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New Types of Pre-θ- Open Sets and Associated Weak Separation Axioms

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ABSTRACT

Pal and Bhattacharyya (1996)introduced the notion of pre-θ-open sets.In this paper, we consider the class of pre-θ-open sets in topological spaces and investigate some of their properties. Also,we present and study some weak separation axioms by involving the notion of pre-θ-open sets.

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1. INTRODUCTION

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms, compactness, etc by utilizing generalized open sets. One of the most well-known notions and also an inspiration source is the notion of preopen set introduced by Mashhour et al. [7] in 1982. Pal and Bhattacharyya [9] used this notion and the preclosure [7] of a set to introduce the pre-θ-open sets by using the notion of the pre-θ-closure of a set. We also study some weak separation axioms defined by using the notion of pre-θ-open sets.

2. PRELIMINARIES

In this paper, spaces (X,τ) and (Y,σ) (or simply X and Y) always assumed to be topological spaces on which no separation axioms are assumed unless explicitly stated, For a subset A of X, the closure, interior and complement of A in X are denoted by $cl(A), int(A)$ and $X \setminus A$, respectively.

Definition 2.1 Let A be a subset of topological space (X,τ). Then the set A is

1. preopen [7], if $A \subseteq int(cl(A))$.
2. preclosed [7], if $X \setminus A$ is preopen or equivalently, if $cl(int(A)) \subseteq A$.

The intersection of all preclosed sets containing A is called the preclosure [4] of A and is denoted by $pc1(A)$. The preinterior [4] of A is the union of all preopen sets contained in A and is denoted by $pint(A)$. A subset A is called preregular [3] if it is both preopen and preclosed. The family of all preopen sets (resp. preregular sets) of (X,τ) is denoted by $PO(X,\tau)$ (resp. $PR(X,\tau)$.

A point x in X is called a θ-adherent [10] (resp. pre-θ-cluster [8]) point of a subset A of X if $c1(U) \bigcap A \neq \phi$ (resp. $pc1(U) \bigcap A \neq \phi$) for every open set (resp. preopen set) U containing x. The pre-θ-closure of A [9], denoted by $pcl_{\theta}(A)$, is defined to be the set of all $x \in X$ such that $pc1(G) \bigcap A \neq \phi$ for every $G \in PO(X,\tau)$ with $x \in G$. A subset A is called pre-θ-closed [9] if $A = pcl_{\theta}(A)$. The complement of a pre-θ-closed set is called pre-θ-open. The family of all pre-θ-open subset of X is denoted by $P\theta O(X,\tau)$.

Lemma 2.2 [2] Let A be a subset of a topological space (X,τ).

1. If $A \in PO(X,\tau)$, then $pcl(A)$ is preregular and $pcl(A) = pcl_{\theta}(A)$.
2. A is preregular if and only if A is pre-θ-closed and pre-θ-open.
3. A is preregular if and only if $A = pint(pcl(A))$.

Lemma 2.3 [2] A subset A of a space X is pre-θ-open if and only if for each $x \in A$, there exists a preopen set W with $x \in W$ such that $x \in W \subseteq pcl(W) \subseteq A$.

Lemma 2.4 Let A be a subset of a topological space (X,τ). Then:

1. $pcl_\theta(X \setminus A) = X \setminus (pint_\theta(A))$.
2. $X \setminus (pcl_\theta(A)) = pint_\theta(X \setminus A)$.

Lemma 2.5 For any subset A of a topological space (X,τ), $pcl_\theta(A)$ is pre-θ-closed for every $A \subset X$.

Lemma 2.6 For any subset A of a topological space (X,τ), $pcl(A) \subseteq pcl_\theta(A)$.

3. PRE-θ-OPEN SETS

Definition 3.1 A set A of a topological space (X,τ) is said to be θ-complement preopen (in short θ-c-preopen) provided there exists a subset G of X for which $X \setminus A = pcl_\theta(G)$. We call a set θ-complement preclosed (in short θ-c-preclosed) if its complement is θ-c-preopen.

Remark 3.2 It should be mentioned that by Lemma 2.5, $X \setminus A = pcl_\theta(G)$ is pre-θ-closed and A is pre-θ-open. Therefore, the equivalence of θ-c-preopen and pre-θ-open is obvious from the definition.

