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Evaluation of real definite Integrals by using mixed quadrature rules over a Triangular Domain

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ABSTRACT

A mixed quadrature rule of higher precision for approximate evaluation of real definite integrals over a triangular domain has been constructed. The relative efficiencies of the proposed mixed quadrature rules have been verified by using suitable test integrals. In this paper,we present a mixed quadrature i.e. mixed quadrature of anti-Lobatto rule and Fejer's first rule in one variable. For real definite integral over the triangular surface: $\{(x,y)|\ 0 \le x,y \le 1,\ \ x+y \le 1\}$ in the Cartesian two-dimensional (x,y) space,mathematical transformation from (x,y) space to $(\xi,\ \eta)$ space maps the standard triangle in (x,y) space to a standard 2-square in $(\xi,\ \eta)$ space:$\{(\xi,\eta)|-1 \le \xi,\ \eta \le 1\}$.

Indexing terms/Keywords

Anti-Lobatto four point rule; Fejer's first rule; mixed quadrature rule; finite element method(FEM); triangular elements; standard 2-square;extended numerical integration.

Academic Discipline And Sub-Disciplines

Numerical Method,Numerical Quadrature

SUBJECT CLASSIFICATION

2000 Mathematics Subject Classification: 65D30, 65D32

TYPE (METHOD/APPROACH)

Mixed quadrature

1.INTRODUCTION

In science and engineering, we observe that some integral problems can't be easily evaluated analytically but it is possible to calculate in numerically.So that numerical techniques are very good tool for solving different integral problems. This paper describes a method for the evaluation of integrals over a triangle using mixed quadrature rules . In particular, they are used for problems involving calculation of mass shell, fluid and mass flows across a surface, electric charge distribution over a surface, plate bending and heat conduction over a plate. The basic problem of integration of an arbitrary function of two variables over the surface of a triangle was first introduced by Hammer et. al. [1,2,3] . In connection of FEM, the triangular elements provide tremendous results. Cowper [4] provided a table of Gaussian quadrature formula for symmetrically placed integration points. Lyness and Jespersen [5] made an elaborate study of symmetric quadrature rules by formulating the problem in polar coordinates. Lannoy [6] discussed the numerical error in integration rule [4]. Laurie [7] derived 7-point integration rule and discussed the numerical error in integrating some functions. Laursen and Gellert [8] gave a detailed table of symmetric integration formulae and suggested some new higher-order formulae of precision up to degree 10. Lether [9], Hillion[10] and Lague and Baldur [11] considered the product formulae derivable from one-dimensional Gaussian quadrature rules. Reddy [12], and Reddy and Shippy [14]derived 3-, 4-, 6 and 7-point formulae which give improved accuracy. The formulation of mixed quadrature rules was first coined by R. N. Das and G. Pradhan [15]. D. P. Laurie [21] is first to coin the idea of anti-Gaussian quadrature formula. An anti-Gaussian quadrature formula is $(n+1)$ point formula of degree$(2n-1)$ which integrates all polynomials of degree up to $(2n+1)$ with an error equal in magnitude but opposite in sign to that of $n-$point Gaussian formula.Many authors [16,17,19] have produced different mixed quadrature rules.

In this paper, we get motivation for successfully forming a mixed quadrature rule of anti-Lobatto rule and Fejer's first rule over a triangular domain. The mixed quadrature rule over a triangular domain so found has been tested and compared with its constituent rules by computing numerically three test integrals.

2. FORMULATION OF INTEGRALS OVER A TRANGULAR AREA

The numerical integration of an arbitrary function f over the triangle T is given by

$$I = \iint_T f(x,y)\, dxdy = \int_0^1 dx \int_0^{1-x} f(x,y) dy = \int_0^1 dy \int_0^{1-y} f(x,y) dx . \quad (2.1)$$

It is now required to find the value of the integral by a quadrature formula:

$$I=\sum_{m=1}^{N} C_m f(x_m, y_m) . \quad (2.2)$$

Where C_m are the weights associated with specific points (x_m, y_m) and N is the number of pivotal points related to the required precision.

The double integral over the triangle surface of equation (2.1) can be transformed to the standard square $\{(u,v) |\ 0 \le u \le 1,\ 0 \le v \le 1\}$ by substitution $x = u$ and $y = (1-u)v$, we have

$$I = \int_0^1 \int_0^{1-x} f(x,y)\, dydx = \int_0^1 \int_0^1 f\big(x(u,v),\ y(u,v)\big)\, J\, dudv . \quad (2.3)$$

Where $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = 1 - u .$

From equation (2.3), we have $I = \int_0^1 \int_0^1 f(u,\ (1-u)v)(1-u) dudv .$ (2.4)

The integral I of equation (2.4) can be transformed further into an integral over the standard 2-square: $\{(\xi,\ \eta) |\ -1 \le \xi \le 1,\ -1 \le \eta \le 1\}$ by substitution $u = \frac{1+\xi}{2}$, $v = \frac{1+\eta}{2}$. (2.5)

Then clearly the determinant of the Jacobian and the differential area are

$$\frac{\partial(u,\ v)}{\partial(\xi,\ \eta)} = \frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta} - \frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi} = \frac{1}{2}\left(\frac{1}{2}\right) - 0 \times 0 = \frac{1}{4}$$

$$dudv = \frac{\partial(u,\ v)}{\partial(\xi,\ \eta)} d\xi d\eta = \frac{1}{4} d\xi d\eta \quad (2.6)$$

