THE MAYER – VIETORIS SEQUENCE CALCULATING DE RHAM COHOMOLOGY

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ABSTRACT

De Rham cohomology it is very obvious that it relies heavily on both topology as well as analysis. We can say it creates a natural bridge between the two. To understand and be able to explain what exactly de Rham cohomology is to the world of mathematics we need to know de Rham groups. This is the reasons to calculate the de Rham cohomology of a manifold. This is usually quite difficult to do directly. We work with manifold. Manifold is a generalization of curves and surfaces to arbitrary dimension. A topological space M is called a manifold of dimension k if :

· M is a topological Hausdorff space .

· M has a countable topological base.

· For all $m \in M$ there is an open neighborhood $U \subset M$ such that U is homeomorphic

 to an open subset V of $\mathbb{R}^k$.

There are many different kinds of manifolds like topological manifolds, $\mathbb{C}^k$ - manifolds, analytic manifolds, and complex manifolds, we concerned in smooth manifolds. A smooth manifold can described as a topological space that is locally like the Euclidian space of a dimension known. An important definion is homeomorphism. Let X, Y be topological spaces, and let f: $X \rightarrow Y$ e a bijection. If both f and the inverse function $f^{-1}: X \rightarrow Y$ are continuous, then f is called a homeomorphism We introduce one of the useful tools for this calculating, the Mayer – Vietoris sequence. Another tool is the homotopy axiom. In this material I try to explain the Mayer – Vietoris sequence and give same examples. A short exact sequence of cochain complexes gives rise to a long exact sequence in cohomology, called the Mayer - Vietories sequence. Cohomology of the circle (S^1), cohomology of the spheres (S^2). Homeomorphism between vector spaces and an open cover of a manifold. we define de Rham cohomology and compute a few examples.

Indexing terms/Keywords : Mayer – Vietoris sequence; De Rham cohomology; manifold; differential forms; diffeomorfic ; exact ; close forms; cohomology class.

1 . INTRODUCTION

Understanding the de Rham cohomology we may understand the types of spaces to work with. We work with manifold. Manifold is a generalization of curves and surfaces to arbitrary dimension. There are many different kinds of manifolds like topological manifolds, $\mathbb{C}^k$ - manifolds, analytic manifolds, and complex manifolds, we concerned in smooth manifolds. A smooth manifold can described as a topological space that is locally like the Euclidian space of a dimension known. The 'smooth' is relate to the differentiability of the maps that connect our space to the matching Euclidian space.

1.1 Some preliminary concept

Definition 1.1.[1] A topological space M is called a manifold of dimension k if:

· M is a topological Hausdorff space .

· M has a countable topological base.

· For all $m \in M$ there is an open neighborhood $U \subset M$ such that U is homeomorphic

 to an open subset V of $\mathbb{R}^k$.

Definition 1. 2. Let X, Y be topological spaces, and let f : $X \rightarrow Y$ be a bijection. If both f and the inverse function f^{-1}: $Y \rightarrow X$ are continuous, then f is called a homeomorphism . The homeomorphism f maps each open set in X to a unique open set in Y. Whenever there exists a homeomorphism between two topological spaces, they are said to be homeomorphic . They have the same topological property (topologically the same).Under a homeomorphism, topological properties are preserved.

Example 1. (i) (0; 1) and (a, b) are homeomorphic, for example via the homeomorphism

$f(x) = a + (b - a)x$

(ii) The interval [0; 1) and the circle $\mathbb{S}^1 = \{z \epsilon \mathbb{C} : |z| = 1\}$ are not homeomorphic. There is a bijection h : $[0; 1) \rightarrow \mathbb{S}^1$ given by $h(x) = e^{2\pi i x}$, and h is also continuous, but h^{-1} is not continuous. That there is no homeomorphism between [0; 1) and $\mathbb{S}^1$.

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J o u r n a l o f A d v a n c e s i n M a t h e m a t i c s

2. DEFINITION OF A SMOOTH MANIFOLD

Definition 2.1 A smooth manifold is a pair $(M, \mathbb{A}_{max})$ where M is a manifold and $\mathbb{A}_{max}$ maximal atlas.

Some time that called differentiable structure of M.