Lemma 3.3 Let A be a subset of a topological space (X,τ). Then A is pre-θ-closed if and only if there is a subset B of X such that $A = pcl_\theta(B)$.

Theorem 3.4 If A is preopen, then $pint(pcl(A))$ is pre-θ-open.

Proof. Since $pint(pc1(A))) = (X \setminus \{pcl(X \setminus pcl(A))\})$, $X \setminus pcl(A)$ ($= B$, say) is preopen, $pcl(B) = pcl_\theta(B)$ (Lemma 2.2). Therefore there exists a subset $B = X \setminus pcl(A)$ for which $X \setminus pint(pcl(A))) = pcl_\theta(B)$. Hence $pint(pcl(A))$ is pre-θ-open.

Corollary 3.5 If A is preregular, then A is pre-θ-open.

Proof. It suffices to observe that, A is preregular if and only if $A = pint(pcl(A))$ (Lemma 2.2).

Theorem 3.6 Preregular is equivalent to pre-θ-open if and only if $pcl_\theta(A)$ is preregular for every set A.

Proof. Let X be a topological space. Assume preregular is equivalent to pre-θ-open and let $A \subset X$. Then by Lemma 2.5, $X \setminus pcl_\theta(A)$ is pre-θ-open which implies that $pcl_\theta(A)$ is preregular. Assume $pcl_\theta(G)$ is preregular for every set G. Suppose U is pre-θ-open and let $A \subset X$ such that $X \setminus U = pcl_\theta(A)$ i,e $U = X \setminus pcl_\theta(A)$. Then, $pcl_\theta(A)$ is preregular and U is preregular. Therefor, preregular is equivalent to pre-θ-open.

Theorem 3.7 If A is pre-θ-open, then A is union of preregular sets.

Proof. Let A be pre-θ-open, $x \in A$. Since A is pre-θ-open, there exists, a set $G \subset X$ such that $A = X \setminus pcl_\theta(G)$. Because $x \notin pcl_\theta(G)$, there exists a preopen set W for which $x \in W$ and $pcl(W) \cap G = \phi$. Hence $x \in pint(pcl(W)) \subset X \setminus pcl_\theta(G)$, where

$pint(pcl((W)))(= Vsay) \in PR(X,\tau)$ i.e $A = \cup\{V : V \subset W, V \in PR(X,\tau)\}$.

Corollary 3.8 If A is pre-θ-closed, there A is the intersection of preregular sets.

4. PRE-θ-D_1 TOPOLOGICAL SPACES

now, we introduce new classes of topological spaces in terms of the concept of pre-θ-open sets.

Definition 4.1 A subset A of a topological spaces X is called pre-θ-D-set if there two $U,V \in p\theta O(X,\tau)$ such that $U \neq X$ and $V = \phi$. It is true that every pre-θ-open set U different from X is a pre-θ-D-set if $A = U$ and $V = \phi$.

Definition 4.2 A topological space (X,τ) is called pre-θ-D_0 if for any distinct pair of points x and y of X there exists a pre-θ-D-set of X containing x but not y or a pre-θ-D-set of X containing y but not x.

Definition 4.3 A topological space (X,τ) is called pre-θ-D_1 if for any distinct pair of points x and y of X there exists a pre-θ-D-set of X containing x but not y and a pre-θ- D-set of X containing y but not x.

Definition 4.4 A topological space (X,τ) is called pre-θ-D_2 if for any distinct pair of points x and y of X there exists a disjoint pre-θ-D-sets G and H of X containing x and y, respectively.

Definition 4.5 A topological space (X,τ) is called pre-θ-T_o if for any distinct pair of points x and y of X there exists a pre-θ-open set containing one of the points but not the other.

Definition 4.6 A topological space (X,τ) is called pre-θ-T_1 if for any distinct pair of points x and y of X there exists a pre-θ-open set U in X containing x but not y and a pre-θ-open set V in X containing y but not x.

Definition 4.7 A topological space (X,τ) is called pre-θ-T_2 if for any distinct pair of points x and y of X there exists a pre-θ-open set U and V in X containing x and y, respectively, such that $U \cap V = \phi$.

