,2\frac{1}{2},3,3\frac{1}{2},4\}$.

1. INTRODUCTION

The notion of fuzzy filter has been introduced by Eklund et al. ([16]). By means of this notion a point-based approach to the fuzzy topology related to usual points has been developed. The more general concept for the fuzzy filter introduced by W. Gähler in [18] and the fuzzy filters are classified by types. However, because of the specific type of the fuzzy filter the approach of Eklund is related only to the fuzzy topologies which are stratified, that is, all constant fuzzy sets are open. The more specific fuzzy filters are considered in the former papers to be homogeneous now.

The notion of fuzzy real numbers are introduced by S. Gähler and W. Gähler in [21], as a convex, normal, compactly supported and upper semi-continuous fuzzy subsets of the set of all real numbers $\mathbb{R}$. The set of all fuzzy real numbers is called the fuzzy real line and will be denoted by $\mathbb{R}_L$, where L is complete chain. On the ordinary topological space (X, T), the operation has been defined by Kasahara ([27]) as the mapping φ from T into 2^X such that $A \subseteq A^{\varphi}$, for all $A \in T$. Abd El-Monsef *et al.* in [8], extend Kasahara operation to the power set $P(X)$ of the set X Kandil *et al*.([25l) extended Kasahars's and Abd El-Monsef's Operations by introducing an operation on the class of all fuzzy subsets endowed with an fuzzy topology τ as a mapping $\varphi: L^X \longrightarrow L^X$ such that $\operatorname{int}\mu \le \mu^{\varphi}$ for all $\mu \in L^X$, where μ^{φ} denotes the value of φ at μ . The notions of the fuzzy filters and the Operations on the class of all fuzzy sets on X endowed with the fuzzy topology τ are applied in [5] to introduce a more general theory including all the weaker and stronger forms of the fuzzy topology. By means of these notions the notion of $\varphi_{1,2}$ - fuzzy interior of the fuzzy set, $\varphi_{1,2}$ - fuzzy convergence and $\varphi_{1,2}$ - fuzzy neighborhood filters are defined. The notion of $\varphi_{1,2}$ - fuzzy interior operator for the fuzzy sets is also defined as a mapping $\varphi_{1,2}.\mathrm{int}: L^X \longrightarrow L^X$ which fulfill (I1) to (15). Since there is a one-to-one correspondence between the class of all $\varphi_{1,2}$ – open fuzzy subsets of X and these operators, then the class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$ – open fuzzy subsets of X is characterized by these operators. Hence, the triple $(X, \varphi_{1,2}OF(X))$ as will as the triple $(X, \varphi_{1,2}.\mathrm{int})$ will be called characterized fuzzy space of all $\varphi_{1,2}$ – open fuzzy subsets. For each characterized fuzzy space $(X, \varphi_{1,2}.\mathrm{int})$ the mapping which assigns to each point x of X the $\varphi_{1,2}$ – fuzzy neighborhood filter at x is called the $\varphi_{1,2}$ – fuzzy filter pre topology ([5]). It can be identified itself with the characterized fuzzy space $(X, \varphi_{1,2}.\mathrm{int})$. The characterized fuzzy spaces are characterized by many of characterizing notions in [6]. for example by: $\varphi_{1,2}$ – fuzzy neighborhood filters, $\varphi_{1,2}$ – fuzzy interior of the fuzzy filters and by the set of all $\varphi_{1,2}$ – inner points of the fuzzy filters. Moreover, the notions of closeness and compactness in characterized fuzzy spaces are introduced and studied in [7].

In this paper, we applied the operations on the fuzzy topological space $(I_L, \Im)$, where I_L is the fuzzy unit interval presented in [21] and $\Im$ is the fuzzy topology defined on I_L, to introduce and study the notions of *characterized fuzzy* $R_{2\frac{1}{2}}$ *space* and of *characterized fuzzy* $T_{3\frac{1}{2}}$ *space*. The characterized fuzzy $R_{2\frac{1}{2}}$ space is defined similar to the characterized fuzzy R_k spaces for $k \in \{0,1,2,3\}$ ({2,3l), by using the ordinary points and the usual subset as a generalization of the existence weaker and stronger forms of the completely regular fuzzy topological spaces, such as, the notions defined by Bayoumi and Abedou in [11], by Hutton in [24], by Katsars in [29] and by Kandil and El-Shafee in [26]. The characterized fuzzy space which is characterized fuzzy T_1 and characterized fuzzy $R_{2\frac{1}{2}}$ space will be called characterized fuzzy $T_{3\frac{1}{2}}$ space or ***characterized Tychonofi fuzzy space***. For the characterized Tychonoff fuzzy space, the Urysohn's Lemma is proved and hence it shown that every characterized fuzzy T_4 space is characterized fuzzy $T_{3\frac{1}{2}}$ space, but the inverse is not true in general. Moreover, every characterized fuzzy $T_{3\frac{1}{2}}$ space is characterized fuzzy T_3 space, but the inverse is not true in general. The implications between all the characterized fuzzy T_s spaces and of all characterized fuzzy R_k spaces are listed in Diagrams 3.1 and 4.1 for all $s \in \{0,1,2,2\frac{1}{2},3,3\frac{1}{2},4\}$ and $k \in \{0,1,2,2\frac{1}{2},3\}$. For each case counter examples will be given. Finally, in Section 5, we shall study the relation between characterized fuzzy $T_{3\frac{1}{2}}$ space and the characterized fuzzy proximity spaces which is presented by Abd-Allah in [1]. We applied Urysohn's Lemma and other results which are proved in Sections 2 and 3 to show many results joining the characterized fuzzy $R_{2\frac{1}{2}}$ spaces in our sense and the characterized fuzzy proximity spaces in sense of Abd-Allah. Specially, we show that the associated characterized fuzzy proximity space is characterized fuzzy $R_{2\frac{1}{2}}$ space in our sense. Moreover, we show that every characterized fuzzy $R_{2\frac{1}{2}}$ space is compatible with a fuzzy proximity space defined by the Φ-separated for the fuzzy sets in the characterized fuzzy spaces.

2. PRELIMINARIES

We begin by recalling some facts on fuzzy filters. Let L be a completely distributive complete lattice with different least and last elements 0 and 1, respectively. Sometimes we will assume more specially that L is complete chain, that is, L is a complete lattice whose partial ordering is a linear one. The closed unit interval $I = [0,1]$ is the standard example for the completely distributive complete lattice L. Consider, $L_0 = L \setminus \{0\}$, $L_1 = L \setminus \{1\}$ and $I_{01} = I \setminus \{0,1\}$. For a set X, let L^X be the set of all fuzzy subsets of X, that is, of all mappings $\mu: X \to L$. Assume that an order-reversing involution $\alpha \mapsto \alpha'$ is fixed. For each fuzzy set μ, let $co\,\mu$ denote the complement of μ defined by: $(co\,\mu)(x) = co\,\mu(x)$ for all $x \in X$. For all $x \in X$ and $\alpha \in L_0$, the fuzzy subset of X hose the value α at x and 0 otherwise is said to be fuzzy point in X. The set of all fuzzy point in a set X will be denoted by $S(X)$. $\sup \mu$ means the supremum of the set of values of μ. Denote by $\bar{\alpha}$ the constant fuzzy subset of X with value or $\alpha \in L$.

The fuzzy filter on X ([18]) is a mapping $\mathcal{M}: L^X \to L$ such that the following conditions are fulfilled:

(F1) $\mathcal{M}(\bar{\alpha}) \le \alpha$ for all $\alpha \in L$ and $\mathcal{M}(\bar{1}) = 1$.

(F2) $\mathcal{M}(\mu \wedge \eta) = \mathcal{M}(\mu) \wedge \mathcal{M}(\eta)$ for all $\mu, \rho \in L^X$.

The fuzzy filter $\mathcal{M}$ is said to be *homogeneous* ([18]) if $\mathcal{M}(\overline{\alpha}) = \alpha$ for all $\alpha \in L$. For each $x \in X$, the mapping $\dot{x} : L^X \to L$ defined by $\dot{x}(\mu) = \mu(x)$ for all $\mu \in L^X$ is a homogeneous fuzzy filter on X . The homogeneous fuzzy filter at a fuzzy set is defined by the same way as follows, for each $\mu \in L^X$, the mapping $\dot{\mu} : L^X \to L$ defined by $\dot{\mu}(\sigma) = \bigwedge_{0<\sigma(x)} \sigma(x)$ for all $\sigma \in L^X$ is also homogeneous fuzzy filter on X , called homogenous fuzzy filter at the fuzzy subset $\mu \in L^X$. Let $\mathcal{F}_L X$ and $F_L X$ will be denote the sets of all fuzzy filters and of all homogeneous fuzzy filters on a set X , respectively. If $\mathcal{M}$ and $\mathcal{N}$ are fuzzy filters on a set X , then $\mathcal{M}$ is said to be *finer than* $\mathcal{N}$ and will be denoted by $\mathcal{M} \leq \mathcal{N}$, provided $\mathcal{M}(\mu) \geq \mathcal{N}(\mu)$ holds for all $\mu \in L^X$. Noting that if L is a complete chain then $\mathcal{M}$ is *not finer than* $\mathcal{N}$ and will be denoted by $\mathcal{M} \not\leq \mathcal{N}$, provided there exists $\mu \in L^X$ such that $\mathcal{M}(\mu) < \mathcal{N}(\mu)$ holds.

Lemma 2.1 [4] If $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{L}$ are fuzzy filters on a set X . Then the following sentences are fulfilled:

$\mathcal{M} \neq \mathcal{L} \geq \mathcal{N}$ implies that $\mathcal{M} \neq \mathcal{N}$ and $\mathcal{M} \geq \mathcal{L} \neq \mathcal{N}$ implies that $\mathcal{M} \neq \mathcal{N}$

Proposition 2.1 [13] For each $\mu, \rho \in L^X$, we have $\mu \leq \rho$ if and only if $\dot{\mu} \leq \dot{\rho}$.

The coarsest fuzzy filter is a fuzzy filter $\mathcal{M}$ on X has the value 1 at $\overline{1}$ and 0 otherwise. The supremum and the infimum of sets of fuzzy filters are meant with respect to the finer relation. An fuzzy filter $\mathcal{M}$ on X is said to be *ultra fuzzy filter* ([18]) if it does not have a properly finer fuzzy filter. For each fuzzy filter $\mathcal{M} \in \mathcal{F}_L X$ there exists a finer ultra fuzzy filter $\mathcal{U} \in \mathcal{F}_L X$ such that $\mathcal{U} \not\leq \mathcal{M}$. Moreover, for each non-empty set $\mathcal{A}$ of the fuzzy filters on X the supremum $\bigvee_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ exists ([15]) and it given by $(\bigvee_{\mathcal{M} \in \mathcal{A}} \mathcal{M})(\mu) = \bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M}(\mu)$ for all $\mu \in L^X$. Will the infimum $\bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ of the set $\mathcal{A}$ does not exists, in general. As shown in [18], the infimum $\bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ of the set $\mathcal{A}$ with respect to the finer relation for fuzzy filters exists if and only if $\mathcal{M}_1(\mu_1) \wedge \ldots \wedge \mathcal{M}_n(\mu_n) \leq \sup(\mu_1 \wedge \ldots \wedge \mu_n)$ holds for all finite subset $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ of $\mathcal{A}$ and $\mu_1, \ldots, \mu_n \in L^X$. In this case, the infimum $\bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ of the set $\mathcal{A}$ with respect to the finer relation for fuzzy filters is given by:

$$(\bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M})(\mu) = \bigvee_{\substack{\mu_1 \wedge \ldots \wedge \mu_n \leq \mu, \\ \mathcal{M}_1, \ldots, \mathcal{M}_n \in \mathcal{A}}} (\mathcal{M}_1(\mu_1) \wedge \ldots \wedge \mathcal{M}_n(\mu_n))$$

for all $\mu \in L^X$. On other hand the homogeneous fuzzy filter $\dot{\mu}$ at the fuzzy set $\mu \in L^X$ is characterized by the homogeneous fuzzy filter $\dot{x}$ at the point $x \in X$ in [1], in the form:

$$\dot{\mu}(\eta) = \bigwedge_{\mu(x) \geq 0} \dot{x}(\eta) \tag{2.1}$$

For all $\eta \in L^X$.

Fuzzy filter bases. The family $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ of non-empty subsets of L^X is called a *valued fuzzy filter base* ([18]) if the following conditions are fulfilled:

(V1) $\mu \in \mathcal{B}_\alpha$ implies $\alpha \leq \sup \mu$.

(V2) For all $\alpha, \beta \in L_0$ with $\alpha \wedge \beta \in L_0$ and all $\mu \in \mathcal{B}_\alpha$ and $\eta \in \mathcal{B}_\beta$ there are $\gamma \geq \alpha \wedge \beta$ and $\sigma \leq \mu \wedge \eta$ such that $\sigma \in \mathcal{B}_\gamma$.

Proposition 2.2 [18] Each valued fuzzy filter base $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ defines a fuzzy filter $\mathcal{M}$ on X by $\mathcal{M}(\mu) = \bigvee_{\eta \in \mathcal{B}_\alpha, \eta \leq \mu} \alpha$ for all $\mu \in L^X$. Conversely, each fuzzy filter $\mathcal{M}$ can be generated by a valued fuzzy filter base, e.g. by $(\alpha\text{-pr}\,\mathcal{M})_{\alpha \in L_0}$ with $\alpha\text{-pr}\,\mathcal{M} = \{\mu \in L^X \mid \alpha \leq \mathcal{M}(\mu)\}$.

The family $(\alpha\text{-pr}\,\mathcal{M})_{\alpha\in L_0}$ is a family of pre filters on X and is called the *large valued base* of $\mathcal{M}$. Recall that a pre filter on X ([31]) is a non-empty proper subset $\mathcal{F}$ of L^X such that: (1) $\mu, \eta \in \mathcal{F}$ implies $\mu \wedge \eta \in \mathcal{F}$ and (2) from $\mu \in \mathcal{F}$ and $\mu \leq \eta$ it follows $\eta \in \mathcal{F}$.

A subset $\mathcal{B}$ of L^X is said to be superior fuzzy filter base ([18]) if the following conditions are fulfilled:

(S1) $\overline{\alpha} \in \mathcal{B}$ for every $\alpha \in L$.

(S2) For all $\mu, \eta \in \mathcal{B}$, there is a fuzzy set $\sigma \in \mathcal{B}$ such that $\sigma \leq \mu$, $\sigma \leq \eta$ and $\sup \sigma = \sup \mu \wedge \sup \eta$.

Each superior fuzzy filter base $\mathcal{B}$ generated a homogeneous fuzzy filter $\mathcal{M}$ on X by $\mathcal{M}(\mu) = \bigvee_{\eta\in\mathcal{B}, \eta\leq\mu} \sup \eta$ for all $\mu \in L^X$ and each homogeneous fuzzy filter $\mathcal{M}$ can be generated by a superior fuzzy filter base. e.g. by $\text{base}\,\mathcal{M} = \{\mu \in L^X \mid \mathcal{M}(\mu) = \sup \mu\} = \{\mu \wedge \overline{\mathcal{M}(\mu)} \mid \mu \in L^X\}$, where $\text{base}\,\mathcal{M}$ will be called the *large superior fuzzy filter base* of $\mathcal{M}$. If X is a non-empty set and μ is a fuzzy subset of X, then $\mathcal{B} = \{\mu \wedge \overline{\alpha} \mid \alpha \in L\} \cup \{\overline{\alpha} \mid \alpha \in L\}$ is a superior fuzzy filter base of a homogeneous fuzzy filter on X, called *superior principal fuzzy filter* generated by μ and will be denoted by $[\mu]$. As shown in [15], for each $\mu \in L^X$, the superior principal fuzzy filter $[\mu]$ is given by:

$$[\mu](\eta) = \bigvee_{\mu\wedge\overline{\alpha}\leq\eta} \sup(\mu \wedge \overline{\alpha}) \vee \bigvee_{\overline{\alpha}\leq\eta} \alpha$$

for all $\eta \in L^X$. In case L is a complete chain and μ is not constant, we have $[\mu](\eta) = \sup \mu$, when $\mu \leq \eta$ and $[\mu](\eta) = \bigwedge_{\eta(x)\leq\mu(x)} \eta(x)$ otherwise for all $\eta \in L^X$. For each ordinary subset M of X, we have $[\chi_M](\eta) = \bigvee_{x\in M} \dot{x}$, where χ_M is the characteristic function of M.

Fuzzy topogenous order and fuzzy topogenous structure. The binary relation $<<$ on L^X is said to be *fuzzy topogenous order* on the set X ([29]) if the following conditions are fulfilled:

(1) $\overline{\alpha} << \overline{\alpha}$ holds for all $\alpha \in \{0,1\}$.

(2) If $\mu << \eta$, then $\mu \leq \eta$ holds for all $\mu, \eta \in L^X$.

(3) If $\mu_1 \leq \mu << \eta \leq \eta_1$, then $\mu_1 << \eta_1$ holds.