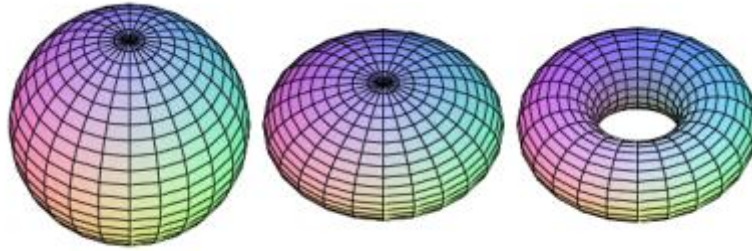**Fig. 1** . Some manifolds

Some examples of smooth manifold

(1) $\mathbb{R}^n$ is a smooth manifold .Atlas : $\phi : U = \mathbb{R}^n \to \mathbb{R}^n$ given by identity map.

(2) Any open subset U of a smooth manifold M is a smooth manifold. Given an atlas
   $\mathbb{A} = \{\phi_\alpha : U_\alpha \to \mathbb{R}^n\}$ for M , an atlas for U is $\{\phi_\alpha : U_\alpha \cap U \to \mathbb{R}^n\}$

(3) $S^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$

(4) (real projective space) $\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{(0, \ldots, 0)\})/\sim$ where

$(x_0, x_{1, \ldots, } x_n) \sim (tx_0, tx_{1, \ldots, } tx_n)$, $t \in \mathbb{R} - \{0\}$

$\mathbb{RP}^n$ is called the real projective space of dimension n . The equivalent class of $(x_0, x_{1, \ldots, } x_n)$ Is denoted by $[x_0, x_{1, \ldots, } x_n]$.

Let $U_0 = \{x_0 \neq 0\}$ with the coordinate chart $\phi_0 : U_0 \to \mathbb{R}^n$ is given by

$[x_0, x_1, \cdots, x_n] = \left[1, \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right] \mapsto (1, \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0})$

In the same way , take $U_i = \{x_i \neq 0\}$ and difine $\phi_i : U_i \to \mathbb{R}^n$

3. THE DE RHAM COMPLEX ON $\mathbb{R}^n$

In this section we difine de Rham cohomology and compute a few examples. This will turn out to be the most important diffeomorphism invariant of a manifold . We define $\mathbf{\Omega}^*$ to be the algebra over $\mathbb{R}$ generated by $dx_1, \ldots, dx_n$ with relation [3]

$$\begin{cases} (dx_i)^2 = 0 \\ dx_i dx_j = -dx_j dx_i, i \neq j \end{cases}$$

As a vector space over $\mathbb{R}$, $\mathbf{\Omega}^*$ has basis

$1, dx_i, dx_i dx_j, dx_i dx_j dx_k, \ldots, dx_1 \ldots dx_n$

    i < j   i < j < k

The $\mathbb{C}^\infty$ differential forms on $\mathbb{R}^n$ are elements of $\mathbf{\Omega}^*(\mathbb{R}^n) = \{\mathbb{C}^\infty \text{ function on } \mathbb{R}^n\} \otimes \mathbf{\Omega}^*$

Defined as follows:

   i) $f \in \mathbf{\Omega}^0(\mathbb{R}^n)$, then $df = \Sigma \frac{\partial f}{\partial x_i} dx_i$

   ii) $\omega = \Sigma df_I dx_I$, then $d\omega = \Sigma df_I dx_I$

EXAMPLE 1.1 If $= x\,dy$, then $d\omega = dxdy$

This d, called the exterior differentiation, is the ultimate abstract extension of the usual gradient, curl, and divergence of the vector calculus on $\mathbb{R}^3$, as below illustrates.

EXAMPLE 1.2 On $\mathbb{R}^3$, $\Omega^0(\mathbb{R}^3)$, and $\Omega^3(\mathbb{R}^3)$ are each 1-dimensional and $\Omega^1(\mathbb{R}^3)$ and $\Omega^2(\mathbb{R}^3)$ are each 3- dimensional over $\mathbb{C}^\infty$ function , so the following identification are possible:[2]

$$\{function\} \cong \{0 - forms\} \cong \{3 - forms\}$$
$$f \leftrightarrow f_1 dx + f_2 dy + f_3 dz \leftrightarrow f\,dx\,dy\,dz$$

And

$$\{vector\ fields\} \cong \{1 - forms\} \cong \{2 - forms\}$$

$$\text{X} = (f_1, f_2, f_3) \leftrightarrow dx + f_2 dy + f_3 dz \leftrightarrow f_1 dydz - f_2 dxdz + f_3 dxdy$$

On functions,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

On 1-forms,

$$d(f_1 dx + f_2 dy + f_3 dz) =$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) dydz - \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) dxdz + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dxdz$$

On 2-forms,

$$d(f_1 dydz - f_2 dxdz + f_3 dxdy) = \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dxdydz$$

So we have

d(0-forms) = gradient,

d(1-forms) = curl,

d(2-forms) = divergence.