Remark 4.8

1. If (X,τ) is pre-θ-T_i then pre-θ-$T_{(i-1)}, i=1,2$.
2. Obviously, if (X,τ) is pre-θ-T_i, then (X,τ) is pre-θ-$D_i, i=0,1,2$.
3. If (X,τ) is pre-θ-D_i, then it is pre-θ-$T_{(i-1)}, i=1,2$.

Theorem 4.9 For a topological (X,τ) the following statement are true:

1. (X,τ) is pre-θ-D_0 if and only if it is pre-θ-T_0.
2. (X,τ) is pre-θ-D_1 if and only if it is pre-θ-D_2.

Proof. (1)**sufficiency:** The sufficiency is stated in Remark 4.8(2).

necessity: To prove necessity, let (X,τ) be pre-$\theta - D_0$. then for each distinct pair $x,y \in X$, at least one of x,y, say x, belong to a pre-$\theta - D$ set G but $y \notin G$. Let $G = U_1 \setminus U_2$ where $U_1 \neq X$ and $U_1, U_2 \in P\theta O(X,\tau)$. Then $x \in U_1$, and for $y \notin G$ we have two cases:

(a) $y \notin U_1$,

(b) $y \in U_1$ and $y \in U_2$.

In case (a), $x \in U_1$ but $y \notin U_1$;

In case (b), $y \in U_2$ but $x \notin U_2$. Hence X is pre-$\theta - T_0$.

(2)**sufficiency:** The sufficiency is stated in Remark 4.8(3).

necessity: Suppose X is pre-θ-D_1. Then for each distinct pair $x,y \in X$, we have pre-θ-D-sets G_1, G_2 such that $x \in G_1, y \notin G_1$ and $y \in G_2, x \notin G_2$. Let $G_1 = U_1 \setminus U_2$, $G_2 = U_3 \setminus U_4$. From $x \in G_2$, it follows that either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. We discuss the two cases separately.

(i) $x \notin U_3$. By $y \notin G_1$ we have two subcases.

(a) $y \notin U_1$. From $x \in (U_1 \setminus U_2)$, it follows that $x \in U_1 \setminus (U_2 \cup U_3)$ and by $y \in U_3 \setminus U_4$ we have $y \in U_3 \setminus (U_1 \cup U_4)$. Therefore, $(U_1 \setminus (U_2 \cup U_3)) \cap ((U_3 \setminus U_1 \cup U_4)) = \phi$.

(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 \setminus U_2, y \in U_2, (U_1 \setminus U_2) \cap U_2 = \phi$.

(ii) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 \setminus U_4, x \in U_4, (U_3 \setminus U_4) \cap U_4 = \phi$. Therefore, X is pre-$\theta - D_2$.

Corollary 4.10 If (X,τ) is pre-θ-D_1, then it is pre-θ-T_0.

Theorem 4.11 A topological space (X,τ) is pre-θ-T_0 if and only if for each pair of distinct points x,y of X, $pcl_\theta(\{x\}) \neq pcl_\theta(\{y\})$.

Proof. Sufficiency: Suppose that $x,y \in X$, $x \neq y$ and $pcl_\theta(\{x\}) \neq pcl_\theta(\{y\})$. Let z be a point of X such that $z \in pcl_\theta(\{x\})$ but $z \notin pcl_\theta(\{y\})$. We claim that $x \notin pcl_\theta(\{y\})$. if $x \in pcl_\theta(\{y\})$ then $pcl_\theta(\{x\}) \subset pcl_\theta(\{y\})$. This contradicts the fact that $z \notin pcl_\theta(\{y\})$. Consequently x belongs to the pre-θ-open set $X \setminus pcl_\theta(\{y\})$ to which y

does not belong.

Necessity: Let (X,τ) be a pre-θ-T_0 space and x,y be any two distinct points of X. There exists a pre-θ-open set G containing x or y, say x but not y. Then $X\setminus G$ is a pre-θ-closed set which does not contain x but contains y. Since $pcl_\theta(\{y\})$ is the smallest pre-θ-closed set containing y, $pcl_\theta(\{y\})\subset X\setminus G$ and therefore $x\notin pcl_\theta(\{y\})$. Consequently $pcl_\theta(\{x\})\neq pcl_\theta(\{y\})$.