(4) If $\mu_1 << \eta_1$ and $\mu_2 << \eta_2$, then $\mu_1 \wedge \mu_2 << \eta_1 \wedge \eta_2$ and $\mu_1 \vee \mu_2 << \eta_1 \vee \eta_2$ are ho1d for all $\mu_i, \eta_j \in L^X$, where $i, j \in \{1, 2\}$.

The fuzzy topogenous order $<<$ is said to be *fuzzy topogenous structure* if it fulfilled the following additional condition:

(5) If $\mu << \eta$,. then there is $\sigma \in L^X$ such that $\mu << \sigma$ and $\sigma << \eta$ are hold for all $\mu, \eta \in L^X$.

The fuzzy topogenous structure $<<$ is said to be *fuzzy topogenous complementarily symmetric* if it fulfilled the condition:

(6) If $\mu << \eta$, then $co\,\eta << co\,\mu$ holds for all $\mu, \eta \in L^X$.

Fuzzy topologies. By the fuzzy topology on a set X ([15,23]), we mean a subset τ of L^X which is closed with respect to all suprema and all finite infima and contains the constant fuzzy sets $\overline{0}$ and $\overline{1}$. The set X equipped with a fuzzy topology τ on X is called *fuzzy topological space*. For each fuzzy topological space (X, τ), the elements of τ are called open fuzzy subsets of this space. If τ_1 and τ_2 are fuzzy topologies on a set X, then τ_2 is said to be finer than τ_1 and τ_1 is said to be coarser than τ_2 provided $\tau_1 \subseteq \tau_2$ holds. The fuzzy topological space (X, τ) and also τ are said to be stratified provided $\overline{\alpha} \in \tau$ holds for all $\alpha \in L$, that is, all constant fuzzy sets are open ([30]). For each fuzzy set $\mu \in L^X$, the strong α-cut and the weak α-cut of μ are the ordinary subsets $S_\alpha(\mu) = \{x \in X \mid \mu(x) > \alpha\}$ and

$W_\alpha(\mu)=\{x\in X \mid \mu(x)\geq\alpha\}$ of X, respectively. For each complete chain, L the α-level topology and the initial topology ([30]) of the fuzzy topology τ on the set X are defined as follows:

$$\tau_\alpha=\{S_\alpha(\mu)\in P(X) \mid \mu\in\tau\} \text{ and } i(\tau)=\inf\{\tau_\alpha \mid \alpha\in L_1\},$$

respectively. where inf is the infimum with respect to the finer relation for topologies. On other hand if (X,T) is an ordinary topological space. then the induced fuzzy topology on X is given by Lowen in [30] as the following:

$$\omega(T)=\{\mu\in L^X \mid S_\alpha(\mu)\in T \text{ for all } \alpha\in L_1\}$$

The fuzzy unit interval. The fuzzy unit interval will be denoted by I_L and it is defined in [21] as the fuzzy subset:

$$\mathrm{I_L}=\{x\in\mathbb{R}_\mathrm{L}^* \mid x\leq\tilde{1}\},$$

Where, $\mathrm{I}=[0,1]$ is the closed unit interval and $\mathbb{R}_\mathrm{L}^*=\{x\in\mathbb{R}_\mathrm{L} \mid x(0)=1 \text{ and } \tilde{0}\leq x\}$ is the set of all positive fuzzy real numbers. Note that, the binary relation $\leq$ is defined on $\mathbb{R}_\mathrm{L}$ as follows:

$$x\leq y \Leftrightarrow x_{\alpha1}\leq y_{\alpha1} \text{ and } x_{\alpha2}\leq y_{\alpha2}, \text{ for all } x,y\in\mathbb{R}_\mathrm{L},$$

Where, $x_{\alpha1}=\inf\{x\in\mathbb{R} \mid x(z)\geq\alpha\}$ and $x_{\alpha2}=\sup\{x\in\mathbb{R} \mid x(z)\geq\alpha\}$ for all $x\in\mathbb{R}_\mathrm{L}$ and $\alpha\in L_0$. Note that the family Ω which is defined by:

$$\Omega=\{R_\delta \mid \mathrm{I_L} \mid \delta\in\mathrm{I}\}\cup\{R^\delta \mid \mathrm{I_L} \mid \delta\in\mathrm{I}\}\cup\{\tilde{0} \mid \mathrm{I_L}\}$$

is a base for a fuzzy topology $\Im$ on $\mathrm{I_L}$ ([21]), where R_δ and R^δ are the fuzzy subsets of $\mathbb{R}_\mathrm{L}$ defined by $R_\delta(x)=\bigvee_{\alpha>\delta}x(\alpha)$ and $R^\delta(x)=(\bigvee_{\alpha\geq\delta}x(\alpha))'$ for all $x\in\mathbb{R}_\mathrm{L}$ and $\delta\in\mathbb{R}$. The restrictions of R_δ and R^δ on $\mathrm{I_L}$ are the fuzzy subsets $R_\delta \mid \mathrm{I_L}$ and $R^\delta \mid \mathrm{I_L}$, respectively. Recall that:

$$R^\delta(x)\wedge R^\gamma(y)\leq R^{\delta+\gamma}(x+y), \tag{2.2}$$

Where, $x+y$ is the fuzzy real number defined by the rule $(x+y)(\xi)=\bigvee_{\varsigma,\zeta\in\mathbb{R},\varsigma+\zeta=\xi}(x(\varsigma)\wedge y(\zeta))$ for all $\xi\in\mathbb{R}$.

The fuzzy function family. Let X be non-empty set. By the fuzzy function family Φ on X, we mean the set of all fuzzy real functions $f:X\to\mathrm{I_L}$ ([21]). Consider $\mu,\eta\in L^X$, then the fuzzy real function $f:X\to\mathrm{I_L}$ is said to be separate μ and η if $\overline{0}\leq f(x)\leq\overline{1}$ holds for all $x\in X$, $x_1\leq\mu$ implies that $f(x)=\overline{1}$ and $y_1\leq\eta$ implies that $f(y)=\overline{0}$, where $x_1,y_1\in S(X)$. Moreover, if Φ is an fuzzy function family on X, then the fuzzy subsets $\mu,\eta\in L^X$ are called Φ-separable or Φ-separated if there exists an fuzzy real function $h\in\Phi$ separating them.

The operation on fuzzy sets. In the sequel, let a fuzzy topological space (X,τ) be fixed. By the *operation* ([25]) on a set X we mean a mapping $\varphi:L^X\to L^X$ such that $\operatorname{int}\mu\leq\mu^\varphi$ holds, for all $\mu\in L^X$, where, μ^φ denotes the value of φ at μ. The class of all operations on X will be denoted by $O_{(L^X,\tau)}$. By the *identity operation* on $O_{(L^X,\tau)}$, we mean the operation $1_{L^X}:L^X\to L^X$ such that $1_{L^X}(\mu)=\mu$, for all $\mu\in L^X$ and the *constant operation* on $O_{(L^X,\tau)}$ is the operation $\mathbf{c}_{L^X}:L^X\to L^X$ such that $\mathbf{c}_{L^X}(\mu)=\overline{1}$, for all $\mu\in L^X$. Consider the binary relation $\leq$ is a partially ordered relation on $O_{(L^X,\tau)}$ defined as follows: $\varphi_1\leq\varphi_2 \Leftrightarrow \mu^{\varphi_1}\leq\mu^{\varphi_2}$ for all $\mu\in L^X$, then the ordered pair $(O_{(L^X,\tau)},\leq)$ is a *completely distributive lattice*. As a directly application on this completely distributive lattice, the operation $\varphi:L^X\to L^X$ is said to be:

(i) *Isotone* if $\mu\leq\eta$ implies $\mu^\varphi\leq\eta^\varphi$, for all $\mu,\eta\in L^X$.

(ii) *Weakly finite intersection preserving* (*wfip*, for short) with respect to $\mathcal{A} \subseteq L^X$ if $\eta \wedge \mu^{\varphi} \leq (\eta \wedge \mu)^{\varphi}$ holds, for all $\eta \in \mathcal{A}$ and $\mu \in L^X$.

(iii) *Idempotent* if $\mu^{\varphi} = (\mu^{\varphi})^{\varphi}$, for all $\mu \in L^X$.

The operations $\varphi, \psi \in O_{(L^X,\tau)}$ are said to be dual if $\mu^{\psi} = co((co\,\mu)^{\varphi}$ or equivalently $\mu^{\varphi} = co((co\,\mu)^{\psi}$ for all $\mu \in L^X$, where $co\,\mu$ denotes the complementation of μ. The dual operation of the operation $\varphi : L^X \to L^X$ will be denoted by $\tilde{\varphi} : L^X \to L^X$. In the classical case of $L = \{0,1\}$, by the operation on a set X, we mean the mapping $\varphi : P(X) \to P(X)$ such that $\operatorname{int} A \subseteq A^{\varphi}$ for all A in the power set $P(X)$. The identity operation on the class of all ordinary operations $O_{(P(X),T)}$ on X will be denoted by $i_{P(X)}$, where $i_{P(X)}(A) = A$ for all $A \in P(X)$.

φ-open fuzzy sets. Let a fuzzy topological space (X,τ) be fixed and $\varphi \in O_{(L^X,\tau)}$. Then the fuzzy set $\mu : X \to L$ is called φ*-open* fuzzy set if $\mu \leq \mu^{\varphi}$ holds. We will denote the class of all φ-open fuzzy sets on X by $\varphi OF(X)$. The fuzzy set μ is called φ*-closed* if its complement $co\,\mu$ is φ-open. The two operations $\varphi, \psi \in O_{(L^X,\tau)}$ are *equivalent* and written $\varphi \approx \psi$ if and only if $\varphi OF(X) = \psi OF(X)$.

$\varphi_{1,2}$-interiors fuzzy sets. Let a fuzzy topological space (X,τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the $\varphi_{1,2}$-interior of the fuzzy set $\mu : X \to L$ is a mapping $\varphi_{1,2}.\operatorname{int}\mu : X \to L$ defined by:

$$\varphi_{1,2}.\operatorname{int}\mu = \bigvee_{\eta \in \varphi_1 OF(X),\, \eta^{\varphi_2} \leq \mu} \eta \tag{2.3}$$

As Shown in [5], $\varphi_{1,2}.\operatorname{int}\mu$ is the greatest φ_1-open fuzzy set η such that η^{φ_2} less than or equal to μ. The fuzzy set μ is said to be $\varphi_{1,2}$*-open* if $\mu \leq \varphi_{1,2}.\operatorname{int}\mu$. The class of all $\varphi_{1,2}$-open fuzzy sets of X will be denoted by $\varphi_{1,2}OF(X)$. The complement $co\,\mu$ of a $\varphi_{1,2}$-open fuzzy subset μ will be called $\varphi_{1,2}$-closed. The class of all $\varphi_{1,2}$-closed fuzzy subsets of X will be denoted by $\varphi_{1,2}CF(X)$. In the classical case of $L = \{0,1\}$, the fuzzy topological space (X,τ) is up to an identification by the ordinary topological space (X,T) and $\varphi_{1,2}.\operatorname{int}\mu$ is the classical one. Hence, in this case the ordinary subset A of X is $\varphi_{1,2}$-open if $A \subseteq \varphi_{1,2}.\operatorname{int}A$. The complement of a $\varphi_{1,2}$-open subset A of X will be called $\varphi_{1,2}$-closed. The classes of all $\varphi_{1,2}$-open and of all $\varphi_{1,2}$-closed subsets of X will be denoted by $\varphi_{1,2}O(X)$ and $\varphi_{1,2}C(X)$, respectively. Clearly, the ordinary subset F is $\varphi_{1,2}$-closed if and only if $\varphi_{1,2}.\operatorname{cl}_T F = F$.

Proposition 2.3 [5]. If (X,τ) is a fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then, for each $\mu, \eta \in L^X$, the mapping $\varphi_{1,2}.\operatorname{int}\mu : X \to L$ fulfills the following axioms:

(i) If $\varphi_2 \geq 1_{L^X}$, then $\varphi_{1,2}.\operatorname{int}\mu \leq \mu$ holds.

(ii) $\varphi_{1,2}.\operatorname{int}$ is isotone, that is, if $\mu \leq \eta$ then $\varphi_{1,2}.\operatorname{int}\mu \leq \varphi_{1,2}.\operatorname{int}\eta$ holds .

(iii) $\varphi_{1,2}.\operatorname{int}\overline{1} = \overline{1}$.

(iv) If $\varphi_2 \geq 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}.\operatorname{int}(\mu \wedge \eta) = \varphi_{1,2}.\operatorname{int}\mu \wedge \varphi_{1,2}.\operatorname{int}\eta$.

(v) If φ_2 is isotone and idempotent operation, then $\varphi_{1,2}.\operatorname{int}\mu \leq \varphi_{1,2}.\operatorname{int}(\varphi_{1,2}.\operatorname{int}\mu)$ holds.

(vi) $\varphi_{1,2}.\operatorname{int}(\bigvee_{i \in I} \mu_i) = \bigvee_{i \in I} \varphi_{1,2}.\operatorname{int}\mu_i$ for all $\mu_i \in \varphi_{1,2}OF(X)$.

Proposition 2.4 [5]. Let (X,τ) be a fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then, the following statements are fulfilled:

(i) If $\varphi_2 \geq 1_{L^X}$, then the class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$-open fuzzy subsets of X forms extended fuzzy topology on X, denoted by $\tau^{\varphi_{1,2}}$ ([19]).

(ii) If $\varphi_2 \geq 1_{L^X}$, then the class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$-open fuzzy subsets of X forms a supra fuzzy topology on X, denoted by $\overline{\tau}^{\varphi_{1,2}}$ ([19]).

(iii) If $\varphi_2 \geq 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}OF(X)$ is a fuzzy pre topology on X, denoted by $\tau^{\wedge}_{\varphi_{1,2}}$ ([19]).

(iv) If $\varphi_2 \geq 1_{L^X}$ is isotone and idempotent operation and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}OF(X)$ forms a fuzzy topology on X, denoted by $\tau_{\varphi_{1,2}}$ ([15,23]).

From Propositions 2.3 and 2.4, if the fuzzy topological space (X,τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then

$$\varphi_{1,2}OF(X) = \{\mu \in L^X \mid \mu \leq \varphi_{1,2}.\text{int}\,\mu\} \tag{2.4}$$

and the following conditions are fulfilled:

(I1) If $\varphi_2 \geq 1_{L^X}$, then $\varphi_{1,2}.\text{int}\,\mu \leq \mu$ holds for all $\mu \in L^X$.

(I2) If $\mu \leq \eta$ then $\varphi_{1,2}.\text{int}\,\mu \leq \varphi_{1,2}.\text{int}\,\eta$ holds for all $\mu,\eta \in L^X$.

(I3) $\varphi_{1,2}.\text{int}\,\overline{1} = \overline{1}$.

(I4) If $\varphi_2 \geq 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}.\text{int}(\mu \wedge \eta) = \varphi_{1,2}.\text{int}\,\mu \wedge \varphi_{1,2}.\text{int}\,\eta$ for all $\mu,\eta \in L^X$.

(I5) If $\varphi_2 \geq 1_{L^X}$ is isotone and idempotent operation, then $\varphi_{1,2}.\text{int}(\varphi_{1,2}.\text{int}\,\mu) = \varphi_{1,2}.\text{int}\,\mu$ for all $\mu \in L^X$.