The wedge product of two differential forms, written $\tau \wedge \omega$ or $\tau \cdot \omega$ is defined as follows: if $\tau = \sum f_I dx_I$ and $\omega = \sum g_J dx_J$ then

$$\tau \wedge \omega = \sum f_I g_J\, dx_I\, dx_J$$

Note that $\tau \wedge \omega = (-1)^{deg\tau\ deg\omega}\ \tau \wedge \omega$.

Proposition 3.1 .d is an antiderivation,

$$d\ (\tau \cdot \omega) = (d\tau) \cdot \omega + (-1)^{deg\tau}\ \tau \cdot d\omega$$

Proof . By linearit it suffices to check on monomials

$\tau = \sum f_I dx_I$, $\omega = \sum g_J dx_J$

$d\ (\tau \cdot \omega) = d(f_I g_J) dx_I dx_J = d(f_I) g_J\, dx_I dx_J + f_I dg_J\, dx_I dx_J$

$= (d\tau) \cdot \omega + (-1)^{deg\tau}\ \tau \cdot d\omega$

On the level of function

$d(fg) = (df)g + f(dg)$ is simply the ordinary product rule.

Proposition 3.2. $d^2 = 0$.

Proof . [4]This is basically a consequence of the fact that the mixed partials are equal. On function,

$d^2 f = d\left(\sum_i \frac{\partial f}{\partial x_i} dx_i\right) = \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j dx_i.$

Here the factor $\frac{\partial^2 f}{\partial x_j \partial x_i}$ are symmetric in i, j while $dx_i dx_j$ are skew- symmetric in i , j ;hence $d^2 f = 0$. On forms $\omega = f_I dx_I$,

$d^2 \omega = d^2(f_I dx_I,) = d(df_I dx_I) = 0$

By the previous computation and the antiderivation property of d.

Now we give same examples exterior differentiation.

EXAMPLE 3.1 Consider function $f = xy$ and differential forms $\sigma = xdy + ydx$ and

$\omega = zdx + xdy$.Calculate differential forms $d(f\omega)$, and $d(\sigma \wedge \omega)$

Solution. $d(f\omega) = df\omega + fd\omega = d(xy)(zdx + xdy)$ +$xyd(zdx + xdy)$=

$\qquad = (ydx + xdy)\wedge(zdx + xdy) + xy(dz\wedge dx + dx\wedge y) = xzdy\wedge dx + yxdx\wedge dy + xydz\wedge dx + xydx\wedge dy$

=$(2xy - xz)dx\wedge dy + xydz\wedge dx.$

$\sigma \wedge \omega = (xdy + ydx)\wedge(zdx + xdy) = xzdy\wedge dx + yxdx\wedge dy = (xy - xz)dx\wedge dy$

$d(\sigma \wedge \omega) = d\big((xy - xz)dx\wedge dy\big) = -xdz\wedge dx\wedge dy$

Note that $\sigma = df$, hence $d(\sigma \wedge \omega) = d(df \wedge \omega) = -df \wedge d\omega$

The comlex $\mathbf{\Omega}^*(\mathbb{R}^n)$ together with the differential operator d is called de Rham complex on $\mathbb{R}^n$

Definition 3. 1 [5] **Let** ω be a $k - form$ in an open subset $\subset \mathbb{R}^n$. If there is a (k-1) –form λ in

U such that $\omega = d\lambda$, then ω is said to be exact in U.

If ω is of class a differentiable functions and $d\omega = 0$, then is said to be closed.

However this Examles tell us that the closed forms wich are not exact.

EXAMPLE 3.2. Consider the 1-forms: $\eta = \frac{xdy - ydx}{x^2+y^2}$. Take $U = \mathbb{R}^2 - \{0\}$

Proved that form is closed to $U = \mathbb{R}^2 - \{0\}$ but it is not exat in

$$U = \mathbb{R}^2 - \{0\}.$$

Solution. Fix $r > 0$, take $\gamma(t) = (r \cos t, r \sin t) \qquad (0 \leq t \leq 2\pi)$

That γ is a curve in $U = \mathbb{R}^2 - \{0\}$. Take the value at point $0 \; and \; 2\pi$ so $\gamma(0) = (r,0) = \gamma(2\pi)$

we get

$$\partial\gamma = 0$$

Computing over $O \;\; and \;\; \int \eta = 2\pi \neq 0 \qquad\qquad (1)$

We can see that η is not exact form in $O = \mathbb{R}^2 - \{0\}$ otherwise since $\partial\gamma = 0$ that imply the integral (1) it going to be 0.