Theorem 4.12 A topological space (X,τ) is pre-θ-T_1 if and only if the singletons are pre-θ-closed sets.

Proof. Let (X,τ) be pre-θ-T_1 and x any point of X. Suppose $y\in X\setminus\{x\}$. Then $x\neq y$ and so there exists a pre-θ-open set U_y such that $y\in U_y$ but $x\notin U_y$. Consequently $y\in U_y\subset X\setminus\{x\}$ i.e, $X\setminus\{x\}\cup\{U_y : y\in X\setminus\{x\}\}$ which is pre-θ-open. Conversely, suppose $\{p\}$ is pre-θ-closed for every $p\in X$. Let $x,y\in X$ with $x\neq y$. Now $x\neq y$ implies $y\in X\setminus\{x\}$.Hence $X\setminus\{x\}$ is a pre-θ-open set containing y but not x. Similarly $X\setminus\{y\}$ is a pre-θ-open set containing x but not y is a pre-θ-T_1 space.

Definition 4.13 A subset A of X is called a pre-θ-neighborhood of a point $x\in X$ if there exists a pre-θ-open set W of X such that $x\in W\subset A$.

Definition 4.14 A point $x\in X$ which has only X as the pre-θ-neighborhood is called a point common to all pre-θ-closed sets (briefly pre-θ-cc)

Theorem 4.15 If a topological space (X,τ) is pre-θ-D_1, then (X,τ) has no pre-θ-cc point .

Proof. Since (X,τ) is pre-θ-D_1, so each point x of X is contained in a pre-θ-D set $W=U\setminus V$ and thus in U. By definition $U\neq X$. This implies that x is not a pre-θ-cc point.

Definition 4.16 A subset A of topological space (X,τ) is called a quasi pre-θ-closed set (briefly qpt-closed) if $pcl_\theta(A)\subset U$ whenever $A\subset U$ and U is pre-θ-open in (X,τ).

Theorem 4.17 For a topological space (X,τ) , the following properties hold:

1. For each points x and y in a topological space (X,τ), $x\in pcl_\theta(\{y\})$ implies $y\in pcl_\theta(\{x\})$.
2. For each $x\in X$, the singleton $\{x\}$ is qpt-closed in (X,τ).

Proof. (1) Let $y\notin pcl_\theta(x)$. This implies that there exists $V\in PO(Y,y)$ such that $pcl(V)\cap\{x\}=\phi$ and $X\setminus pcl(V)\in PR(X,x)$ which means that $x\notin pcl_\theta(\{y\})$.

(2) Suppose that $\{x\}\subset U\in P\theta O(X)$. This implies that there exists $V\in PO(X,x)$ such that $x\in V\subset pcl(V)\subset U$. Now, we have $pcl_\theta(\{x\})\subset pcl_\theta(V)=pcl_\theta(V)\subset U$.

Definition 4.18 A topological space (X,τ) is said to be pre-θ-$T_{\frac{1}{2}}$ if every qpt-closed set is pre-θ-closed.

Theorem 4.19 For a topological space (X,τ), the following are equivalent:

1. (X,τ) is pre-θ- $T_{\frac{1}{2}}$
2. (X,τ) is pre-θ-T_1.

Proof. $(1)\Rightarrow(2)$: For distinct points x,y of X, $\{x\}$ is qpt-closed by Theorem 4.17. By hypothesis, $X\setminus\{x\}$ is pre-θ-open and $y\in X\setminus\{x\}$. By the same token, $x\in X\setminus\{y\}$ and $X\setminus\{y\}$ is pre-θ-open. Therefore (X,τ) is pre-θ-T_1.

$(2)\Rightarrow(1)$: Suppose that A is qpt-closed set which is not pre-θ-closed. There exists $x\in pcl_\theta(A)\setminus A$. For each $a\in A$, there exists a pre-θ-open set V_a such that $a\in V_a$ and $x\notin V_a$. Since $A\subset\bigcup\{V_a : a\in V_a\}$ and $\bigcup\{V_a : a\in V_a\}$ is pre-θ-open, we have $pcl_\theta(A)\subset\bigcup\{V_a : a\in V_a\}$. Since $x\in pcl_\theta(A)$, there exists $a_0\in A$ such that $x\in V_{a_0}$. But this is a contradiction. Recall that a topological space (X,τ) is called pre-T_2 [5] if for any distinct pair of points x and y in X, there exist preopen subsets U and V of X containing x and y, respectively, such that $U\bigcap V=\phi$.