The characterized fuzzy spaces. Independently on the fuzzy topologies, the notion of $\varphi_{1,2}$-interior operator for the fuzzy sets can be defined as a mapping $\varphi_{1,2}.\text{int}: L^X \to L^X$, which fulfills **(I1)**, to **(I5)**. It is well-known that (2.3) and (2.4) give a one-to-one correspondence between the class of all $\varphi_{1,2}$-open fuzzy sets and these operators, that is, $\varphi_{1,2}OF(X)$ can be characterized by $\varphi_{1,2}$-interior operators. In this case $(X, \varphi_{1,2}.\text{int})$ as well as $(X, \varphi_{1,2}OF(X))$ will be called *characterized fuzzy space* ([5]) of all the $\varphi_{1,2}$-open fuzzy subsets of X. The characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is said to be *stratified* if and only if $\varphi_{1,2}.\text{int}\,\overline{\alpha} = \overline{\alpha}$ for all $\alpha \in L$. As shown in [5], the characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is stratified if the related fuzzy topological space (X,τ) is stratified. Moreover, the characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is said to have the *weak infimum property* ([19]), provided that $\varphi_{1,2}.\text{int}(\mu \wedge \overline{\alpha}) = \varphi_{1,2}.\text{int}\,\mu \wedge \varphi_{1,2}.\text{int}\,\overline{\alpha}$ for all $\mu \in L^X$ and all $\alpha \in L$. The characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is said to be *strongly stratified* ([19]), provided $\varphi_{1,2}.\text{int}$ is stratified and have the weak infimum property. If $(X, \varphi_{1,2}.\text{int})$ and $(X, \psi_{1,2}.\text{int})$ are two characterized fuzzy spaces, then $(X, \varphi_{1,2}.\text{int})$ is said to be finer than $(X, \psi_{1,2}.\text{int})$ and denoted by $\varphi_{1,2}.\text{int} \leq \psi_{1,2}.\text{int}$ provided $\varphi_{1,2}.\text{int}\,\mu \geq \psi_{1,2}.\text{int}\,\mu$ holds for all $\mu \in L^X$. If τ is a fuzzy topology on a set X and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$, then by the $\alpha-$*level characterized space* and the *initial characterized space* of the fuzzy topological space (X,τ), we mean the characterized spaces $(X,(\varphi_{1,2}OF(X))_\alpha)$ and $(X,i(\varphi_{1,2}OF(X)))$, respectively where $(\varphi_{1,2}OF(X))_\alpha$ and $i(\varphi_{1,2}OF(X))$ are defined as follows:

$$(\varphi_{1,2}OF(X))_\alpha = \{S_\alpha(\mu) \in P(X) \mid \mu \in \varphi_{1,2}OF(X)\} \text{ and } i(\varphi_{1,2}OF(X)) = \inf\{(\varphi_{1,2}OF(X))_\alpha \mid \alpha \in L_1\}.$$

Sometimes, we denote to the α – level characterized space and the initial characterized space of the fuzzy topological space (X,τ) by $(X,\varphi_{1,2}.\text{int}_\alpha)$ and $(X,\varphi_{1,2}.\text{int}_i)$, respectively. On other hand if (X,T) is an ordinary topological space and $\varphi_1, \varphi_2 \in O_{(P(X),T)}$, then the *induced characterized fuzzy space* will be denoted by $(X,\omega(\varphi_{1,2}O(X)))$ or by $(X,\varphi_{1,2}.\text{int}_\omega)$ and it defined by:

$$\omega(\varphi_{1,2}O(X)) = \{\mu \in L^X \mid S_\alpha(\mu) \in \varphi_{1,2}O(X) \text{ for all } \alpha \in L_1\}.$$

If $\varphi_1 = \text{int}$ and $\varphi_2 = 1_{L^X}$, then the class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$-open fuzzy subset of X coincide with the fuzzy topology τ and hence the characterized fuzzy space $(X,\varphi_{1,2}.\text{int})$ coincide with the fuzzy topological space (X,τ) presented in [15,23]. Another special choices for the operations φ_1 and φ_2 are obtained in Table (1).

The $\varphi_{1,2}$-fuzzy neighborhood filters. An important notion in the characterized fuzzy space $(X,\varphi_{1,2}.\text{int})$ is that of the $\varphi_{1,2}$-fuzzy neighborhood filter at the point and at the ordinary subset in this space. Let (X,τ) be a fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. As follows by **(I1)** to **(I5)** for each, $x \in X$ the mapping $\mathcal{N}_{\varphi_{1,2}}(x): L^X \to L$ which is defined by:

$$\mathcal{N}_{\varphi_{1,2}}(x)(\mu) = (\varphi_{1,2}.\text{int}\,\mu)(x) \tag{2.5}$$

for all $\mu \in L^X$ is a fuzzy filter on X, called $\varphi_{1,2}$*-fuzzy neighborhood filter* at x ([5]). Moreover, if $\varphi \neq F \subseteq P(X)$, then the $\varphi_{1,2}$-fuzzy neighborhood filter at the ordinary subset F will be denoted by $\mathcal{N}_{\varphi_{1,2}}(F)$ and it will be defined by:

$$\mathcal{N}_{\varphi_{1,2}}(F) = \bigvee_{x\in F} \mathcal{N}_{\varphi_{1,2}}(x).$$

Since $\mathcal{N}_{\varphi_{1,2}}(x)$ is a fuzzy filter for all $x \in X$, then $\mathcal{N}_{\varphi_{1,2}}(F)$ is also fuzzy filter on X. Moreover, because of $[\chi_F] = \bigvee_{x\in F} \dot{x}$, then we have $\mathcal{N}_{\varphi_{1,2}}(F) \geq [\chi_F]$ holds.

More generally, if the related $\varphi_{1,2}$-interior operator fulfill the axioms **(I1)** and **(I2)** only, then the mapping $\mathcal{N}_{\varphi_{1,2}}(x): L^X \to L$, which is defined by (2.5) is a fuzzy stack ([19]), called $\varphi_{1,2}$*-fuzzy neighborhood stack* at x. Moreover, if the $\varphi_{1,2}$-interior operator fulfill the axioms **(I1)**, **(I2)** and **(I4)** such that in **(I4)** instead of, $\eta \in L^X$ we take $\overline{\alpha}$, then the mapping $\mathcal{N}_{\varphi_{1,2}}(x): L^X \to L$, is a fuzzy stack with the cutting property, called $\varphi_{1,2}$*-fuzzy neighborhood stack with the cutting property* at x. Obviously, the $\varphi_{1,2}$-fuzzy neighborhood filters fulfill the following axioms:

(N1) $\dot{x} \leq \mathcal{N}_{\varphi_{1,2}}(x)$ holds for all $x \in X$.

(N2) $\mathcal{N}_{\varphi_{1,2}}(x)(\mu) \leq \mathcal{N}_{\varphi_{1,2}}(x)(\eta)$ holds for all $\mu,\eta \in L^X$ and $\mu \leq \eta$.

(N3) $\mathcal{N}_{\varphi_{1,2}}(x)(y \mapsto \mathcal{N}_{\varphi_{1,2}}(y)(\mu)) = \mathcal{N}_{\varphi_{1,2}}(x)(\mu)$, for all $x \in X$ and $\mu \in L^X$.

Clearly, $y \mapsto \mathcal{N}_{\varphi_{1,2}}(y)(\mu)$ is the fuzzy set $\varphi_{1,2}.\text{int}\,\mu$.

The characterized fuzzy space $(X,\varphi_{1,2}.\text{int})$ of all $\varphi_{1,2}$-open fuzzy subsets of a set X is characterized as a fuzzy filter pre topology ([5]), that is, as a mapping $\mathcal{N}_{\varphi_{1,2}}(x): X \to \mathcal{F}_L X$ such that the axioms **(N1)** to **(N3)** are fulfilled.

The valued $\varphi_{1,2}$-fuzzy neighborhoods. Let (X,τ) be a fuzzy topological space and $\varphi_1,\varphi_2\in O_{(L^X,\tau)}$. Then the fuzzy subset μ of X will be called $\varphi_{1,2}\alpha$ *-fuzzy neighborhood* at the point $x\in X$ if

$$\alpha\le(\varphi_{1,2}.\mathrm{int}\,\mu)(x) \tag{2.6}$$

holds for some $\alpha\in L_0$ ([5]). Because of Proposition 2.2. the fuzzy subset μ of X is $\varphi_{1,2}\alpha$-fuzzy neighborhood at x if and only if $\mu\in\alpha\text{-pr}\,(\mathcal{N}_{\varphi_{1,2}}(x))$ holds, where $\mathcal{N}_{\varphi_{1,2}}(x)$ is given by (2.5). By a *valued* $\varphi_{1,2}$*-fuzzy neighborhood* at x, we mean an $\varphi_{1,2}\alpha$-fuzzy neighborhood at x for some $\alpha\in L_0$. For each $\alpha\in L_0$ and $x\in X$, let $\mathrm{N}_\alpha(x)$ be the set of all $\varphi_{1,2}\alpha$-fuzzy neighborhood at x, that is, $\mathrm{N}_\alpha(x)=\{\mu\in L^X \mid \alpha\le(\varphi_{1,2}.\mathrm{int}\,\mu)(x)\}$. Then the family $(\mathrm{N}_\alpha(x))_{\alpha\in L_0}$ is the *large valued base* of $\mathcal{N}_{\varphi_{1,2}}(x)$ at the point $x\in X$. The $\varphi_{1,2}$-open fuzzy sets is characterized by the valued $\varphi_{1,2}$-fuzzy neighborhood at x as follows:

$$\mu\in L^X \text{ is } \varphi_{1,2}\text{-open} \Leftrightarrow \text{for all } x\in X \text{ with } \mu(x)>0 \text{ there is an } \varphi_{1,2}\mu(x)\text{-fuzzy neighborhood } \eta \text{ at } x \text{ with } \eta\le\mu. \tag{2.7}$$

Remark 2.1 If $\varphi_1=\mathrm{int}$ and $\varphi_2=1_{L^X}$, then the notion of the valued $\varphi_{1,2}$-fuzzy neighborhood at the ordinary point x is closely related to that of fuzzy neighborhood at the fuzzy point, used in the fuzzy topology (cp. [32]). If $x_\alpha\in S(X)$ is a fuzzy point, then the *fuzzy neighborhood* at x_α is nothing else than the α-fuzzy neighborhood at x.

The $\varphi_{1,2}$-fuzzy convergence. Let a topological L-spaces (X,τ) be fixed and $\varphi_1,\varphi_2\in O_{(L^X,\tau)}$. If x is a point in the characterized fuzzy space $(X,\varphi_{1,2}.\mathrm{int})$, $F\subseteq X$ and $\mathcal{M}$ is a fuzzy filter on X. Then $\mathcal{M}$ is said to be $\varphi_{1,2}$-fuzzy convergence ([2]) to x and written $\mathcal{M}\xrightarrow[\varphi_{1,2}.\mathrm{int}]{}x$, provided $\mathcal{M}$ is finer than the $\varphi_{1,2}$-fuzzy neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(x)$. Moreover, $\mathcal{M}$ is said to be $\varphi_{1,2}$-convergence to the ordinary subset F and written $\mathcal{M}\xrightarrow[\varphi_{1,2}.\mathrm{int}]{}F$, provided $\mathcal{M}$ is finer than the $\varphi_{1,2}$-fuzzy neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(x)$ for all $x\in F$, that is, $\mathcal{M}$ is finer than the $\varphi_{1,2}$-fuzzy neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(F)$.

The $\varphi_{1,2}$-closure operator and internal $\varphi_{1,2}$-closure of fuzzy sets. Let a fuzzy topological space (X,τ) be fixed and $\varphi_1,\varphi_2\in O_{(L^X,\tau)}$. The *internal* $\varphi_{1,2}$*-closure* ([7]) of the fuzzy set $\mu:X\to L$ is the mapping $\varphi_{1,2}.\mathrm{cl}\,\mu:X\to L$ defined by:

$$(\varphi_{1,2}.\mathrm{cl}\,\mu)(x)=\bigvee_{\mathcal{M}\le\mathcal{N}_{\varphi_{1,2}}(x)}\mathcal{M}(\mu) \tag{2.8}$$

for all $x\in X$. In (2.8), the fuzzy filters $\mathcal{M}$ my have additional properties, e.g, we my assume that they are homogeneous or even that they are ultra fuzzy filters. Obviously, $\varphi_{1,2}.\mathrm{cl}\,\mu\ge\mu$ holds for all $\mu\in L^X$. The mapping $\varphi_{1,2}.\mathrm{cl}:\mathcal{F}_LX\to\mathcal{F}_LX$ which assigns $\varphi_{1,2}.\mathrm{cl}\,\mathcal{M}$ to each fuzzy filter $\mathcal{M}$ on X, that is,

$$(\varphi_{1,2}.\mathrm{cl}\,\mathcal{M})(\mu)=\bigvee_{\varphi_{1,2}.\mathrm{cl}\,\rho\le\mu}\mathcal{M}(\rho) \tag{2.9}$$

is called $\varphi_{1,2}$*-closure operator* ([7]) of the characterized fuzzy space $(X,\varphi_{1,2}.\mathrm{int})$ with respect to the related fuzzy topology τ. Obviously, the $\varphi_{1,2}$-closure operator $\varphi_{1,2}.\mathrm{cl}$ is isotone hull operator, that is, for all $\mathcal{M},\mathcal{N}\in\mathcal{F}_LX$ we have

$\mathcal{M}\le\mathcal{N}$ **implies** $\varphi_{1,2}.\mathrm{cl}\,\mathcal{M}\le\varphi_{1,2}.\mathrm{cl}\,\mathcal{N}$ and that $\mathcal{M}\le\varphi_{1,2}.\mathrm{cl}\,\mathcal{M}$ holds.

Lemma 2.2 [2]. Let (X,τ) be a fuzzy topological space and $\varphi_1,\varphi_2\in O_{(L^X,\tau)}$. Then for each $x\in X$, we have that $\varphi_{1,2}.\mathrm{cl}\,\dot{x}=\dot{x}$ implies $\varphi_{1,2}.\mathrm{cl}\{x\}=\{x\}$.

The $\varphi_{1,2}\,\psi_{1,2}$- fuzzy continuity. Let now the fuzzy topological spaces (X,τ_1) and (Y,τ_2) are fixed, $\varphi_1,\varphi_2 \in O_{(L^X,\tau_1)}$ and $\psi_1,\psi_2 \in O_{(L^Y,\tau_2)}$. Then the mapping $f:(X,\varphi_{1,2}.\text{int}) \to (Y,\psi_{1,2}.\text{int})$ is said to be $\varphi_{1,2}\,\psi_{1,2}$-*fuzzy continuous* ([5]) if and only if

$$(\psi_{1,2}.\text{int}\,\eta)\circ f \le \varphi_{1,2}.\text{int}\,(\eta\circ f) \tag{2.10}$$

holds for all $\eta \in L^Y$. If an order reversing involution $\alpha \mapsto \alpha'$ of L is given, then we have that f is $\varphi_{1,2}\,\psi_{1,2}$-fuzzy continuous if and only if $\varphi_{1,2}.\text{cl}(\eta\circ f) \le (\psi_{1,2}.\text{cl}\,\eta)\circ f$ for all $\eta \in L^Y$, where $\varphi_{1,2}.\text{cl}$ and $\psi_{1,2}.\text{cl}$ are the closure operators related to $\varphi_{1,2}.\text{int}$ and $\psi_{1,2}.\text{int}$, respectively. Obviously if f is $\varphi_{1,2}\,\psi_{1,2}$-fuzzy continuity mapping, then the inverse mapping $f^{-1}:(Y,\psi_{1,2}.\text{int}) \to (X,\varphi_{1,2}.\text{int})$ is $\varphi_{1,2}\,\psi_{1,2}$-fuzzy continuous mapping, that is, $(\varphi_{1,2}.\text{int}\,\mu)\circ f^{-1} \le \psi_{1,2}.\text{int}\,(\mu\circ f^{-1})$ holds for all $\mu \in L^X$. By means of the $\varphi_{1,2}$-fuzzy neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(x)$ of $\varphi_{1,2}.\text{int}$ at x and the $\psi_{1,2}$-fuzzy neighborhood filter $\mathcal{N}_{\psi_{1,2}}(x)$ of $\psi_{1,2}.\text{int}$ at x, the $\varphi_{1,2}\,\psi_{1,2}$-fuzzy continuity of f is also characterized as follows:

The mapping $f:(X,\varphi_{1,2}.\text{int}) \to (Y,\psi_{1,2}.\text{int})$ is $\varphi_{1,2}\,\psi_{1,2}$-fuzzy continuous if for each $x \in X$, the inequality

$$\mathcal{N}_{\psi_{1,2}}(f(x)) \ge \mathcal{F}_L f\,(\mathcal{N}_{\varphi_{1,2}}(x)) \tag{2.11}$$

holds. Obviously, in case of $L=\{0,1\}$, $\varphi_1=\psi_1=\text{int}$, $\varphi_2=1_{L^X}$ and $\psi_2=1_{L^Y}$, then the $\varphi_{1,2}\,\psi_{1,2}$-fuzzy continuity of f is coincides with the usual continuity.

The characterized fuzzy R_k and T_s spaces. The notions of characterized fuzzy R_k and characterized fuzzy T_s spaces are investigated and studied in [2,3,4] for all $k \in \{0,1,2,3\}$ and $s \in \{0,1,2,2\frac{1}{2},3,4\}$. These characterized fuzzy spaces depend only on the usual points and the operation defined on the class of all fuzzy subsets of X endowed with a fuzzy topological space (X,τ). The characterized fuzzy R_k and the characterized fuzzy T_s spaces will be denoted by *characterized* FR_k and *characterized* FT_s, respectively for short. Let a fuzzy topological space (X,τ) be fixed and $\varphi_1,\varphi_2 \in O_{(L^X,\tau)}$. Then the characterized fuzzy space $(X,\varphi_{1,2}.\text{int})$ is said to be:

(1) Characterized FR_0 space (resp. FR_1 space), if for all $x,y \in X$ such that $x \ne y$ and $\dot{x} \not\le \varphi_{1,2}.\text{cl}\,\dot{y}$ implies $\dot{y} \not\le \varphi_{1,2}.\text{cl}\,\dot{x}$ (resp. $\mathcal{N}_{\varphi_{1,2}}(x) \wedge \mathcal{N}_{\varphi_{1,2}}(y)$ does not exists). The related fuzzy topological space (X,τ) is said to be $F\varphi_{1,2}$-R_0 (resp. $F\varphi_{1,2}$-R_1), if for all $x,y \in X$ such that $x \ne y$ and $\dot{x} \not\le \varphi_{1,2}.\text{cl}\,\dot{y}$, we have $\varphi_{1,2}.\text{cl}\,\dot{x} \not\le \dot{y}$ (resp. $\dot{x} \not\le \varphi_{1,2}.\text{cl}(\mathcal{N}_{\varphi_{1,2}}(y))$ and $\dot{y} \not\le \varphi_{1,2}.\text{cl}(\mathcal{N}_{\varphi_{1,2}}(x))$).