Now on using equations (2.5) and (2.6) in equation(2.4) we have

$$I = \int_0^1 \int_0^{1-x} f(x,y)\, dydx = \int_0^1 \int_0^1 f(u,\ (1-u)v)(1-u)\, dudv$$

$$= \int_{-1}^1 \int_{-1}^1 f\left(\frac{1+\xi}{2},\ \frac{(1-\xi)(1+\eta)}{4}\right)\left(\frac{1-\xi}{8}\right) d\xi d\eta \quad (2.7)$$

Equation(2.7) represents an integral over the surface of standard 2-square:

$\{(\xi,\ \eta) |\ -1 \le \xi \le 1, -1 \le \eta \le 1\}$. Efficiently quadrature coefficients are readily obtained from [20].

From equation(2.7), we can write:

$$I = \int_{-1}^1 \int_{-1}^1 f\big(x(\xi,\eta),\ y(\xi,\eta)\big)\left(\frac{1-\xi}{8}\right) d\xi d\eta .$$

$$I \cong \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{1-\xi_i}{8}\right) w_i w_j f\big(x(\xi_i,\ \eta_j),\ y(\xi_i,\ \eta_j)\big) . \quad (2.8)$$

Where ξ_i, η_j are Gaussian points in the $\xi,\ \eta$ directions respectively, and w_i, w_j are the corresponding weights.

We can write equation(2.8) as:

$$I \cong \sum_{k=1}^{N=n\times n} C_k f(x_k,\ y_k) . \quad (2.9)$$

Where C_k, x_k, y_k are obtained from the relation

$$\left.\begin{aligned} C_k &= \left(\frac{1-\xi_i}{8}\right) w_i w_j \\ x_k &= \frac{1+\xi_i}{2} \\ y_k &= \frac{(1-\xi_i)(1+\eta_j)}{4} \end{aligned}\right\} \tag{2.10}$$

Where $k = 1,2, \ldots, n$

$i = 1,2, \ldots, n$

$j = 1,2, \ldots, n$

The weighting coefficients C_k and sampling points $(x_k,\ y_k)$ of various orders can now be easily computed by formulae (2.9) and (2.10). We have tabulated a sample of these weight coefficients and sampling points in Table-1.

3. MIXED QUADRATURE OF ANTI-LOBATTO RULE AND FEJER'S 1ST RULE IN ONE VARIABLE

The mixed quadrature (I_{mix}) of Anti-Lobatto rule (which has been framed by us in the light of anti-Gaussian rule) and Fejer 's first rule is given below:

$$I_{mix} = \int_{-1}^{1} f(x)\,dx$$

$$\approx \frac{30}{99}\left[f\left(\sqrt{\frac{2}{5}}\right) + f\left(-\sqrt{\frac{2}{5}}\right)\right] - \frac{3}{99}[f(-1) + f(1)] + \frac{32}{99}\left[f\left(\frac{\sqrt{3}}{2}\right) + f\left(-\frac{\sqrt{3}}{2}\right)\right] + \frac{80}{99} f(0). \tag{3.1}$$

Where $w_1 = \frac{30}{99},\ w_2 = \frac{30}{99},\ w_3 = -\frac{3}{99},\ w_4 = -\frac{3}{99},\ ,w_5 = \frac{32}{99},\ w_6 = \frac{32}{99},\ w_7 = \frac{80}{99}$.

Applying the mixed rule(3.1) to double integral

$$I_{mix} = \int_{-1}^{1}\int_{-1}^{1} f\big(x(\xi,\eta),\ y(\xi,\eta)\big)\left(\frac{1-\xi}{8}\right) d\xi\, d\eta$$

we have

$$I_{mix} \approx \sum_{k=1}^{49} C_k f(x_{ij},\ y_{ij}) = \sum_{k=1}^{49} C_k f(x_k,\ y_k) \tag{3.2}$$

Where
$$\left.\begin{aligned} C_k = C_{ij} &= \left(\frac{1-\xi_i}{8}\right) w_i w_j \\ x_k = x_{ij} &\ and\ y_k = y_{ij} \end{aligned}\right\} \tag{3.3}$$

The weighting coefficients C_k and sampling points $(x_k,\ y_k)$ of various orders can be easily computed using the equation (2.10).

4. NUMERICAL VERIFICATIONS

Integrals	*Exact Value*	*Approximate Values*	*|Error|*
$I_1 = \int_0^1\int_0^{1-y} (x+y)^{\frac{1}{2}}\,dx\,dy$	0.400000000	0.401243	0.001243
$I_2 = \int_0^1\int_0^{1-y} (x+y)^{-\frac{1}{2}}\,dx\,dy$	0.666666667	0.641119	0.025547667
$I_3 = \int_0^1\int_0^{1-x} e^{-y^2}\cos(xy)\,dx\,dy$	0.4284998849	0.429191	0.0006911151

5. CONCLUSIONS

From the numerical verification we conclude that the mixed quadrature rule used in this paper gives better result than those obtained in the previous papers [13, 18, 19] on integration of real functions over triangles.

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