Theorem 4.20 For a topological space (X,τ), the following are equivalent:

1. (X,τ) is pre-θ-T_2,
2. (X,τ) is pre-T_2.

Proof. $(1)\to(2)$: This is obvious since every pre-θ-open set is preopen [11]

$(2)\Rightarrow(1)$: Let x and y be distinct points of X. There exist preopen sets U and V such that $x\in U, y\in V$, and $pcl(U)\cap pcl(V)=\varnothing$,[11, Theorem 4.13]. Since $pcl(U)$ and $pcl(V)$ and preregular they are pre-θ -open and hence (X,τ) is pre-θ-T_2.

Definition 4.21 A function $f:(X,\tau)\to(Y,\sigma)$ is said to be qusi-preirrasolute if for each $x\in X$ and each $V\in PO(Y, f(x))$, there is $U\in PO(X,x)$ such that $f(U)\subset pcl(V)$.

Remark 4.22 A function $f:(X,\tau)\to(Y,\sigma)$ is qusi-preirrasolute if and only if $f^{-1}(V)$ is pre-θ-closed (resp.pre-θ -open) in (X,τ) for every pre-θ-closed (resp.pre-θ-open) set V in (Y,σ) .

Theorem 4.23 If $f:(X,\tau)\to(Y,\sigma)$ is qusi-preirrasolute surjective and E is a pre θ-D-set in Y , then the inverse image of E is a pre θ-D-set in X.

Proof. Let E be a pre-θ-D set in Y. Then there are pre-θ-open sets U and V in Y such that $E=U\setminus V$ and $U\neq Y$. By qusi-preirrasoluteness of f , $f^{-1}(U)$ and $f^{-1}(V)$ are pre- θ -open in X. Scince $U\neq Y$, we have $f^{-1}(U)\neq X$. Hence $f^{-1}(E)=f^{-1}(U)\setminus f^{-1}(V)$ is a pre-θ-D-set in X.

Theorem 4.24 If (Y,σ) is pre-θ-D_1 and $f:(X,\tau)\to(Y,\sigma)$ is a qusi-preirrasolute injection, then (X,τ) is a pre-θ-D_1.

Proof. Suppose that Y is a pre-θ-D_1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is pre-θ-D_1, there exist pre-θ-D-sets U and V of Y containing $f(x)$ and $f(y)$, respectively, such that $f(y)\in U$ and $f(x)\in V$. By the above theorem, $f^{-1}(U)$ and $f^{-1}(V)$ are pre-θ-D-sets in X containing x and y, respectively . This implies that X is a pre-θ-D_1 space.

Theorem 4.25 For a topological space (X,τ) the following statement are equivalent:

1. (X,τ) is pre-θ-D_1,
2. For each pair of distinct points, x,y in X, there exists a quasi-preirresolute subjective function $f:(X,\tau)\to(Y,\sigma)$, where Y is pre-θ-D_1 space such that $f(x)$ and $f(y)$ are distinct.

Proof. $(1)\Rightarrow(2)$: For every pair of distinct points of X, it suffices to take the identity function on X

$(2)\Rightarrow(1)$: Let x and y be any pair of distinct points in X. By hypothesis, there exists a surjective quasi-preirresolute function f of space X into pre-θ-D_1 space Y such that $f(x)\neq f(y)$. therefor, there exist disjoint pre-θ-D-sets U and V of Y containing $f(x)$ and $f(y)$, respectively. Since f is quasi-preirresolute and surjective, by Theorem 4.23, $f^{-1}(U)$ and $f^{-1}(V)$ are pre-θ-D set in X containing x and y, respectively. Hence X is pre-θ-D_1 space.