(2) Characterized FR_2 space (resp. FR_3 space), if for all $x \in X$, $F \in \varphi_{1,2}C(X)$ such that $x \notin F$ (resp. $F_1,F_2 \in \varphi_{1,2}C(X)$ such that $F_1 \cap F_2 = \varphi$), the infimum $\mathcal{N}_{\varphi_{1,2}}(x) \wedge \mathcal{N}_{\varphi_{1,2}}(F)$ (resp. $\mathcal{N}_{\varphi_{1,2}}(F_1) \wedge \mathcal{N}_{\varphi_{1,2}}(F_2)$) does not exists). The related fuzzy topological space (X,τ) is said to be $F\varphi_{1,2}$-R_2 (resp. $F\varphi_{1,2}$-R_3) if for all $x \in X$, (resp. $F \in \varphi_{1,2}C(X)$) and $\mathcal{M} \in \mathcal{F}_L X$ such that $\mathcal{M} \xrightarrow[\varphi_{1,2}.\text{int}]{} x$ (resp. $\mathcal{M} \xrightarrow[\varphi_{1,2}.\text{int}]{} F$) we have $\varphi_{1,2}.\text{cl}\,\mathcal{M} \xrightarrow[\varphi_{1,2}.\text{int}]{} x$ (resp. $\varphi_{1,2}.\text{cl}\,\mathcal{M} \xrightarrow[\varphi_{1,2}.\text{int}]{} F$).

(3) Characterized FT_0 space (resp. FT_1 space), if for all $x,y \in X$ such that $x \ne y$, there exists $\mu,\eta \in L^X$ and $\alpha,\beta \in L_0$ such that $\mu(x) < \alpha \le (\varphi_{1,2}.\text{int}\,\mu)(y)$ or (resp. and) $\eta(y) < \beta \le (\varphi_{1,2}.\text{int}\,\eta)(x)$ are hold. The related fuzzy topological space (X,τ) is said to be $F\varphi_{1,2}$-T_0 (resp. $F\varphi_{1,2}$-T_1) if for all $x, y \in X$ such that $x \neq y$, we have $\dot{x} \not\leq \mathcal{N}_{\varphi_{1,2}}(y)$ or (resp. and) $\dot{y} \not\leq \mathcal{N}_{\varphi_{1,2}}(x)$ are hold.

(4) Characterized FT_2 space (resp. $FT_{2\frac{1}{2}}$ space), if for all $x, y \in X$ such that $x \neq y$, the infimum $\mathcal{N}_{\varphi_{1,2}}(x) \wedge \mathcal{N}_{\varphi_{1,2}}(y)$ (resp. $\varphi_{1,2}.\mathrm{cl}\,\mathcal{N}_{\varphi_{1,2}}(x) \wedge \varphi_{1,2}.\mathrm{cl}\,\mathcal{N}_{\varphi_{1,2}}(y)$) does not exists. The related fuzzy topological space (X,τ) is said to be $F\varphi_{1,2}$-T_2 (resp. $F\varphi_{1,2}$-$T_{2\frac{1}{2}}$), when $\mathcal{M} \xrightarrow[\varphi_{1,2}.\mathrm{int}]{} x, y$ (resp. $\varphi_{1,2}.\mathrm{cl}\mathcal{M} \xrightarrow[\varphi_{1,2}.\mathrm{int}]{} x, y$) implies $x = y$ for all $\mathcal{M} \in \mathcal{F}_L X$ and all $x, y \in X$.

(5) Characterized FT_3 space (resp. FT_4 space), i if and only if it Characterized FR_2 space (resp. FR_3 space) and Characterized FT_1 space. The related fuzzy topological space (X,τ) is said to be $F\varphi_{1,2}$-T_3 (resp. $F\varphi_{1,2}$-T_4) if and only if it is $F\varphi_{1,2}$-R_2 (resp. $F\varphi_{1,2}$-R_3) and $F\varphi_{1,2}$-T_1.

Proposition 2.5 [4]. Let (X,τ) be a fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the characterized fuzzy space $(X, \varphi_{1,2}.\mathrm{int})$ is characterized FT_1 space if and only if $\varphi_{1,2}.\mathrm{cl}\,\dot{x} = \dot{x}$ for all $x \in X$.

3. THE NOTIONS OF CHARACTERIZED FUZZY $R_{2\frac{1}{2}}$ AND $T_{3\frac{1}{2}}$ SPACES

The notion of $\varphi_{1,2}\,\psi_{1,2}$-fuzzy continuity between the characterized fuzzy spaces is applied to introduce and study the notion of characterized fuzzy $R_{2\frac{1}{2}}$ spaces or the characterized $FR_{2\frac{1}{2}}$, for short. However, the related notion for the fuzzy topological space is introduce as a generalization to the weaker and stronger forms of the fuzzy completely regular introduced in [11, 24, 26, 29].

Let a fuzzy topological space (X,τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the characterized fuzzy space $(X, \varphi_{1,2}.\mathrm{int})$ is said to be *characterized fuzzy* $R_{2\frac{1}{2}}$ *space* or (*characterized* $FR_{2\frac{1}{2}}$ *space*, for short) if for all $x \in X$, $F \in \varphi_{1,2}C(X)$ such that $x \notin F$, there exists a $\varphi_{1,2}\,\psi_{1,2}$-fuzzy continuous mapping $f:(X,\varphi_{1,2}.\mathrm{int}) \to (\mathrm{I}_L, \psi_{1,2}.\mathrm{int}_{\Im})$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in F$. The related fuzzy topological space (X,τ) is said to be *completely regular fuzzy* $\varphi_{1,2}$-space or ($F\varphi_{1,2}$-$R_{2\frac{1}{2}}$ *space*, for short) if and only if $(X, \varphi_{1,2}.\mathrm{int})$ is characterized $FR_{2\frac{1}{2}}$ space. The characterized fuzzy space $(X, \varphi_{1,2}.\mathrm{int})$ is said to be *characterized fuzzy* $T_{3\frac{1}{2}}$ or (*characterized* $FT_{3\frac{1}{2}}$ *space*, for short) if and only if it is characterized $FR_{2\frac{1}{2}}$ space and characterized FT_1 space.

In the classical case of $L = \{0,1\}$, $\varphi_1 = \mathrm{int}_\tau, \psi_1 = \mathrm{int}_\Im$, $\varphi_2 = 1_{L^X}$ and $\psi_2 = 1_{L^{\mathrm{I}}}$. the $\varphi_{1,2}\,\psi_{1,2}$-fuzzy continuity of f is up to an identification the usual fuzzy continuity of f. Then, in this case the notion of characterized $FR_{2\frac{1}{2}}$ space is coincide with the notion of fuzzy completely regular space defined in [11]. Another special chooses for the operations $\varphi_1, \psi_1, \varphi_2$ and ψ_2 obtained in Table (1).

1n the following proposition we give an equivalent characterization for the characterized $FR_{2\frac{1}{2}}$ spaces.

Proposition 3.1 Let (X,τ) be a fuzzy topological space, $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$ and Ω is a subbase for the characterized fuzzy space $(X,\varphi_{1,2}.\text{int})$. Then, the characterized fuzzy space $(X,\varphi_{1,2}.\text{int})$ is characterized $FR_{2\frac{1}{2}}$ space if and only if for all $F\in\Omega'$ and $x\in X$ such that $x\notin F$, there exists a $\varphi_{1,2}\,\psi_{1,2}$-fuzzy continuous mapping $f:(X,\varphi_{1,2}.\text{int})\to(\text{I}_\text{L},\psi_{1,2}.\text{int}_\Im)$ such that $f(x)=\overline{1}$ and $f(y)=\overline{0}$ for all $y\in F$.

Proof. Let $(X,\varphi_{1,2}.\text{int})$ is a characterized $FR_{2\frac{1}{2}}$ space, Ω is a subbase for $(X,\varphi_{1,2}.\text{int})$ and $F\in\Omega'$, $x\in X$ such that $x\notin F$, then obviously there exists a $\varphi_{1,2}\,\psi_{1,2}$-fuzzy continuous mapping $f:(X,\varphi_{1,2}.\text{int})\to(\text{I}_\text{L},\psi_{1,2}.\text{int}_\Im)$ such that $f(x)=\overline{1}$ and $f(y)=\overline{0}$ for all $y\in F$.

Conversely. let $x\in X$, $F\in\varphi_{1,2}C(X)$ such that $x\notin F$. Then, $x\in F'$ with $F'\in\varphi_{1,2}O(X)$ and therefore there are $V_1,\ldots,V_n\in\Omega$ such that $x\in V_1\cap\ldots\cap V_n\subseteq\Omega'$, that is, $x_i\in V_i$ for all $i\in\{1,2,\ldots,n\}$. Hence. $x_i\notin V_i'$ for all $i\in\{1,2,\ldots,n\}$ and therefore there is a $\varphi_{1,2}\,\psi_{1,2}$-fuzzy continuous mappings $f_i:(X,\varphi_{1,2}.\text{int})\to(\text{I}_\text{L},\psi_{1,2}.\text{int}_\Im)$ such that $f_i(x)=\overline{1}$ and $f_i(y)=\overline{0}$ for all $y\in V_i'$ and $i\in\{1,2,\ldots,n\}$, which implies that $f_i(x)=\overline{1}$ and $f_i(y)=\overline{0}$ for all $y\in(V_1'\cup V_2'\cup\ldots\cup V_n')\subseteq F$. Taking any one of the functions f_i, gives the required $\varphi_{1,2}\,\psi_{1,2}$-fuzzy continuous mapping $f:(X,\varphi_{1,2}.\text{int})\to(\text{I}_\text{L},\psi_{1,2}.\text{int}_\Im)$ for which $f(x)=\overline{1}$ and $f(y)=\overline{0}$ for all $y\in F$. Consequently. $(X,\varphi_{1,2}.\text{int})$ is characterized $FR_{2\frac{1}{2}}$ space. ■

Corollary 3.1 Let (X,τ) be a fuzzy topological space, $\varphi_1,\varphi_2\in O_{(L^X,\tau)}$ and Ω is a subbase for the characterized fuzzy space $(X,\varphi_{1,2}.\text{int})$. Then, (X,τ) is $F\varphi_{1,2}\text{-}R_{2\frac{1}{2}}$ space if and only if for all $F\in\Omega'$ and $x\in X$ such that $x\notin F$, there exists a $\varphi_{1,2}\,\psi_{1,2}$-fuzzy continuous mapping $f:(X,\varphi_{1,2}.\text{int})\to(\text{I}_\text{L},\psi_{1,2}.\text{int}_\Im)$ such that $f(x)=\overline{1}$ and $f(y)=\overline{0}$ for all $y\in F$.

Proof. Immediate from Proposition 3.1 and the definition of the $F\varphi_{1,2}\text{-}R_{2\frac{1}{2}}$ spaces. ■

The following example is an example of characterized $FR_{2\frac{1}{2}}$ space and characterized FT_1 space, that is, an example of characterized $FT_{3\frac{1}{2}}$ space.

Example 3.1 Let $L=\{0,\frac{1}{2},1\}$, $X=\{x,y\}$ and $\tau=\{\overline{1},\overline{0},x_1,y_1\}$ is a fuzzy topology on X. Choose $\varphi_1=\text{int}_\tau$, $\psi_1=\text{int}_\Im$, $\varphi_2=\text{cl}_\tau$ and $\psi_2=\text{cl}_\Im$. Hence, $x\neq y$ and there is only two cases, the first is $x\notin F=\{y\}\in\varphi_{1,2}C(X)$ and the second is $y\notin F=\{x\}\in\varphi_{1,2}C(X)$. We shall consider the first case and the second case is similar.

Consider the mapping $f:(X,\varphi_{1,2}.\text{int})\to(\text{I}_\text{L},\psi_{1,2}.\text{int}_\Im)$ defined by $f(x)=\overline{1}$ and $f(y)=\overline{0}$, then f is $\varphi_{1,2}\,\psi_{1,2}$-fuzzy continuous and therefore $(X,\varphi_{1,2}.\text{int}_\tau)$ is characterized $FR_{2\frac{1}{2}}$ space and obviously $(X,\varphi_{1,2}.\text{int}_\tau)$ is also characterized FT_1 space, that is, $(X,\varphi_{1,2}.\text{int}_\tau)$ is characterized $FT_{3\frac{1}{2}}$ space. ■

In the following proposition, we give the relation between the class of all characterized FT_0 spaces introduced in [2] and our class of all characterized $FT_{3\frac{1}{2}}$ spaces.

Proposition 3.2 Let (X,τ) is a fuzzy topological space and $\varphi_1,\varphi_2\in O_{(L^X,\tau)}$. Then, every characterized $FT_{3\frac{1}{2}}$ space is characterized FT_3 space.

Proof. Let $(X,\varphi_{1,2}.\text{int})$ is characterized $FT_{3\frac{1}{2}}$ space, $x\in X$, $F\in\varphi_{1,2}C(X)$ such that $x\notin F$. Hence, $(X,\varphi_{1,2}.\text{int})$ is characterized FT_1 and characterized $FR_{2\frac{1}{2}}$ space, therefore there exists a $\varphi_{1,2}\,\psi_{1,2}$-fuzzy continuous mapping $f:(X,\varphi_{1,2}.\text{int})\rightarrow(\mathrm{I_L},\psi_{1,2}.\text{int}_{\Im})$ such that $f(x)=\overline{1}$ and $f(y)=\overline{0}$ for all $y\in F$. Consider $R_{\frac{1}{2}},R^{\frac{1}{2}}\in\psi_{1,2}O(\mathrm{I_L})$, then we have $(R_{\frac{1}{2}}\circ f)(x)=R_{\frac{1}{2}}(\overline{1})=\bigvee_{\alpha>\frac{1}{2}}\overline{1}(\alpha)=1$ and $(R^{\frac{1}{2}}\circ f)(y)=R^{\frac{1}{2}}(\overline{0})=(\bigvee_{\alpha>\frac{1}{2}}\overline{0}(\alpha))'=1$ for all $y\in F$. Hence, $\mu=R_{\frac{1}{2}}\circ f$ and $\eta=R^{\frac{1}{2}}\circ f$ are two fuzzy subsets of X such that $\mathcal{N}_{\varphi_{1,2}}(x)(\mu)\wedge\mathcal{N}_{\varphi_{1,2}}(F)(\eta)=1$. On other hand because of $\bigwedge_{s<t}f(s)\geq\bigvee_{r>t}f(r)$ holds for all $s,r\in X$, then for all $z\in X$, we have that

$$\begin{aligned}(\mu\wedge\eta)(z)&=\big((R_{\frac{1}{2}}\circ f)\wedge(R^{\frac{1}{2}}\circ f)\big)(z)\\&=\bigvee_{\alpha>\frac{1}{2}}f(z)(\alpha)\wedge\big(\bigvee_{\alpha\geq\frac{1}{2}}f(z)(\alpha)\big)'\\&\leq\bigwedge_{\alpha\leq\frac{1}{2}}f(z)(\alpha)\wedge\bigwedge_{\alpha<\frac{1}{2}}f(z)(\alpha)'<1.\end{aligned}$$

Hence, $\sup(\mu\wedge\eta)<\mathcal{N}_{\varphi_{1,2}}(x)(\mu)\wedge\bigwedge_{y\in F}\mathcal{N}_{\varphi_{1,2}}(y)(\eta)$ and therefore the $\mathcal{N}_{\varphi_{1,2}}(x)\wedge\mathcal{N}_{\varphi_{1,2}}(F)$ does not exists, that is, $(X,\varphi_{1,2}.\text{int})$ is characterized FR_2 space. Using that $(X,\varphi_{1,2}.\text{int})$ is characterized FT_1, we get that $(X,\varphi_{1,2}.\text{int})$ is characterized FT_3 space. ■

Corollary 3.2 Let a fuzzy topological space (X,τ) be fixed and $\varphi_1,\varphi_2\in O_{(L^X,\tau)}$. Then, every $F\varphi_{1,2}$-$T_{3\frac{1}{2}}$ topological space is $F\varphi_{1,2}$-T_3 topological space and every $F\varphi_{1,2}$-$R_{2\frac{1}{2}}$ topological space is $F\varphi_{1,2}$-R_2 topological space.

Proof. Follows immediately from Proposition 3.2. ■

Because of the Theorems 3.2 and 4.2 in [3], Propositions 4.1 and 4.2 and Corollary 4.1 in [2] and Proposition 3.2 and Corollary 3.2, we have the following diagram of implications:

$$\begin{array}{ccccccccc}FT_{3\frac{1}{2}}&\Rightarrow&FT_3&\Rightarrow&FT_{2\frac{1}{2}}&\Rightarrow&FT_2&\Leftrightarrow&FT_1\wedge FR_2\\\Downarrow&&\Downarrow&&\Downarrow&&\Downarrow&&\\FT_0\wedge FR_{2\frac{1}{2}}&\Rightarrow&FT_0\wedge FR_2&\Rightarrow&FT_1&\Leftrightarrow&FT_0\wedge FR_0&&\end{array}$$

Diagram 3.1

The following examples shows that the inverse of the implications in Diagram 3.1 are not true in general.