Definition. 3. 2 [3] The $k - th$ de Rham cohomology of $\mathbb{R}^n$ is the vector space .

$$H_{DR}^k(\mathbb{R}^n) = \frac{\{closed \; k - forms\}}{\{exact \; k - forms\}}$$

We sometime write that $H^k(\mathbb{R}^n)$

To distinguish between a form ω and its cohomology class,we denote the cohology class by $[\omega]$

All definitions so far work equally well for any open subset $U \; of \; \mathbb{R}^n$

EXAMPLE 3.3

　　(a) n = 0

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

(b) n = 1

Since $(ker\, d) \cap \Omega^0(\mathbb{R}^1)$ are the constant function

$$H^0(\mathbb{R}^1) = \mathbb{R}$$

On $\Omega^0(\mathbb{R}^1)$, $ker\, d$ are all 1- forms.

If $\omega = g(x)dx$ is a 1- form,then by taking

$$f = \int_0^x g(u)\,du$$

we find that

$$df = g(x)dx$$

Therefore every 1-form on $\mathbb{R}^1$ is exact so

$$H^1(\mathbb{R}^1) = 0$$

4. THE MAYER – VIETORIS SEQUENCE

Let M be a manifold and U and V both open subsets of M . A short exact sequences of a chain complexes give a rise a long exact sequence to the group cohomology correspond.

We take inclusions $i_1 : U \longrightarrow M$ and $i_2 : V \longrightarrow M$ gjithashtu $j_1 : U \cap V \longrightarrow U$ and $j_2 : U \cap V \longrightarrow V$

$h = (i_1{}^*, i_2{}^*)$ is a injective

$g = j_1{}^* - j_2{}^*$ is a syrjective

Now the sequences is short exact Mayer –Vietoris sequence

$$0 \longrightarrow \Omega^*(U \cup V) \overset{k}{\to} \Omega^*(U) \oplus \Omega^*(V) \overset{h}{\to} \Omega^*(U \cap V) \longrightarrow 0$$

From Homological Algebra we know that we get a long exact sequences of de Rham cohomology groups

$$\longrightarrow H^n(U \cup V) \overset{k}{\to} H^n(U) \oplus H^n(V) \overset{h}{\to} H(U \cap V) \overset{l}{\to} H^{n+1}(U \cup V) \longrightarrow$$

Where l is the homomorphism, defined as . For $\omega \in \Omega^n(U \cup V)$ that have a property $d\omega = 0$

Exist $\omega_1 \in \Omega^n(U)$ and $\omega_2 \in \Omega^n(V)$ that $\omega = j_1{}^*\omega_1 - j_2{}^*\omega_2$ from that we get

$j_1{}^*d\omega_1 = j_2{}^*d\omega_2 \Longrightarrow$ there is $\eta \in H^{n+1}(U \cup V)$ such that $i_1{}^*\eta = d\omega_1$ and $i_2{}^*\eta = d\omega_2$ .So we get $l[\omega] = [\eta]$

We take a smooth manifold $M \times \mathbb{R}$, and let see a function $p_1 : M \times \mathbb{R} \longrightarrow M$ and

$p_0 : M \longrightarrow M \times \mathbb{R}$ such that $m \mapsto (m, 0)$

Lemma $[6]$. Given M And a maps defined above we have that

$p_1{}^* : H^*(M \times \mathbb{R}) \longrightarrow H^*(M)$ and $p_0{}^* : H^*(M) \longrightarrow H^*(M \times \mathbb{R})$ are mutual invers , and in particular , $H^*(M \times \mathbb{R}) \cong H^*(M)$

Corrollary .

$$H^K(point) = H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

Proof .We see that the statement is truue for $H^K(point)$. Than by inductions we get the corally

$$H^K(point) \cong H^k(point \times \mathbb{R}) = H^K(\mathbb{R})$$
$$\cong H^K(\mathbb{R} \times \mathbb{R}) = H^K(\mathbb{R}^2)$$
$$\cong$$
$$\cong H^K(\mathbb{R}^{n-1} \times \mathbb{R}) = H^K(\mathbb{R}^n)$$

Conclusion

Smooth manifolds can be characterized by an atlas, however these atlases can be very complex and hard to understand, so ideally we would like a simple characteristic of manifolds that we can easily work with. For the same cohomological

space we can use a property to one for another. The linear approximation can be viewed as a generalizations of the linear approach to the graph of a function (tangent line of the graph of a function can be viewed as a linear approximation).

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