5. ADDITIONAL PROPERTIES

Let A be a subset of a topological space (X,τ). The pre-θ -kernal of A, denoted by $PKer_\theta(A)$, is defined to be the set $\bigcap\{G\in P\theta O(X,\tau): A\subset G\}$, or equivelently to be the set $\{x\in X: pcl_\theta(x)\cap A\}\neq\phi$

Definition 5.1 A topological space (X,τ) is said to be sober pre-$\theta-R_0$ if $\bigcap\{pcl_\theta(\{x\}): x\in X\}=\phi$

Theorem 5.2 A topological space (X,τ) is sober pre-$\theta-R_0$ if and only if $PKer_\theta(\{x\})\neq X$ for any $x\in X$.

Proof. Necessity Let the space (X,τ) be sober pre-$\theta-R_0$. Assume that there is a point y in X such that

$PKer_{\theta}(\{y\}) = X$. then $y \notin G$ which G is some proper pre- θ -open subset of X . this implies that $y \in \bigcap\{pcl_{\theta}(\{x\}) : x \in X\}$. But this is a contradiction.

Sufficiency: Now assume that $PKer_{\theta}(\{x\}) \neq X$ for any $x \in X$. If there exists a point y in X such that $y \in \bigcap\{pcl_{\theta}(\{x\}) : x \in X\}$, then every pre-$\theta$-open set containing y must contain every point of X. This implies that the space X is the unique pre-θ-open set containing y, Hence $PKer_{\theta}(\{y\}) = X$ which is a contradiction. Therefore (X,τ) is sober pre-θ-R_0.

Theorem 5.3 If the topological space X is sober pre-θ-R_0 and Y is any topological space, then the product $X \times Y$ is sober pre-θ-R_0.

Proof. By showing that $\bigcap\{pcl_{\theta}(\{x,y\}) : (x,y) \in X \times Y\} = \phi$ we are done . We have:

$$\bigcap\{pcl(\{x,y\}) : (x,y) \in X \times Y\} \subseteq \bigcap\{pcl_{\theta}(\{x\}) \times pcl_{\theta}(\{y\}) : (x,y) \in X \times Y\}$$
$$= \bigcap\{pcl_{\theta}(\{x\}) : x \in X\} \times \bigcap\{pcl_{\theta}(\{y\}) : y \in Y\} \subseteq \phi \times Y = \phi$$

Definition 5.4 A function $f : (X,\tau) \to (Y,\sigma)$ is called:

1. R-continuous [6] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open subset U of X containing x such that $cl(f(U)) \subset V$.
2. θ-R-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open subset U of X containing x such that $pcl_{\theta}(f(U)) \subset V$.
3. R-precontincuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open subset U of X containing X such that $pcl(f(U)) \subset V$.
4. Preopen [7] if $f(U)$ is preopen in Y for every open set U of X.

Remark 5.5 For a subset A of a topological space (X,τ)

1. $A \subset pcl(A) \subset cl_{\theta}(A)$ since for any set A, θ-R-preconuity implies R-preconuity.
2. Since the preclosure and pre-θ-closure operators agree on preopen on preopen sets Lemma 2.2, if follows that if $f : (X,\tau) \to (Y,\sigma)$ is R-precontinuous and preopen, then f is θ-R-precontinuous.

Definition 5.6 The graph $G(f)$ of a function $f : (X,\tau) \to (Y,\sigma)$ is said to be pθ-c-preclosed if for each point $(x,y) \in (X \times Y) \setminus G(f)$, there exists subsets $U \in PO(X,x)$ and $V \in P\theta O(Y,y)$ such that $(pcl(U) \cap V) \cap G(f) = \phi$.

Lemma 5.7 The graph $G(f)$ of a function $f : (X,\tau) \to (y,\sigma)$ is pθ-c-preclosed in $X \times Y$ if and only if for each point $(x,y) \in (X \times Y) \setminus G(f)$. there exist $U \in PO(X,x)$ and $V \in P\theta O(Y,y)$ such that $f(pcl(U)) \cap V = \phi$.

Proof. It follow immediately from Definition 5.6.

In [6, Theorem 4.1], it is shown that the graph of a R-continuous function into a T_1 - space is θ-closed with respect to the domain. Here an analogous result is proved for θ-R-precontinuous functions.