Example 3.2 Let $L=\{0,\frac{1}{4},1\}$, $X=\{x,y\}$ and $\tau=\{\overline{1},\overline{0},\mu\}$ is a fuzzy topology on X, where $\mu:X\rightarrow L$ is the fuzzy subset defined by $\mu(x)=1$ and $\mu(y)=0$. Choose, $\varphi_1=\text{cl}_\tau\circ\text{int}_\tau$ and $\varphi_2=\text{cl}_\tau$, then the characterized fuzzy space $(X,\varphi_{1,2}.\text{int})$ is characterized FR_0 space and characterized FT_0 space, but it is neither characterized FR_1 space nor characterized FT_1 space. Hence. $(X,\varphi_{1,2}.\text{int})$ is not characterized FT_2 space. Indeed for $x\neq y$ in X, we have $\dot{y}(\rho)\geq\mathcal{N}_{\varphi_{1,2}}(x)(\rho)$ holds for all $\rho\in L^X$, that is, the infimum $\mathcal{N}_{\varphi_{1,2}}(x)\wedge\mathcal{N}_{\varphi_{1,2}}(y)$ does not exists. ■

Example 3.3 Let $L=\{0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1\}$, $X=\{x,y\}$ and $\tau=\{\overline{1},\overline{0},y_{\frac{1}{2}},y_1,x_{\frac{3}{4}}\vee y_{\frac{1}{2}},x_{\frac{3}{4}}\vee y_1\}$ is a fuzzy topology on X. Choose $\varphi_1=\text{int}_\tau$ and $\varphi_2=\text{int}_\tau\circ\text{cl}_\tau$, then the characterized fuzzy space $(X,\varphi_{1,2}.\text{int}_\tau)$ is characterized FT_3, because there is only the case of $y \in X$ and $F = \{x\} \in \varphi_{1,2}C(X)$ such that $y \notin F$ and there exists $\mu, \eta \in L^X$ defined $\mu = x_{\frac{3}{4}} \vee y_{\frac{1}{2}}$, $\eta = y_1$ and

$$\mathcal{N}_{\varphi_{1,2}}(x)(\mu) \wedge \mathcal{N}_{\varphi_{1,2}}(y)(\eta) = \varphi_{1,2}.\text{int}_\tau\, \mu(x) \wedge \varphi_{1,2}.\text{int}_\tau\, \eta(y)$$
$$= \tfrac{3}{4} > \tfrac{1}{2} = \sup(\mu \wedge \eta).$$

Hence, the $\mathcal{N}_{\varphi_{1,2}}(y) \wedge \mathcal{N}_{\varphi_{1,2}}(F)$ does not exists and therefore $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized FR_2 space. Obviously, the characterized fuzzy space $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized FT_1 space and therefore, $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized FT_3 space. On other hand $(X, \varphi_{1,2}.\text{int}_\tau)$ is not characterized $FR_{2\frac{1}{2}}$ space. Indeed for $y \in X$ and $F = \{x\} \in \varphi_{1,2}C(X)$ such that $y \notin F$ all the mappings $f_i : (X, \varphi_{1,2}.\text{int}_\tau) \rightarrow (\mathrm{I_L}, \psi_{1,2}.\text{int}_\Im)$ which fulfilled that $f_i(y) = \overline{1}$ and $f_i(x) = \overline{0}$ for all $x \in F$ are not $\varphi_{1,2}\, \psi_{1,2}$-fuzzy continuous for $i \in \{1, 2, ..., n\}$, $\psi_1 = \text{int}_\Im$ and $\psi_2 = \text{int}_\Im \circ \text{cl}_\Im$. Therefore, $(X, \varphi_{1,2}.\text{int}_\tau)$ is neither characterized $FT_{3\frac{1}{2}}$ space nor characterized FT_0 space. ■

4. NEW RELATIONS BETWEEN CHARACTERIZED $FR_{2\frac{1}{2}}$, CHARACTERIZED $FT_{3\frac{1}{2}}$ AND SOME CHARACTERIZED FUZZY SPACES

In this section. we are going to introduce and study difference relations between the characterized $FR_{2\frac{1}{2}}$ and the characterized $FT_{3\frac{1}{2}}$ with other characterized FR_k and characterized FT_s spaces which are listed in Section 2 for some special choices of k and s. To find these relations, we try to introduce generalization to the Urysohn's Lemma for the characterized FR_3 spaces with help of the characterized fuzzy proximity spaces presented in [1]. So, we at first applied the relation between the farness and the finer relation on the fuzzy sets to introduce the notions of $\varphi_{1,2}\delta$-fuzzy neighborhood at the point x in the characterized fuzzy proximity space and of $\varphi_{1,2}\psi_{1,2}\delta$-fuzzy continuity between the characterized fuzzy proximity spaces. The concepts of fuzzy function family and the Φ-separable are applied to introduce important properties for the concept of the $\varphi_{1,2}\psi_{1,2}\delta$-continuity. Moreover, we show that the $\varphi_{1,2}\psi_{1,2}$-fuzzy continuity of the mapping f between characterized fuzzy spaces is more general than of the $\varphi_{1,2}\psi_{1,2}\delta$-fuzzy continuity of f between the characterized fuzzy proximity spaces. An important result., we show that if the fixed fuzzy topological space (X, τ) is normal, then the characterized fuzzy space $(X, \varphi_{1,2}.\text{int}_\tau)$ is finer than the associated characterized fuzzy proximity space $(X, \varphi_{1,2}.\text{int}_\delta)$ and they identical if $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized FT_4 space.

The binary relation δ on L^X is said to be *fuzzy proximity* on a set X ([28]), provided it fulfill the following conditions:

(P1) $\mu\, \overline{\delta}\, \rho$ implies $\rho\, \overline{\delta}\, \mu$ for all $\mu, \rho \in L^X$, where $\overline{\delta}$ is the negation of δ.

(P2) $(\mu \vee \rho)\, \overline{\delta}\, \eta$ if and only if $\mu\, \overline{\delta}\, \eta$ and $\rho\, \overline{\delta}\, \eta$ for all $\mu, \rho, \eta \in L^X$.

(P3) $\mu = \overline{0}$ or $\rho = \overline{0}$ implies $\mu\, \overline{\delta}\, \rho$ for all $\mu, \rho \in L^X$.

(P4) $\mu\, \overline{\delta}\, \rho$ implies $\mu \le \rho'$ for all $\mu, \rho \in L^X$.

(P5) If $\mu\, \overline{\delta}\, \rho$, then there is an $\eta \in L^X$ such that $\mu\, \overline{\delta}\, \eta$ and $\eta\, \overline{\delta}\, \rho$.

The set X equipped with a fuzzy proximity δ on X is called a *fuzzy proximity space* and will be denoted by (X, δ). Every fuzzy proximity δ on a set X is associated a fuzzy topology on X denoted by τ_δ. The fuzzy proximity δ on a set X is said to be *separated* if and only if for all $x, y \in X$ such that $x \neq y$ we have $x_\alpha\, \overline{\delta}\, y_\beta$ for all $\alpha, \beta \in L_0$.

As shown in [1], the fuzzy proximity will be identified with the finer relation on the fuzzy filters, especially with the finer relation on the $\varphi_{1,2}$- fuzzy neighborhood filters in the characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$.

Proposition 4.1[1] Let (X, τ) be a fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then the binary relation δ on L^X which is defined by $\mu \,\overline{\delta}\, \rho$ if and only if $\mathcal{N}_{\varphi_{1,2}}(\rho) \leq \dot{\mu}'$ for all $\mu, \rho \in L^X$ is fuzzy proximity on X.

Consider the fuzzy topological space (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then each fuzzy proximity δ on X is associated a set of all $\varphi_{1,2}$-open fuzzy subsets of X with respect to δ denoted by $\varphi_{1,2}OF(X)_\delta$. In this case, the triple $(X, \varphi_{1,2}OF(X)_\delta)$ as will as the triple $(X, \varphi_{1,2}.\text{int}_\delta)$ is said to be *characterized fuzzy proximity space* ([1]). The related $\varphi_{1,2}$-interior and $\varphi_{1,2}$-closure operators will be denoted by $\varphi_{1,2}.\text{int}_\delta$ and $\varphi_{1,2}.\text{cl}_\delta$, respectively and they are given by:

$$\varphi_{1,2}.\text{int}_\delta\, \mu = \bigvee_{\mu' \overline{\delta} \rho} \rho \quad \text{and} \quad \varphi_{1,2}.\text{cl}_\delta \mu = \bigwedge_{\rho' \overline{\delta} \mu} \rho \tag{4.1}$$

for all $\mu \in L^X$. Consider the characterized fuzzy proximity space $(X, \varphi_{1,2}.\text{int}_\delta)$ be fixed and $\mu \in L^X$, then μ is said to be $\varphi_{1,2}\delta$- fuzzy neighborhood at the point $x \in X$ if and only if $x_1 \,\overline{\delta}\, \mu'$.

Moreover, the mapping $f : (X, \varphi_{1,2}.\text{int}_\delta) \to (Y, \psi_{1,2}.\text{int}_{\delta^*})$ is said to be $\varphi_{1,2}\psi_{1,2}\delta$- *fuzzy continuous*, provided $\eta \,\overline{\delta}^*\, \rho$ implies $(\eta \circ f)\, \overline{\delta}\, (\rho \circ f)$ for all $\eta, \rho \in L^Y$. Obviously, there is an identification between the fuzzy proximity δ and the complementarily symmetric fuzzy topogenous structure $<<$ on the same set X given by:

$$\mu << \eta' \iff \mu \,\overline{\delta}\, \eta \tag{4.2}$$

for all $\mu, \eta \in L^X$.

Now. let $\{<<_n\}_{n=1}^{\infty}$ is a sequence of fuzzy topogenous structure on a set X and $\{\prec_n\}_{n=1}^{\infty}$ is a sequence of fuzzy topogenous structure on $\mathrm{I_L}$. Then, the fuzzy real function $f : X \to \mathrm{I_L}$ is said to be associated with the sequence $\{<<_n\}_{n=1}^{\infty}$ if and only if $\eta \prec_n \rho$ implies that $(\eta \circ f) <<_{n+1} (\rho \circ f)$ holds for all $\eta, \rho \in L^{\mathrm{I_L}}$ and $n \in \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of all positive integer numbers.

Remark 4.1. Given that $\{<<_n\}_{n=1}^{\infty}$ and $\{\prec_n\}_{n=1}^{\infty}$ are two sequence of complementarily symmetric fuzzy topogenous structures $<<$ and $\prec$ on X and $\mathrm{I_L}$, respectively. If δ and δ^* are two fuzzy proximities on X and $\mathrm{I_L}$ identified with δ and δ^* by the equation (4.2), then the associated fuzzy real function $f : (X, \varphi_{1,2}.\text{int}_\delta) \to (Y, \psi_{1,2}.\text{int}_{\delta^*})$ with the complementarily symmetric fuzzy topogenous structures $<<$ is $\varphi_{1,2}\psi_{1,2}\delta$- fuzzy continuous, because from (4.2) we get that $\eta \,\overline{\delta}^*\, \rho$ implies $(\eta \circ f)\, \overline{\delta}\, (\rho \circ f)$ for all $\eta, \rho \in L^{\mathrm{I_L}}$.

Lemma 4.1 [11] Consider $<<_n$ for $n \in \{0, 1, \ldots, \infty\}$ are complementarily symmetric fuzzy topogenous structures on a set X. Then, for each $F, G \in P(X)$ such that $\chi_F <<_0 \chi_G$, there exists a fuzzy real function $f : X \to \mathrm{I_L}$ associated with the sequence $\{<<_n\}_{n=0}^{\infty}$ for which $f(x) = \overline{0}$ for all $x \in F$ and $f(y) = \overline{1}$ for all $y \in G'$.

Because of equation (4.2), Remark 4.1 and Lemma 4.1, we can easily deduce the following proposition.

Proposition 4.2 Let $(X, \varphi_{1,2}.\text{int}_\delta)$ is a characterized fuzzy proximity space and $F, G \in P(X)$ such that $\chi_F \,\overline{\delta}\, \chi_G$. If Φ is the family of all $\varphi_{1,2}\psi_{1,2}\delta$- fuzzy continuous mappings $f : (X, \varphi_{1,2}.\text{int}_\delta) \to (Y, \psi_{1,2}.\text{int}_{\delta^*})$ for which $x \in X$ implies $\overline{0} \leq f(x) \leq \overline{1}$, then χ_F and χ_G are Φ- separable.

Proof. Let $<<$ be a complementarily symmetric fuzzy topogenous structure identified with δ. Because of (4.2), $\chi_F \,\overline{\delta}\, \chi_G$ implies that $\chi_F <<_0 \chi_G$. Since $f \in \Phi$ is $\varphi_{1,2}\psi_{1,2}\delta$- fuzzy continuous, then because of Remark 4.1, we have that f is associated with $<<$. Hence, Lemma 4.1 implies that χ_F and χ_G are separated by f. Therefore, χ_F and χ_G are Φ-separable. ∎

Proposition 4.3 Let $(X, \varphi_{1,2}.\mathrm{int}_\delta)$ and $(Y, \psi_{1,2}.\mathrm{int}_{\delta^*})$ are characterized fuzzy proximity spaces. If the mapping $f:(X, \varphi_{1,2}.\mathrm{int}_\delta) \to (Y, \psi_{1,2}.\mathrm{int}_{\delta^*})$ is $\varphi_{1,2}\psi_{1,2}\delta$-fuzzy continuous, then the mapping $f:(X, \varphi_{1,2}.\mathrm{int}) \to (Y, \psi_{1,2}.\mathrm{int})$ is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous.

Proof. Similar to the proof of Proposition 11.2 in [20]. ∎

Proposition 4.4 [1] Let (X, τ) be a fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$ such that $\varphi_2 \geq 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$. If (X, τ) is a fuzzy normal space and L is a complete chain, then the binary relation δ on X which is defined by:

$$\mu \overline{\delta} \rho \Leftrightarrow \mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\mu) \leq (\varphi_{1,2}.\dot{\mathrm{cl}}\,\rho)' \tag{4.3}$$

for all $\mu, \rho \in L^X$ is a fuzzy proximity on X and (X, δ) is a fuzzy proximity space. On other hand, if (X, δ) is a fuzzy proximity space and δ fulfills (4.3.), then the associated characterized fuzzy proximity space $(X, \varphi_{1,2}.\mathrm{int}_\delta)$ is characterized FR_3 space.

In the following, we are going to show an important relation between the associated fuzzy proximity space $(X, \varphi_{1,2}.\mathrm{int}_\delta)$ by the fuzzy proximity 6 defined by (4.3) and the associated characterized fuzzy space $(X, \varphi_{1,2}.\mathrm{int})$ that introduced form the fuzzy normal topological space (X, τ).

Proposition 4.5 Let (X, τ) is a fuzzy normal topological space and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$ such that $\varphi_2 \geq 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$. If δ is a fuzzy proximity on X defined by (4.3) and L is a complete chain, then $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is finer than $(X, \varphi_{1,2}.\mathrm{int}_\delta)$. Moreover, $(X, \varphi_{1,2}.\mathrm{int}_\tau) = (X, \varphi_{1,2}.\mathrm{int}_\delta)$ if and only if $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is characterized FT_4 space.

Proof. Let (X, τ) is fuzzy normal topological space and μ is $\varphi_{1,2}\delta$-fuzzy neighborhood for the point $x \in X$, then $x_1 \overline{\delta} \mu'$ and because of (4.3), we have $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}_\tau(x_1)) \leq (\varphi_{1,2}.\dot{\mathrm{cl}}_\tau \mu')'$. Therefore, $\dot{x} \leq \mathcal{N}_{\varphi_{1,2}}(x) \leq \mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}_\tau\{x\}) = \mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}_\tau(x_1)) \leq (\varphi_{1,2}.\dot{\mathrm{cl}}_\tau \mu')' \leq \dot{\mu}$.. Because of Proposition 2.1, we get $x_1 \leq (\varphi_{1,2}.\mathrm{cl}_\tau \mu')' \leq \dot{\mu}$ and $(\varphi_{1,2}.\mathrm{cl}_\tau \mu')' \in \varphi_{1,2}OF(X)$. Then μ is $\varphi_{1,2}$-fuzzy neighborhood of x and therefore the family $(\varphi_{1,2}OF(X))_\delta$ is coarser than the family $(\varphi_{1,2}OF(X))$, that is, $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is finer than $(X, \varphi_{1,2}.\mathrm{int}_\delta)$.

Now, let $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is characterized FT_4 space, $\mathcal{N}_{\varphi_{1,2}}(x)$ and $\mathcal{N}^\delta_{\varphi_{1,2}}(x)$ denote for the $\varphi_{1,2}$-fuzzy neighborhood filters at x in the characterized fuzzy space $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ and in the associated characterized fuzzy proximity space $(X, \varphi_{1,2}.\mathrm{int}_\delta)$, respectively. Then, $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is characterized FR_3 and FT_1 space. Therefore, $(\varphi_{1,2}OF(X))_\delta \subseteq (\varphi_{1,2}OF(X))$ and $\mathcal{N}_{\varphi_{1,2}}(x) \not\leq \dot{y}$ holds for all $y \neq x$ in X. Hence, $\mathcal{N}_{\varphi_{1,2}}(x) \leq \mathcal{N}^\delta_{\varphi_{1,2}}(x)$ holds for all $x \in X$ and then $\mathcal{N}^\delta_{\varphi_{1,2}}(x) \geq \mathcal{N}_{\varphi_{1,2}}(x) \not\leq \dot{y}$ holds for all $y \neq x$ in X. Because of Lemma 2.1, we have that $\mathcal{N}^\delta_{\varphi_{1,2}}(x) \not\leq \dot{y}$ holds for all $y \neq x$ in X and therefore $(X, \varphi_{1,2}.\mathrm{int}_\delta)$ is characterized FT_1 space. Because of Proposition 2.5 and Lemma 2.2, we get $\varphi_{1,2}.\mathrm{cl}_\tau(x_1) = x_1$ for all $x \in X$ and therefore $x \in (\varphi_{1,2}CF(X))_\delta$ for all $x \in X$. Consider μ is $\varphi_{1,2}$-fuzzy neighborhood of x in $(X, \varphi_{1,2}.\mathrm{int}_\tau)$, then $\mu' \leq x_1'$ and since $x_1' \in (\varphi_{1,2}OF(X))_\delta$, then x_1' is $\varphi_{1,2}$-fuzzy neighborhood for every $y \in X$ such that $y_1 \leq \mu'$. Thus, $\mu' \overline{\delta} x_1$ and hence μ is $\varphi_{1,2}\delta$-fuzzy neighborhood of x in $(X, \varphi_{1,2}.\mathrm{int}_\delta)$. Thus,

$(\varphi_{1,2}OF(X)) \subseteq (\varphi_{1,2}OF(X))_\delta$, that is, $\mathcal{N}^{\delta}_{\varphi_{1,2}}(x) \leq \mathcal{N}_{\varphi_{1,2}}(x)$ holds for all $x \in X$ and therefore $(X, \varphi_{1,2}.\mathrm{int}_\delta)$ is finer than the characterized fuzzy space $(X, \varphi_{1,2}.\mathrm{int}_\tau)$. Consequently, $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is characterized FT_4 space implies that, $(X, \varphi_{1,2}.\mathrm{int}_\tau) = (X, \varphi_{1,2}.\mathrm{int}_\delta)$.

Conversely, let $((X, \varphi_{1,2}.\mathrm{int}_\tau) = (X, \varphi_{1,2}.\mathrm{int}_\delta)$, $x \in X$ and μ is $\varphi_{1,2}$- fuzzy neighborhood of x in the characterized fuzzy space $(X, \varphi_{1,2}.\mathrm{int}_\tau)$. Then, $\mu \in (\varphi_{1,2}OF(X))_\delta$ and $x_1 \leq \mu$, this means that

$$(\varphi_{1,2}.\dot{\mathrm{cl}}_\tau(x_1)) \leq \mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}_\tau(x_1)) \leq (\varphi_{1,2}.\dot{\mathrm{cl}}_\tau \mu')' \leq \dot{\mu}.$$

Because of Proposition 2.1, we get $\varphi_{1,2}.\mathrm{cl}_\tau(x_1) \leq \mu$ and therefore $\varphi_{1,2}.\mathrm{cl}_\tau(x_1) \leq x_1$ holds for all $x \in X$. Thus, $\varphi_{1,2}.\mathrm{cl}_\tau(x_1) = x_1$ for all $x \in X$. Hence. Proposition 2.5, implies that, $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is characterized FT_1 space. Because of Proposition 4.3, $(X, \varphi_{1,2}.\mathrm{int}_\delta)$ is characterized FR_3 space and the hypothesis that $(X, \varphi_{1,2}.\mathrm{int}_\tau) = (X, \varphi_{1,2}.\mathrm{int}_\delta)$ implies that $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is characterized FR_3 space, Consequently, $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is characterized FT_4 space. ■

Now. we are going to introduce and study a generalization of Urysohn's Lemma for the characterized FR_3 spaces to prove the relation between the characterized FR_3 spaces and the characterized $FR_{2\frac{1}{2}}$ spaces in general case. The relation between the characterized $FT_{3\frac{1}{2}}$ spaces and the characterized FT_4 spaces is also introduced by the generalization of Urysohn's Lemma.

Lemma 4.2 (Generalized Urysohn's Lemma) Let a fuzzy topological space (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$ such that $\varphi_2 \geq 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$. If L is a complete chain, then $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is characterized FR_3 space if and only if for all $F_1, F_2 \in \varphi_{1,2}\mathrm{C(X)}$ such that $F_1 \cap F_2 = \varphi$, there exists a $\varphi_{1,2}\ \psi_{1,2}$-fuzzy continuous mapping $f : (X, \varphi_{1,2}.\mathrm{int}_\tau) \to (\mathrm{I_L}, \psi_{1,2}.\mathrm{int}_\Im)$ such that $f(x) = \overline{0}$ for all $x \in F_1$ and $f(y) = \overline{1}$ for all $y \in F_2$.

Proof. Let $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is characterized FR_3 space, then the infimum $\mathcal{N}_{\varphi_{1,2}}(F_1) \wedge \mathcal{N}_{\varphi_{1,2}}(F_2) = \varphi$ does not exists for all $F_1, F_2 \in \varphi_{1,2}\mathrm{C(X)}$ such that $F_1 \cap F_2 = \varphi$. Therefore, $\mathcal{N}_{\varphi_{1,2}}(F_1) \leq \dot{F}_2'$. Consider δ is a fuzzy proximity on X defined by (4.3), then we have $\chi_{F_1}\ \overline{\delta}\ \chi_{F_2}$. Because of Proposition 4.2, there exists $\varphi_{1,2}\psi_{1,2}\delta$- fuzzy continuous mapping $f : (X, \varphi_{1,2}.\mathrm{int}_\delta) \to (\mathrm{I_L}, \psi_{1,2}.\mathrm{int}_{\delta^*})$ for which χ_{F_1} and χ_{F_2} are Φ- separated by f, where δ^* is a fuzzy proximity on $\mathrm{I_L}$ defined by (4.3). Hence, because of Proposition 4.3, we have that $f : (X, \varphi_{1,2}.\mathrm{int}_\tau) \to (\mathrm{I_L}, \psi_{1,2}.\mathrm{int}_\Im)$ is $\varphi_{1,2}\ \psi_{1,2}$ -fuzzy continuous mapping and from Proposition 4.5, the characterized fuzzy space $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is finer than the associated characterized fuzzy proximity space $(X, \varphi_{1,2}.\mathrm{int}_\delta)$. Therefore, the mapping $f : (X, \varphi_{1,2}.\mathrm{int}_\tau) \to (\mathrm{I_L}, \psi_{1,2}.\mathrm{int}_\Im)$ is $\varphi_{1,2}\ \psi_{1,2}$ -fuzzy continuous such that $f(x) = \overline{0}$ for all $x \in F_1$ and $f(y) = \overline{1}$ for all $y \in F_2$.

Conversely, let there exists a $\varphi_{1,2}\ \psi_{1,2}$ -fuzzy continuous mapping $f : (X, \varphi_{1,2}.\mathrm{int}_\tau) \to (\mathrm{I_L}, \psi_{1,2}.\mathrm{int}_\Im)$ such that $f(x) = \overline{0}$ for all $x \in F_1$ and $f(y) = \overline{1}$ for all $y \in F_2$, where $F_1, F_2 \in \varphi_{1,2}\mathrm{C(X)}$ and $F_1 \cap F_2 = \varphi$. Consider $R_{\frac{1}{2}}$ and $R^{\frac{1}{2}}$ are the restricted of F_1 and F_2 on $\mathrm{I_L}$. Then, $\mu = R^{\frac{1}{2}} \circ f$ and $\eta = R_{\frac{1}{2}} \circ f$ are $\varphi_{1,2}$- open fuzzy sets on X such that

$$\mathcal{N}_{\varphi_{1,2}}(F_1)(\mu) = \bigwedge_{x \in F_1} \mu(x) = \bigwedge_{x \in F_1} R^{\frac{1}{2}}(f(x)) = \bigwedge_{x \in F_1} (\bigvee_{\alpha \geq \frac{1}{2}} f(x)(\alpha))' = 1,$$

and

$$\mathcal{N}_{\varphi_{1,2}}(F_2)(\eta)=\bigwedge_{y\in F_2}\eta(y)=\bigwedge_{y\in F_2}R_{\frac{1}{2}}(f(y))=\bigwedge_{y\in F_2}(\bigvee_{\alpha>\frac{1}{2}}f(y)(\alpha))=1.$$

Therefore, $\mathcal{N}_{\varphi_{1,2}}(F_1)\wedge\mathcal{N}_{\varphi_{1,2}}(F_2)=1.$ Since

$$(\mu\wedge\eta)(z)=\bigvee_{\alpha>\frac{1}{2}}f(z)(\alpha)\wedge(\bigvee_{\alpha\geq\frac{1}{2}}f(z)(\alpha))'\leq\bigwedge_{\alpha>\frac{1}{2}}f(z)(\alpha)\wedge\bigwedge_{\alpha\geq\frac{1}{2}}f(z)(\alpha)'<1$$

for all $z\in X$. Hence, the infimum $\mathcal{N}_{\varphi_{1,2}}(F_1)\wedge\mathcal{N}_{\varphi_{1,2}}(F_2)$ does not exists and therefore $(X,\varphi_{1,2}.\text{int}_\tau)$ is characterized FR_3 space. ■

Corollary 4.1 Let a fuzzy topological space (X,τ) be fixed and $\varphi_1,\varphi_2\in O_{(L^X,\tau)}$ such that $\varphi_2\geq 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$. If L is a complete chain, then every characterized FR_3 space is characterized $FR_{2\frac{1}{2}}$ space.

Proof. Follows directly from Lemma 4.2. ■

In the following proposition, we show that the characterized $FT_{3\frac{1}{2}}$ spaces are more general than the characterized FT_4 spaces.

Proposition 4.6 Let a fuzzy topological space (X,τ) be fixed and $\varphi_1,\varphi_2\in O_{(L^X,\tau)}$ such that $\varphi_2\geq 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$. If L is a complete chain, then every characterized FT_4 space is characterized $FT_{3\frac{1}{2}}$ space.

Proof. Let $(X,\varphi_{1,2}.\text{int}_\tau)$ is characterized FT_4 space and let $x\in X$, $F\in\varphi_{1,2}C(X)$ such that $x\notin F$. Then, $(X,\varphi_{1,2}.\text{int}_\tau)$ is characterized FR_3 and FT_1 space. Because of Proposition 4.5, we have $\{x\}\in\varphi_{1,2}C(X)$ and $\{x\}\cap F=\varphi$, therefore because of Generalized Urysohn's Lemma, there is a $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping $f:(X,\varphi_{1,2}.\text{int}_\tau)\to(\text{I}_\text{L},\psi_{1,2}.\text{int}_\Im)$ such that $f(x)=\overline{1}$ and $f(y)=\overline{0}$ for all $y\in F$. Hence, $(X,\varphi_{1,2}.\text{int}_\tau)$ is characterized $FR_{2\frac{1}{2}}$ space. Consequently, $(X,\varphi_{1,2}.\text{int}_\tau)$ is characterized $FT_{3\frac{1}{2}}$ space. ■

Because of the Theorems 3.2 and 4.2 and Proposition 4.1 in [3], Propositions 2.1 and 4.3 and Corollary 4.1 in [2] and Proposition 4.5 and Corollary 4.1, we have the following diagram of implications:

$$\begin{array}{ccccccc} FT_4 & \Rightarrow FT_{3\frac{1}{2}} & \Rightarrow FT_3 & \Rightarrow FT_{2\frac{1}{2}} \Rightarrow & FT_2 & \Rightarrow FT_1 & \Rightarrow FT_0 \\ \Downarrow & \Downarrow & \Downarrow & & \Downarrow & \Downarrow & \Downarrow \\ FR_3\wedge FT_0 & \Rightarrow FR_{2\frac{1}{2}}\wedge FT_0 \Rightarrow & FR_2\wedge FT_0 & \Rightarrow FR_2 & \Rightarrow FR_1 & \Rightarrow FR_0 & \Rightarrow \text{any chara. fuzzy space} \end{array}$$

Diagram 4.1

The inverse of the implication in Diagram 4.1 are not true in general as shown in Examples 3.2, 3.3 and 4.3 in [2] and the following example.

Example 4.1. Let $X=\{(x,y)\in\mathbb{R}^2 \mid y\geq 0\}\subseteq\mathbb{R}^2$, L is a complete chain and τ is the fuzzy topology on X defined as follows:

For each $p\in\{(x,y)\in\mathbb{R}^2 \mid y>0\}$, the basic fuzzy neighborhoods will be the usual open disks and at

$q \in X \setminus \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$, the basic fuzzy neighborhoods will be the sets $\{q\} \cup O$, where O is the open disk in $\{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ and tangent to the x-axis at q. Consider $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$ for which $\varphi_1 = \text{int}_\tau \circ \text{cl}_\tau$ and $\varphi_2 = 1_{L^X}$. Hence, $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized FT_0 space, because for all $p = (r,0), q = (s,0) \in X$ with $p \neq q$ for all $r \in \mathbb{Q}$ and $s \in \mathbb{Q}'$, there exist $\mu = \mathbb{Q}, \eta = \mathbb{Q}' \in L^X$ such that $\mu(p) < \alpha \leq (\varphi_{1,2}.\text{int}_\tau\, \mu)(q)$ and $\eta(q) < \beta \leq (\varphi_{1,2}.\text{int}_\tau\, \eta)(p)$ are hold for some $\alpha, \beta \in L_0$, but $(X, \varphi_{1,2}.\text{int}_\tau)$ is not characterized FR_3 space. Therefore, $(X, \varphi_{1,2}.\text{int}_\tau)$ is not characterized FT_4 space. Moreover, $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized $FR_{2\frac{1}{2}}$ space, because if $p \in X$ and Ω is the $\varphi_{1,2}$-fuzzy neighborhood of p, then Ω is either $\varphi_{1,2}$-open fuzzy disk has centered at p or else p together with an $\varphi_{1,2}$-fuzzy open tangent to p and depending on the placement of p. Consider $f : (X, \varphi_{1,2}.\text{int}_\tau) \to (\mathrm{I}_L, \psi_{1,2}.\text{int}_\Im)$ is the mapping defined by: $f(p) = \overline{1}$ and $f(q) = \overline{0}$ for all $q \notin \Omega$ and let f is linearly along the straight line passing through the point p and the points on the boundary of Ω, where $\psi_1 = \text{int}_\Im \circ \text{cl}_\Im$ and $\psi_2 = 1_{L^{\mathrm{I}_L}}$. Then, f is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous such that $f(p) = \overline{1}$ and $f(q) = \overline{0}$ for all $q \in \Omega'$. Therefore, $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized $FR_{2\frac{1}{2}}$ space. Hence, $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized $FT_{3\frac{1}{2}}$ space, but it is not characterized FT_4 space. ■

5. NEW CHARACTERIZATIONS FOR THE CHARACTERIZED FUZZY PROXIMITY SPACES BY CHARACTERIZED $FR_{2\frac{1}{2}}$ SPACES

In this section, we are going to introduce and study some important relations joining and characterized the characterized fuzzy proximity spaces introduced by, Abd-Allah in [1] and our characterized $FR_{2\frac{1}{2}}$ spaces and the characterized $FT_{3\frac{1}{2}}$ spaces, which are present in Section 3.

One of these relations at the beginning, we shall prove that the associated characterized fuzzy proximity space $(X, \varphi_{1,2}.\text{int}_\delta)$ is characterized $FR_{2\frac{1}{2}}$ space in our sense.

Proposition 5.1 Let (X, τ) be a fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. If δ is a fuzzy proximity on X, then the associated characterized fuzzy proximity space $(X, \varphi_{1,2}.\text{int}_\delta)$ is characterized $FR_{2\frac{1}{2}}$ space.

Proof. Let $x \in X$ and $F \in \varphi_{1,2}C(X)$ such that $x \notin F$. Since $\chi_{F'}$ is $\varphi_{1,2}\delta$-fuzzy neighborhood of x, then $x_1 \overline{\delta}\, \chi_F$. Because of Proposition 4.1, we get that x_1 and χ_F are Φ-separated by the $\varphi_{1,2}\psi_{1,2}\delta$-fuzzy continuous mapping $f : (X, \varphi_{1,2}.\text{int}_\delta) \to (\mathrm{I}_L, \psi_{1,2}.\text{int}_{\delta^*})$ for which $\overline{0} \leq f(x) \leq \overline{1}$, that is, $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in F$. Consequently, $(X, \varphi_{1,2}.\text{int}_\delta)$ is characterized $FR_{2\frac{1}{2}}$ space. ■

To examine for a given characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$, when the fuzzy proximity δ on X is compatible with the $\varphi_{1,2}$-interior operator $\varphi_{1,2}.\text{int}$, we need the following proposition. It will be shown that, this happens if and only if $(X, \varphi_{1,2}.\text{int})$ is characterized $FR_{2\frac{1}{2}}$ space.