A space (X,τ) is pre-T_1 ([5]), if to each pair of distinct points x and y of X, there exists a pair of preopen sets one containing x but not y and the other containing y but not x.

Theorem 5.8 If $f : (X,\tau) \to (Y,\sigma)$ is θ-R-precontinuous quasi-preirresolute and Y is pre-T_1, then $G(f)$ is Pθ-c-preclosed.

Proof. Assume that $(x,y) \in (X \times Y) \setminus G(f)$ and Y is preopen, there exists an preopen subset V of Y such that $f(x) \in V$ and $y \notin V$. The θ-R-precontinuity of f implies the existance of an open subset U of X containing x such that $pcl_{\theta}(f(U) \subset V$. Therefore, $(X,y) \in pcl(U) \times (Y \setminus pcl_{\theta}(f(U))$ which is disjoint from $G(f)$ because if $x \in pcl(U)$, then since f is quasi-preirresolute, $f(x) \in f(pcl(U)) \subset pcl_{\theta}(f(U))$. Note that $Y \setminus pcl_{\theta}(f(U))$ is pre-θ-open. It is proved in [6, Theorem 3.1] that a function $f : (X,\tau) \to (Y,\sigma)$ is R-continuous if and only if for each $x \in X$ and each closed subset F of Y with $f(x) \notin F$, there exist open subsets $U \subset X$ and $V \subset Y$ such that $x \in U$, $F \subset V$ and $f(U) \cap V = \phi$. The following theorem is an analogous result for θ-R-precontinuous functions.

Theorem 5.9 Let $f:(X,\tau) \to (Y,\sigma)$ be a quasi-preirresolute function. Then f is θ-R-precontinuous if and only if for each $x \in X$ and each preclosed subset F of Y with $f(x) \notin F$, there exists an open subset U of X containing x and a pre-θ-open subset V of Y with $F \subset V$ such that $f(pcl(U)) \bigcap V = \phi$.

Proof. Necessity: Let $x \in X$ and F be a preclosed subset of Y with $f(x) \in Y \setminus F$. Since f is θ-R-precontinuous there exists an open subset U of X containing x such that $f(pcl_\theta(f(U))) \subset Y \setminus F$. Let $V = Y \setminus (f(U))$, then V is pre-θ-open and $F \subset V$. Since f is quasi-preirresolute, $f(pcl(U)) \subset pcl_\theta(f(U))$. Therefore, $f(pcl(U)) \bigcap V = \phi$.

Sufficiency: Let $x \in X$, V be a preopen subset of Y with $f(x) \in V$ and let $F = Y \setminus V$. Since $f(x) \notin F$, there exists an open subset U of X containing X and a pre-θ-open subset W of Y with $F \subset W$ such that $f(pcl(U)) \cap W = \varnothing$. Then $f(pcl(U) \subset Y \setminus W$ and

$pcl_\theta(f(U) \subset pcl_\theta(f(Y \setminus W)) = Y \setminus W \subset Y \setminus F = V$. Therefore, f is θ-R-precontinuous.

Corollary 5.10 Let X and Y are a topological spaces and $f:(X,\tau) \to (Y,\sigma)$ be a quasi-preirresolute function. Then f is θ-R-precontinuous if and only if for each $x \in X$ and each preopen subset V of Y containing $f(x)$, there exists an open subset U of X containing X such that $pcl_\theta(f(pcl(U))) \subset V$.

Proof. Assume f is θ-R-precontinuous. Let $x \in X$ and let V an preopen subset of Y with $f(x) \in V$. Then there exists an open subset U of X containing X such that $pcl_\theta(f(U)) \subset V$. Since f is quasi-preirresolute, we have $pcl_\theta(f(pcl(U))) \subset cl_\theta(pcl_\theta(f(U))) = pcl_\theta(f(U)) \subset V$ Thus $pcl_\theta(f(pcl(U))) \subset V$. The converse implication is immediate.Recall that a topological space (X,τ) is said to be pre-R_1 ([7]) if for $x,y \in X$ with $pcl(\{x\}) \neq pcl(\{y\})$, there exist disjoint preopen sets U and V such that $pcl(\{x\}) \subset U$ and $pcl(\{y\}) \subset V$.