Proposition 5.2 Let (X, τ) be a fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. If Φ is the fuzzy function family of the $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mappings $f_k : (X, \varphi_{1,2}.\text{int}_k) \to (\mathrm{I}_L, \psi_{1,2}.\text{int}_\Im)$, $k \in K$, where K is any class, then

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$(X, \varphi_{1,2}.\text{int})$ is characterized $FR_{2\frac{1}{2}}$ space if and only if $\varphi_{1,2}.\text{int}$ coincide with the coarsest $\delta_{1,2}$- interior operator $\delta_{1,2}.\text{int}$ on the set X for which each member of Φ is $\delta_{1,2}\psi_{1,2}$-fuzzy continuous.

Proof. Let $(X, \varphi_{1,2}.\text{int})$ is a characterized $FR_{2\frac{1}{2}}$ space. Then, there exists a $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping $f:(X, \varphi_{1,2}.\text{int}) \to (\mathrm{I}_L, \psi_{1,2}.\text{int}_{\Im})$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in F$, where $F \in \varphi_{1,2}C(X)$ and $x \notin F$. If $\delta_{1,2}.\text{int}$ is the coarsest $\delta_{1,2}$-interior operator on X for which each member of Φ is $\delta_{1,2}\psi_{1,2}$-fuzzy continuous, then $(X, \varphi_{1,2}.\text{int}_{\delta})$ is one of the family $((X, \varphi_{1,2}.\text{int}_k))_{k\in K}$ and therefore $\delta_{1,2}.\text{int} \geq \varphi_{1,2}.\text{int}$, that is, $\delta_{1,2}OF(X) \subseteq \varphi_{1,2}OF(X)$. Consider $x \in X$ and $\mu \in \varphi_{1,2}OF(X)$ such that $x_1 \leq \mu$. Then, there exists a $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping $f:(X, \varphi_{1,2}.\text{int}) \to (\mathrm{I}_L, \psi_{1,2}.\text{int}_{\Im})$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in S_0(\mu')$. From the hypothesis that $\delta_{1,2}.\text{int}$ is the coarsest $\delta_{1,2}$-interior operator on X for which each member of Φ is $\delta_{1,2}\psi_{1,2}$-fuzzy continuous mapping, we get that f is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping and therefore $\eta = f^{-1}(R_{\frac{1}{2}}) \in \delta_{1,2}OF(X)$ such that $\eta(x) = R_{\frac{1}{2}}(f(x)) = R_{\frac{1}{2}}(\overline{1}) = 1$ and $\eta(y) = R_{\frac{1}{2}}(f(y)) = R_{\frac{1}{2}}(\overline{0}) = 0$ for all $y_1 \leq \eta$. This means that $x_1 \leq \eta$ and $\mu' \leq \eta'$, that is, $x_1 \leq \eta$ and $\eta \in \delta_{1,2}OF(X)$ with $x_1 \leq \eta \leq \mu$. Hence, $\mu \in \delta_{1,2}OF(X)$ and then $\varphi_{1,2}.\text{int} \geq \delta_{1,2}.\text{int}$, that is, $\varphi_{1,2}OF(X) \subseteq \delta_{1,2}OF(X)$. Thus, $\varphi_{1,2}.\text{int}$ coincide with the coarsest $\delta_{1,2}$- interior operator $\delta_{1,2}.\text{int}$ on the set X.

Conversely, let $\varphi_{1,2}.\text{int}$ coincide with the coarsest $\delta_{1,2}$- interior operator $\delta_{1,2}.\text{int}$ on X for which each member of Φ is $\delta_{1,2}\psi_{1,2}$-fuzzy continuous, then each member of Φ is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous. Since $\Omega = \{R_\alpha \circ f \mid f \in \Phi, \alpha \in I\} \cup \{R^\alpha \circ f \mid f \in \Phi, \alpha \in I\} \cup \{\overline{0}, \overline{1}\}$ is a base for the characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$, then we can define the mapping $g:(X, \varphi_{1,2}.\text{int}) \to (\mathrm{I}_L, \psi_{1,2}.\text{int}_{\Im})$ by: $g(y)(s) = 1 - f(y)(1-s)$ for all $f \in \Phi, s \in I_{01}$ and $y \in X$. Hence, $g^{-1}(R_\alpha) = f^{-1}(R^{1-\alpha})$ and $g^{-1}(R^\alpha) = f^{-1}(R_{1-\alpha})$, therefore the base Ω for $(X, \varphi_{1,2}.\text{int})$ is in the form:

$$\Omega = \{f^{-1}(R_\alpha) \mid f \in \Phi, \alpha \in I_{01}\} \cup \{\overline{0}, \overline{1}\}.$$

Now, let $\mu \in \Omega$ and $x \in X$ such that $x \in \mu$. Then, there exists $f \in \Phi$ and $\alpha \in I_{01}$ such that $\chi_\mu = f^{-1}(R_{\alpha_0})$. On other hand for each $y \in X$, define the mapping $g(y): I \to L$ by: $g(y)(\alpha) = f(y)(\alpha_0 + \alpha(1-\alpha_0))$, then $g^{-1}(R_\alpha) = f^{-1}(R_{\alpha_0+\alpha(1-\alpha_0)})$ and $g^{-1}(R^\alpha) = f^{-1}(R^{\alpha_0+\alpha(1-\alpha_0)})$, therefore $g:(X, \varphi_{1,2}.\text{int}) \to (\mathrm{I}_L, \psi_{1,2}.\text{int}_{\Im})$ is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous. Since $R_0(g(y)) = R_{\alpha_0}(f(y)) = f^{-1}(R_{\alpha_0})(y) = \chi_\mu(y)$ for all $y \in X$, $\alpha_0 \in I_{01}$ and $f \in \Phi$, then $R_0(g(x)) = 1$ and $R_0(g(y)) = 0$ for all $y \in \mu'$, that is, $g(y) = \overline{0}$ for all $y \in \mu'$ and $g(x)(\alpha) = 1$ for some $\alpha \in I_{01}$. Thus, there exists $\gamma \in I_{01}$ such that $R^\gamma(g(x)) = \bigwedge_{k\geq\gamma}(g(x)(k))' = 1$ holds. Hence, we define the mapping $h:(X, \varphi_{1,2}.\text{int}) \to (\mathrm{I}_L, \psi_{1,2}.\text{int}_{\Im})$ as follows $h(z)(s) = g(z)(rs)$ for all $z \in X$ and $s \in I_{01}$, then h is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous and $R_0(h(y)) = R_0(g(y)) = 0$ for all $y \in \mu'$ and $R_0(h(x)) = R_0(g(x)) = 1$. Moreover, since $R^1(h(x)) = R^\gamma(g(x)) = 1$, then we have $h(x) = \overline{1}$ and $h(y) = \overline{0}$ for all $y \in \mu'$. Hence, because of Proposition 3.1, we get that $(X, \varphi_{1,2}.\text{int})$ is characterized $FR_{2\frac{1}{2}}$ space. ■

From Diagram 4.1, we note that, every characterized $FT_{3\frac{1}{2}}$ space is characterized FT_1 space and because of Propositions 2. 5 and 5. 2, we can deduce the following result.

Corollary 5.1 Let a fuzzy topological space (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. If $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized

$FT_{3\frac{1}{2}}$ space and Φ is the fuzzy function family of all the $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mappings, $f:(X,\varphi_{1,2}.\mathrm{int}_{\tau})\to(\mathrm{I}_{\mathrm{L}},\psi_{1,2}.\mathrm{int}_{\Im})$, then every two distinct points in X are Φ-separated.

Proof. Easily seen. ■

We should notice that Proposition 4.3 gives us fuzzy proximity δ that is compatible with the characterized FT_4 space from Proposition 4.4. Now, we have the following important result, which shows that there is also other fuzzy proximity δ on L^X, which is compatible with the characterized $FR_{2\frac{1}{2}}$ space.

Proposition 5.3 Let a fuzzy topological space (X,τ) be fixed and $\varphi_1,\varphi_2\in O_{(L^X,\tau)}$. If $(X,\varphi_{1,2}.\mathrm{int}_{\tau})$ is characterized $FR_{2\frac{1}{2}}$ space and Φ is a fuzzy function family of all $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mappings, then the binary relation δ on L^X which is defied by:

$$\mu\,\overline{\delta}\,\rho \Leftrightarrow \mu \text{ and } \rho \text{ are } \Phi\text{-separated,}$$

for all $\mu,\rho\in L^X$ is a fuzzy proximity on X compatible with the family of all $\varphi_{1,2}$-open fuzzy subsets $\varphi_{1,2}OF(X)$, that is, $(X,\varphi_{1,2}.\mathrm{int}_{\tau})=(X,\varphi_{1,2}.\mathrm{int}_{\delta})$.

Proof. Let $\mu,\rho\in L^X$ such that $\mu\,\overline{\delta}\,\rho$, then there exists $g\in\Phi$ such that $g(x)=\overline{1}$ for all $x_1\le\mu$ and $g(y)=\overline{0}$ for all $y_1\le\rho$. Consider $f:(X,\varphi_{1,2}.\mathrm{int}_{\tau})\to(\mathrm{I}_{\mathrm{L}},\psi_{1,2}.\mathrm{int}_{\Im})$ is the mapping defied by: $f(x)(s)=1-g(x)(1-s)$ for all $x\in X$ and $s\in I$, then f is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous such that $f(x)(s)=0$ for all $x_1\le\mu$ and $f(y)(s)=1$ for all $y_1\le\rho$, that is, $\rho\,\overline{\delta}\,\mu$. Hence, condition (P1) is fulfilled. Consider $\mu\vee\eta$ and ρ are Φ-separated, then μ,ρ and η,ρ are Φ-separated. Hence, $(\mu\vee\eta)\,\overline{\delta}\,\rho$ implies that $\mu\,\overline{\delta}\,\rho$ and $\eta\,\overline{\delta}\,\rho$. On other hand $\mu\,\overline{\delta}\,\rho$ and $\eta\,\overline{\delta}\,\rho$ means that there exist $f_1,g_1\in\Phi$ such that $f_1(x)=\overline{1}$ for all $x_1\le\mu$ and $f_1(y)=\overline{0}$ for all $y_1\le\rho$ and $g_1(x)=\overline{1}$ for all $x_1\le\eta$ and $g_1(y)=\overline{0}$ for all $y_1\le\rho$. Consider $h:(X,\varphi_{1,2}.\mathrm{int}_{\tau})\to(\mathrm{I}_{\mathrm{L}},\psi_{1,2}.\mathrm{int}_{\Im})$ is the mapping defied by: $h(x)(s)=\max\{f_1(x)(s),g_1(x)(s)\}$ for all $x\in X$ and $s\in I$, then h is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous such that $h(x)=\overline{1}$ for all $x_1\le\mu$ or $x_1\le\eta$ and $h(y)=\overline{0}$ for all $y_1\le\rho$. Then, $(\mu\vee\eta)\,\overline{\delta}\,\rho$ and therefore (P2) is fulfilled. To prove (P3), let $k:(X,\varphi_{1,2}.\mathrm{int}_{\tau})\to(\mathrm{I}_{\mathrm{L}},\psi_{1,2}.\mathrm{int}_{\Im})$ is the mapping defined by $k(x)(s)=0$ for all $x\in X$ and $s\in I$, then we get $k(x)=\overline{0}$ for all $x\in X$ and therefore k is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous. So, we can easily say that $k(x)=\overline{1}$ for all $x_1\le\overline{0}$ and $k(y)=\overline{0}$ for all $\eta\in L^X$ and $y_1\le\eta$. That is, $\overline{0}$ and η are Φ-separated for all $\eta\in L^X$ and hence $\mu=\overline{0}$ or $\eta=\overline{0}$ implies that $\mu\,\overline{\delta}\,\eta$. Thus, (P3) is fulfilled. Obviously, from the definition of δ, it is clear to see that $\mu\,\overline{\delta}\,\rho$ implies that $\mu\,\overline{\delta}\,\rho'$ and therefore (P4) is fulfilled. Consider $\mu,\rho\in L^X$ such that $\mu\,\overline{\delta}\,\rho$, then there exists a mapping $f_2\in\Phi$ such that $f_2(x)=\overline{1}$ for all $x_1\le\mu$ and $f_2(y)=\overline{0}$ for all $y_1\le\rho$. Consider $g,l:(X,\varphi_{1,2}.\mathrm{int}_{\tau})\to(\mathrm{I}_{\mathrm{L}},\psi_{1,2}.\mathrm{int}_{\Im})$ are the mappings defined by:

$$g(x)(s)=\frac{(1+s)}{2}f_2(x) \quad \text{and} \quad l(x)(s)=\frac{s}{2}f_2(x)$$

for all $x\in X$ and $s\in I_{01}$. Since $f_2\in\Phi$, then g and l are $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous and $g(x)(s)=1$ for all $x_1\le\mu$ and $l(x)(s)=0$ for all $x_1\le\rho$. Since $R^1(g(x))=R^1(f_2(x))\ge R^{\frac{1}{2}}(f_2(x))\ge R_{\frac{1}{2}}(f_2(x))=R_0(g(x))$ and $R^1(l(x))=R^{\frac{1}{2}}(f_2(x))\le R_0(f_2(x))=R_0(l(x))$, then if we consider $\eta=\left(f_2^{-1}(R^{\frac{1}{2}})\right)'$ where $f_2^{-1}(R^{\frac{1}{2}})\in L^X$, we get

$R_0(g(x)) \leq \eta'(x) \leq R_0(l(x))$ and therefore $g(x)(s)=0$ for all $x_1 \leq \eta$ and $s \in I_{01}$ and $l(x)(s)=1$ for all $x_1 \leq \eta'$ and $s \in I_{01}$. That is, $g(x)=\overline{1}$ for all $x_1 \leq \mu$ and $g(y)=\overline{0}$ for all $y_1 \leq \eta$ and moreover $l(x)=\overline{1}$ for all $x_1 \leq \eta'$ and $l(y)=\overline{0}$ for all $y_1 \leq \rho$. Hence, $\mu \,\overline{\delta}\, \eta$ and $\eta' \,\overline{\delta}\, \rho$, therefore (P5) is fulfilled. Consequently, δ is a fuzzy proximity on X.

Now, let $\eta \in (\varphi_{1,2}CF(X))_\delta$ and $x \in X$ such that $\eta'(x)=1$. Since $\eta(y)=\varphi_{1,2}.\mathrm{cl}_\delta\eta(y)=\bigwedge_{\eta\overline{\delta}\rho'}\rho(y)$, then there exists $\rho \in L^X$ with $\eta \,\overline{\delta}\, \rho'$ such that $\rho(x)=0$. Hence, $\eta \,\overline{\delta}\, \rho'$ implies there exists $f_3 \in \Phi$ such that $f_3(x)=\overline{1}$ for all $x_1 \leq \eta$ and $f_3(y)=\overline{0}$ for all $y_1 \leq \rho'$. Consider $\mu = f_3^{-1}(R^{\frac{1}{2}})$, then we get $\mu(y)=R^{\frac{1}{2}}(f_3(y)) \geq (R_0(f_3(y)))'=1$ holds for all $y_1 \leq \eta'$ and then $\eta \,\overline{\delta}\, \rho'$ implies that $\rho' \leq \eta'$. Moreover, $\mu(y)=R^{\frac{1}{2}}(f_3(y)) \leq R^1(f_3(y)) \leq \eta'(y)$ holds for. all $y \in X$. Thus, $\mu \in \varphi_{1,2}OF(X)$ with $x_1 \leq \mu$ and $\mu \leq \eta'$ which means that $\eta' \in \varphi_{1,2}OF(X)$ and therefore $\eta \in \varphi_{1,2}CF(X)$. Hence, $(\varphi_{1,2}CF(X))_\delta \subseteq \varphi_{1,2}CF(X)$ which implies that $\varphi_{1,2}OF(X) \subseteq (\varphi_{1,2}OF(X))_\delta$. Thus, $\varphi_{1,2}.\mathrm{int}_\delta \leq \varphi_{1,2}.\mathrm{int}_\tau$ holds. Consequently, $(X, \varphi_{1,2}.\mathrm{int}_\delta)$ is finer than $(X, \varphi_{1,2}.\mathrm{int}_\tau)$.