Proposition 5.11 A space X is pre-R_1 if and only if for each preopen set A and each $x \in A$, $pcl_\theta(\{x\}) \subset A$.

Proof. Necessity: Assume X is pre-R_1. Suppose that A is a preopen subset of X and let $x \in A$, y be arbitrary element of $X \setminus A$. Since X is pre-R_1. $pcl_\theta(\{y\}) = pcl(\{y\}) \subset X \setminus A$. Hence, we have $x \notin pcl_\theta(\{y\})$ and $y \notin pcl_\theta(\{x\})$. It follows that $pcl_\theta(\{x\}) \subset A$.

Sufficiency: Assume now that, $y \in pcl_\theta(\{x\}) \setminus pcl(\{x\})$ for some $x \in X$. Then there exists a preopen set A containing y such that $pcl(A) \cap \{x\} \neq \phi$. but $A \cap \{x\} = \phi$. Then $pcl_\theta(\{y\}) \subset A$ and $pcl_\theta(\{y\}) \cap \{x\} = \phi$. Hence $x \notin pcl_\theta(\{y\})$. Thus $y \notin pcl_\theta(\{x\})$. By this contradiction, we obtain $pcl_\theta(\{x\}) = pcl(\{x\})$ for each $x \in X$. Thus X is pre-R_1.

Now, we show that the range of a θ-R-precontinuous function satisfies the stronger pre-R_1 condition.

Theorem 5.12 If $f:(X,\tau) \to (Y,\sigma)$ is a θ-R-precontinuous surjection, then (Y,σ) is a pre-R_1 space.

Proof. Let V be a preopen subset of Y and $y \in V$, $x \in X$ such that $y = f(x)$. Since f is θ-R-precontinuous, there exists an open subset U of X containing x such that $pcl_\theta(f(U)) \subset V$. Then $pcl_\theta(\{y\}) \subset pcl_\theta(f(U)) \subset V$. Therefore by Proposition 5.11, Y is pre-R_1.

We close this paper with a sample of the basic properties of θ-R-precontinuous function concerning composition and restriction.

Theorem 5.13 If $f:(X,\tau) \to (Y,\sigma)$ is continuous and $g:(y,\sigma) \to (Z,\eta)$ is θ-R-precontinuous, then $f \circ g:(X,\tau) \to (Z,\eta)$ is θ-R-precontinuous

Proof. Let $x \in X$ and W be a preopen subset of Z containing $g(f(x))$. Since g is θ-R-precontinuous, there exists an open subset V of Y containing f (x) such that $pcl_\theta(g(V)) \subset W$. Since f is continuous, there exists an open subset U of X containing x with $f(U) \subset V$, hence $pcl_\theta(g(f(U))) \subset W$. Therefore, $g \circ f$ is θ-R-precontinuous.

Theorem 5.14 Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(y,\sigma)\to(Z,\eta)$ be functions. if $g\circ f:(X,\tau)\to(Z,\eta)$ is θ-R-precontinuous and f is an open surjection, then g is θ-R-precontinuous.

Proof. Let $y\in Y$ and W be a preopen subset of Z containing $g(y)$. Since f is surjective, there exists $x\in X$ such that $y=f(x)$. Since $g\circ f$ is θ-R-precontinuous, there exists an open subset U of X containing x such that $pcl_\theta(g(f(U)))\subset W$. Note that $f(U)$ is an open set containing y. Therefore, g is θ-R-precontinuous.

Theorem 5.15 If $f:(X,\tau)\to(Y,\sigma)$ is θ-R-precontinuous, $A\subset X$ and $f(A)\subset B\in PO(Y,\sigma)$, then $f\setminus A:A\to B$ is θ-R-precontinuous.

Proof. Let $x\in A$ and V be a preopen subset of B containing $f(x)$ (not that $f(A)\subset B$). Hence V is be a preopen subset of Y containing $f(x)$. Science f is θ-R-precontinuous, there exists an open subset U of X containing x such that $pcl_\theta(f(U))\subset V$. Let $G=U\bigcap A$, then an open subset of A containing x such that $pcl_\theta(G)\subset pcl_\theta(U)\subset A$. Therefore, $f\setminus A:A\to B$ is θ-R-precontinuous.

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