Conversely, let $\eta \in \varphi_{1,2}CF(X)$ and $\eta \neq \varphi_{1,2}.\mathrm{cl}_\delta$, then there exists $x \in X$ such that $\varphi_{1,2}.\mathrm{cl}_\delta\eta(x)>0$ and $\eta(x)=0$. Since, $x \in S_0\eta' \in \varphi_{1,2}OF(X)$ and $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is characterized $FR_{2\frac{1}{2}}$ space, then there exists $f_4 \in \Phi$ such that $f_4(x)=\overline{1}$ and $f_4(y)=\overline{0}$ for all $y \in S_0\eta$. Consider $\mu \in L^X$ is the fuzzy set defined by $\mu(y)=(R^1(f_4(y)))'=\bigvee_{\alpha\geq 1} f_4(y)(\alpha)$ for all $y \in X$, then $\mu(y) \leq R_0(f_4(y)) \leq \eta'(y)$ holds for all $y \in X$. This means that $\bigvee_{\alpha\geq 1} f_4(x)(\alpha)=1$ for all $x_1 \leq \mu$ and $\bigvee_{\alpha>0} f_4(y)(s)=0$ for all $y_1 \leq \eta$, that is, $f_4(x)=\overline{1}$ for all $x_1 \leq \mu$ and $f_4(y)=\overline{0}$ for all $y_1 \leq \eta$. Hence, μ and η are Φ-separated which implies that $\mu \,\overline{\delta}\, \eta$. Therefore,

$$\varphi_{1,2}.\mathrm{cl}_\delta\eta(x)=\bigwedge_{\eta\overline{\delta}\rho'}\rho(x) \leq \mu'(x)=R^1(f_4(x))=0$$

Hence, $\varphi_{1,2}.\mathrm{cl}_\delta\eta(x)=0$ which is a contradiction and therefore we have $\eta \in (\varphi_{1,2}CF(X))_\delta$. Thus, $(\varphi_{1,2}OF(X))_\delta \subseteq \varphi_{1,2}OF(X)$, which implies that $\varphi_{1,2}.\mathrm{int}_\tau \leq \varphi_{1,2}.\mathrm{int}_\delta$ holds and therefore $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is finer than $(X, \varphi_{1,2}.\mathrm{int}_\delta)$. Consequently, $(X, \varphi_{1,2}.\mathrm{int}_\tau)=(X, \varphi_{1,2}.\mathrm{int}_\delta)$ and then δ is compatible with $\varphi_{1,2}OF(X)$. ■

Now, we introduce an example of a fuzzy proximity δ on a set X and show that it is induces a characterized $FR_{2\frac{1}{2}}$ space.

Example 5.1 Let $L=\{0,\frac{1}{2},1\}$, $X=\{x,y\}$ and $\tau=\{\overline{1},\overline{0},x_1,y_1\}$ is a fuzzy topology on X. Choose $\varphi_1=\mathrm{int}_\tau$, $\varphi_2=\mathrm{cl}_\tau$, $\psi_1=\mathrm{int}_\Im$ and $\psi_2=\mathrm{cl}_\Im$. Hence, $x \neq y$ and because of Example 3.1, $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is characterized $FT_{3\frac{1}{2}}$ space. Now, consider δ is a binary relation on L^X defined as follows:

$$\mu \,\overline{\delta}\, \eta \Leftrightarrow \exists\ \varphi_{1,2}\psi_{1,2}\text{-fuzzy continuous mapping } f:(X, \varphi_{1,2}.\mathrm{int}_\tau) \to (\mathrm{I_L}, \psi_{1,2}.\mathrm{int}_\Im) \ni$$
$$f(x)=\overline{1} \text{ for all } x \in X \text{ with } x_1 \leq \mu \text{ and } f(y)=\overline{0} \text{ for all } y_1 \leq \eta,$$

for all $\mu, \eta \in L^X$. Hence, because of Proposition 5.3, δ is a fuzzy proximity on X and it is compatible with $\varphi_{1,2}OF(X)$, that is, the associated characterized fuzzy proximity space $(X, \varphi_{1,2}.\text{int}_\delta)$ with δ is characterized $FR_{2\frac{1}{2}}$ space. ■

Proposition 5.4 Let (X,τ) be a fuzzy topological space, $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$ and δ is a fuzzy proximity on X. If $\mu \,\bar{\delta}\, \eta$ for some $\mu,\eta \in L^X$ and Φ is a fuzzy function family of all $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mappings $f:(X,\varphi_{1,2}.\text{int}_\delta) \to (\mathrm{I}_L, \psi_{1,2}.\text{int}_{\delta^*})$, then μ and η are Φ-separated by the $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping $f:(X,\varphi_{1,2}.\text{int}_\tau) \to (\mathrm{I}_L, \psi_{1,2}.\text{int}_\Im)$

Proof. Because of (4.2). Lemma 4.1 and Remark 4.1, we can deduce that μ and η are Φ-separated and therefore because of Proposition 4.2, we deduce that they are Φ-separated by the $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping $f:(X,\varphi_{1,2}.\text{int}_\tau) \to (\mathrm{I}_L, \psi_{1,2}.\text{int}_\Im)$. ■

Corollary 5.2 Let (X,δ) be a fuzzy proximity space, $\varphi_1, \varphi_2 \in O_{(L^X,\tau_\delta)}$ and $\mu,\eta \in L^X$ such that $\mu \,\bar{\delta}\, \eta$. If Φ is the fuzzy function family of all proximity fuzzy continuous mappings $f:(X,\delta) \to (\mathrm{I}_L, \delta^*)$, then μ and η are Φ-separated by the $\varphi_{1,2}\psi_{1,2}\delta$-fuzzy continuous mapping $f:(X,\varphi_{1,2}.\text{int}_\delta) \to (\mathrm{I}_L, \psi_{1,2}.\text{int}_{\delta^*})$.

Proof. Immediate from (4.2). Lemma 4.1 and Proposition 5.4. ■

As shown in [20], if δ and δ^* are two fuzzy proximity on a set X, then δ is finer than δ^* or δ^* is coarser than δ, provided $\mu \,\bar{\delta}^*\, \eta$ implies that $\mu \,\bar{\delta}\, \eta$ for all $\mu,\eta \in L^X$. Because this fact we can deduce the following result.

Proposition 5.5 Let (X,τ) and (X,σ) are two fuzzy topological spaces, $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$ and $\delta_1, \delta_2 \in O_{(L^X,\sigma)}$. Consider $(X,\varphi_{1,2}.\text{int}_\tau)$ and $(X,\delta_{1,2}.\text{int}_\sigma)$ are characterized $FR_{2\frac{1}{2}}$ spaces and δ is a fuzzy proximity on X compatible with the class of all $\varphi_{1,2}$-open fuzzy subsets $\varphi_{1,2}OF(X)$. If δ^* is the fuzzy proximity on X defined by:

$$\mu \,\bar{\delta}^*\, \eta \Leftrightarrow \mu \text{ and } \eta \text{ are } \Phi\text{-separated in } (X,\delta_{1,2}.\text{int}_\sigma)$$

for all $\mu,\eta \in L^X$, then $\delta_{1,2}.\text{int}_\sigma \le \varphi_{1,2}.\text{int}_\tau$ implies that δ^* is finer than δ.

Proof. Suppose that $\mu,\eta \in L^X$ such that $\mu \,\bar{\delta}\, \eta$. Because of Proposition 5.3, there exists $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping $f:(X,\varphi_{1,2}.\text{int}_\tau) \to (\mathrm{I}_L, \psi_{1,2}.\text{int}_\sigma)$ such that $f(x) = \bar{1}$ for all $x_1 \le \mu$ and $f(y) = \bar{0}$ for all $y_1 \le \eta$. Since $\delta_{1,2}.\text{int}_\sigma \le \varphi_{1,2}.\text{int}_\tau$, then $\varphi_{1,2}OF(X) \subseteq \delta_{1,2}OF(X)$ and therefore f is $\delta_{1,2}\psi_{1,2}$-fuzzy continuous, that is, μ and η are Φ-separated in $(X,\delta_{1,2}.\text{int}_\sigma)$. Hence, $\mu \,\bar{\delta}^*\, \eta$ and therefore δ^* is finer than δ. ■

6. CONCLUSION

In this paper, we introduced and studied two new types of characterized fuzzy spaces named characterized $FR_{2\frac{1}{2}}$ spaces and characterized $FT_{3\frac{1}{2}}$ spaces by using the real fuzzy function family of all $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mappings presented in [5] as a generalization of all the weaker and stronger forms of the notion of completely regular fuzzy topological spaces introduced in [11,24 ,26 ,29]. The characterized $FT_{3\frac{1}{2}}$ space or characterized Tychonoff space is the characterized fuzzy space for which it is characterized FT_1 and characterized $FR_{2\frac{1}{2}}$ space in this sense. We introduced and studied many difference relations between the characterized $FR_{2\frac{1}{2}}$ spaces and the characterized

$FT_{3\frac{1}{2}}$ spaces with other characterized FR_k and characterized FT_S spaces, which are presented in [2, 3, 4]. To find these relations, we introduced generalization to the Urysohn's Lemma for the characterized FR_3 spaces with help of the characterized fuzzy proximity spaces presented in [1]. So, we applied the relation between the farness and the finer relation on the fuzzy sets to introduced the notions of $\varphi_{1,2}\delta$- fuzzy neighborhood at the point x in the characterized fuzzy proximity space and of $\varphi_{1,2}\psi_{1,2}\delta$-fuzzy continuity between the characterized fuzzy proximity spaces. Moreover, the characterized fuzzy space is finer than the associated characterized fuzzy proximity space that is present in [1]. The concepts of fuzzy function family and of the Φ-separable are applied to introduce important properties for the concept of the $\varphi_{1,2}\psi_{1,2}\delta$-continuity. The $\varphi_{1,2}\psi_{1,2}$-fuzzy continuity of the mapping f between characterized fuzzy spaces is more general than of the $\varphi_{1,2}\psi_{1,2}\delta$-continuity of f between the characterized fuzzy proximity spaces. An important result, we show that if the fixed fuzzy topological space (X, τ) is normal, then the characterized fuzzy space $(X, \varphi_{1,2}.\text{int}_\tau)$ is finer than the associated characterized fuzzy proximity space $(X, \varphi_{1,2}.\text{int}_\delta)$ and they identical if $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized FT_4 space with help of the complementarily symmetric fuzzy topogenous structure that identified with the fuzzy proximity δ. More generally, the fuzzy function family of all $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mappings are used to show that the characterized $FR_{2\frac{1}{2}}$ spaces and the associated characterized fuzzy proximity spaces are identical. Many new special classes from the $\varphi_{1,2}$-open fuzzy sets, valued $\varphi_{1,2}$-fuzzy neighborhoods, characterized $FR_{2\frac{1}{2}}$ spaces, characterized FT_1 spaces, characterized $FT_{3\frac{1}{2}}$ spaces and characterized FT_4 spaces are listed in Table (1).

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	Operations	$\varphi_{1,2}$- open fuzzy sets	Valued $\varphi_{1,2}$- Fuzzy neigh.	Char. $FR_{2\frac{1}{2}}$ Space	Char. FT_1 space	Char. $FT_{3\frac{1}{2}}$ space	Char. FT_4 Space
1	$\varphi_1 = \text{int}_\tau$, $\varphi_2 = 1_{L^X}$ $\psi_1 = \text{int}_\Im$, $\psi_2 = 1_{L^I}$	τ [16,19]	Valued fuzzy neighbor.[18]	Fuzzy $R_{2\frac{1}{2}}$ space [11,12]	Fuzzy T_1 space [14]	Fuzzy $T_{3\frac{1}{2}}$ Space [11,12]	Fuzzy T_4 space [14]
2	$\varphi_1 = \text{int}_\tau$, $\varphi_2 = \text{cl}_\tau$ $\psi_1 = \text{int}_\Im$, $\psi_2 = \text{cl}_\Im$	τ_θ [31]	Valued θ-Fuzzy neigh.	Fuzzy θ-$R_{2\frac{1}{2}}$ space	Fuzzy θ-T_1 space	Fuzzy θ-$T_{3\frac{1}{2}}$ space	Fuzzy θ-T_4 space
3	$\varphi_1 = \text{int}_\tau$, $\varphi_2 = \text{int}_\tau \circ \text{cl}_\tau$ $\psi_1 = \text{int}_\Im$, $\psi_2 = \text{int}_\Im \circ \text{cl}_\Im$	τ_δ [22]	Valued δ-Fuzzy neigh.	Fuzzy δ-$R_{2\frac{1}{2}}$ space	Fuzzy δ-T_1 space	Fuzzy δ-$T_{3\frac{1}{2}}$ space	Fuzzy δ-T_4 space
4	$\varphi_1 = \text{cl}_\tau \circ \text{int}_\tau$, $\varphi_2 = 1_{L^X}$ $\psi_1 = \text{int}_\Im$, $\psi_2 = 1_{L^I}$	$SOF(X)$ [10]	Valued Semi-Fuzzy neigh.	Fuzzy semi-$R_{2\frac{1}{2}}$ space	Fuzzy semi-T_1 space	Fuzzy semi-$T_{3\frac{1}{2}}$ space	Fuzzy semi-T_4 space
5	$\varphi_1 = \text{cl}_\tau \circ \text{int}_\tau$, $\varphi_2 = \text{cl}_\tau$ $\psi_1 = \text{int}_\Im$, $\psi_2 = \text{cl}_\Im$	$\tau_{(\theta.S)}$	Valued θ-semi Fuzzy neigh.	Fuzzy θ semi-$R_{2\frac{1}{2}}$ space	Fuzzy θ semi-T_1 space	Fuzzy θ semi-$T_{3\frac{1}{2}}$ space	Fuzzy θ semi-T_4 space
6	$\varphi_1 = \text{cl}_\tau \circ \text{int}_\tau$, $\varphi_2 = \text{int}_\tau \circ \text{cl}_\tau$ $\psi_1 = \text{int}_\Im$, $\psi_2 = \text{int}_\Im \circ \text{cl}_\Im$	$\tau_{(\delta.S)}$	Valued δ-semi Fuzzy neigh.	Fuzzy δ semi-$R_{2\frac{1}{2}}$ space	Fuzzy δ semi-T_1 space	Fuzzy δ semi-$T_{3\frac{1}{2}}$ space	Fuzzy δ semi-T_4 space
7	$\varphi_1 = \text{int}_\tau \circ \text{cl}_\tau$, $\varphi_2 = 1_{L^X}$ $\psi_1 = \text{int}_\Im$, $\psi_2 = 1_{L^I}$	$POF(X)$ [17]	Valued pre-Fuzzy neigh.	Fuzzy pre-$R_{2\frac{1}{2}}$ space	Fuzzy pre-T_1 space	Fuzzy pre-$T_{3\frac{1}{2}}$ space	Fuzzy pre-T_4 space
8	$\varphi_1 = \text{cl}_\tau \circ \text{int}_\tau$, $\varphi_2 = S.\text{cl}_\tau$ $\psi_1 = \text{int}_\Im$, $\psi_2 = S.\text{cl}_\Im$	$\tau_{(S.\theta)}$	Valued semi θ-Fuzzy neigh.	Fuzzy semi θ-$R_{2\frac{1}{2}}$ space	Fuzzy semi θ-T_1 space	Fuzzy semi θ-$T_{3\frac{1}{2}}$ space	Fuzzy semi θ-T_4 space
9	$\varphi_1 = \text{cl}_\tau \circ \text{int}_\tau, \varphi_1 = S.\text{int}_\tau \circ S.\text{cl}_\tau$ $\psi_1 = \text{int}_\Im$, $\psi_2 = S.\text{int}_\Im \circ S.\text{cl}_\Im$	$\tau_{(S.\delta)}$	Valued semi δ-Fuzzy neigh.	Fuzzy semi δ-$R_{2\frac{1}{2}}$ space	Fuzzy semi δ-T_1 space	Fuzzy semi δ-$T_{3\frac{1}{2}}$ space	Fuzzy semi δ-T_4 space
10	$\varphi_1 = \text{cl}_\tau \circ \text{int}_\tau \circ \text{cl}_\tau$, $\varphi_2 = 1_{L^X}$ $\psi_1 = \text{cl}_\Im \circ \text{int}_\Im \circ \text{cl}_\Im$, $\psi_2 = 1_{L^I}$	$\beta OF(X)$ [9]	Valued β- Fuzzy neigh.	Fuzzy β-$R_{2\frac{1}{2}}$ space	Fuzzy β-T_1 space	Fuzzy β-$T_{3\frac{1}{2}}$ space	Fuzzy β-T_4 space
11	$\varphi_1 = \text{int}_\tau \circ \text{cl}_\tau \circ \text{int}_\tau$, $\varphi_2 = 1_{L^X}$ $\psi_1 = \text{int}_\Im \circ \text{cl}_\Im \circ \text{int}_\Im, \psi_2 = 1_{L^I}$	$\lambda OF(X)$ [17]	Valued λ-Fuzzy neigh.	Fuzzy λ-$R_{2\frac{1}{2}}$ space	Fuzzy λ-T_1 space	Fuzzy λ-$T_{3\frac{1}{2}}$ space	Fuzzy λ-T_4 space
12	$\varphi_1 = S.\text{cl}_\tau \circ \text{int}_\tau$, $\varphi_2 = 1_{L^X}$ $\psi_1 = S.\text{cl}_\Im \circ \text{int}_\Im$, $\psi_2 = 1_{L^I}$	$fOF(X)$	Valued feebly-Fuzzy neigh.	Fuzzy feebly-$R_{2\frac{1}{2}}$ space	Fuzzy feebly -T_1 space	Fuzzy feebly-$T_{3\frac{1}{2}}$ space	Fuzzy feebly-T_4 space

Table (1) : Some special classes of $\varphi_{1,2}$- open fuzzy sets, Valued $\varphi_{1,2}$- fuzzy neighborhoods, Char. $FR_{2\frac{1}{2}}$ spaces, Char. T_1 spaces, Char. $T_{3\frac{1}{2}}$ spaces, Char. T_4 spaces.

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