Exponentiated Rayleigh Distribution: A Bayes Study Using MCMC Approach Based on Unified Hybrid Censored Data

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ABSTRACT

This paper aims to estimate the unknown parameters, survival and hazard functions for exponentiated Rayleigh distribution based on unified hybrid censored data. The maximum likelihood estimators (MLEs), Bayes, and parametric bootstrap methods are used for estimating the two unknown parameters as well as survival and hazard functions. We propose to apply Markov chain Monte Carlo (MCMC) technique to carry out a Bayesian estimation procedure. Approximate confidence intervals for the unknown parameters moreover survival and hazard functions are constructed based on the s-normal approximation to the asymptotic distribution of MLEs. The approximate Bayes estimators have been obtained under the assumptions of informative and non-informative priors depending on symmetric and asymmetric loss functions via the Gibbs within Metropolis-Hasting samplers procedure. Finally, the proposed methods can be understood through illustrating the results of the real data analysis.

KEYWORDS

Exponentiated Rayleigh distribution; Unified hybrid censoring scheme; Maximum likelihood estimators; Bootstrap method; Bayesian estimation; MCMC method.

1. INTRODUCTION

Supposing there are n identical units put to test in the same time, when the experiment is terminated at a pre-fixed time T, we refer to Type-I censoring scheme. The disadvantage of Type-I censoring scheme is very few failures may occur before time T. The experiment is terminated when the rth failures occur, we refer to Type-II censoring scheme. Furthermore, the disadvantage of Type-II censoring scheme is that the termination time is unknown prior to the experiment. For more details on Type-I and Type-II censoring, and related inferential issues, see Nelson [23] and Balakrishnan and Cohen [4]. A hybrid censoring scheme (HCS) is a mixture of Type-I and Type-II censoring schemes. This HCS was introduced by Epstein [10], to avoid the disadvantages of Type-I and type-II censored scheme, and it has been used in reliability acceptance test in MILSTD-781-C [21], see also Jeong et al. [16], Gupta and Kundu [12], Park and Balakrishnan [24]. Epstein [10] is considered a hybrid censored scheme in which the life-testing experiment is terminated at a random time $T^*=\min\{X_{r:n}, T\}$, this refer to Type-I HCS. The disadvantage of Type-I HCS is very few failures occurring until the pre-fixed time T. To overcome this disadvantage, Childs et al. [8] introduced a substitute Type-I HCS that would terminate the experiment at the random time $T^*=\max\{X_{r:n}, T\}$. This HCS is called Type-II HCS. In the same respect, to avoid the disadvantages in these schemes, Chandrasekar et al. [7] proposed two new schemes which are called generalized Type-I and Type-II HCS. In generalized Type-I HCS, fix k, $r \in (1, 2, …, n)$ and $T \in (0,\infty)$ such that $k < r < n$. If the kth failure occurs before time T, the experiment is terminated at $\min\{X_{r:n}; T\}$. If the kth failure occurs after time T, the experiment is terminated at $X_{k:n}$, so, it is clear that this HCS modifies the Type-I HCS by allowing the experiment to continue after time T if very few failures had observed until that time point. In generalized Type-II HCS, fix $r \in (1, 2, …, n)$ and T_1, $T_2 \in (0,\infty)$ such that $T_2 > T_1$. If the rth failure occurs before time point T_1, the experiment is terminated at T_1. If the rth failure occurs between T_1 and T_2, the experiment is terminated at $X_{r:n}$. If the rth failure occurs after T_2, the experiment is terminated at T_2. There are some drawbacks in generalized HCS, such as, in generalized Type-I HCS. Moreover, because the experiment is terminated at the same time or before T, we cannot guarantee observing r failures. While in the generalized Type-II HCS, we cannot observe any failure at all or observe only few number of failures until the pre-fixed time T_2. To avoid these drawbacks, Balakrishnan et al. [5] introduced an unified hybrid censoring scheme (UHCS), which can be described as follows, fix r, $k \in \{1, …, n\}$ where $k < r < n$ and T_1, $T_2 \in (0,\infty)$ where $T_2 > T_1$. If the kth failure occurs before time T_1, the experiment is terminated at $\min\{\max\{X_{r:n}, T_1\}, T_2\}$. If the kth failure occurs between T_1 and T_2, the experiment is terminated at $\min\{X_{r:n}, T_2\}$. and if the kth failure occurs after time T_2, the experiment is terminated at $X_{k:n}$. Under this censoring scheme, we can guarantee that the experiment would be completed at most in time T_2 with at least k failure and if not, we can guarantee exactly k failures.

Therefore, under the UHCS, we have one of the following six cases:

Case I: $0 < x_{k:n} < x_{r:n} < T_1 < T_2$; the experiment is terminated at T_1.

Case II: $0 < x_{k:n} < T_1 < x_{r:n} < T_2$, the experiment is terminated at $x_{r:n}$.

Case III: $0 < x_{k:n} < T_1 < T_2 < x_{r:n}$, the experiment is terminated at T_2.

Case IV: $0 < T_1 < x_{k:n} < x_{r:n} < T_2$, the experiment is terminated at $x_{r:n}$.

Case V: $0 < T_1 < x_{k:n} < T_2 < x_{r:n}$, the experiment is terminated at T_2.

Case VI: $0 < T_1 < T_2 < x_{k:n} < x_{r:n}$, the experiment is terminated at $x_{k:n}$.

The Rayleigh distribution is an important distribution in statistics and operations research. It has a wide range of applications in several areas such as health, agriculture, biology, analyzing wind speed data and other. Burr [6] introduced twelve different forms of cumulative distribution functions for modeling lifetime data. The exponentiated Rayleigh distribution was derived by Vodă [35] to generalize the Rayleigh distribution originally developed by Rayleigh [28]. One of the important properties of the Rayleigh distribution is that its failure rate is an increasing function of time. Exponentiated Rayleigh distribution was studied by Sartawi and Abu-Salih [29], Jaheen [15, 14], Ahmad et al. [2], Raqab [25] and Surles and Padgett [31]. Recently, Surles and Padgett [32], introduced two parameter Burr Type X distribution and correctly named as the exponentiated Rayleigh distribution. The two parameter exponentiated Rayleigh distribution is a special case of the Weibull distribution originally suggested by Mudholkar and Srivastava [22]. Kundu and Raqab [17] estimated the parameters of ER(α,β) distribution by using different methods. Raqab and Madi [26] studied exponentiated Rayleigh distribution in Bayesian framework. Abd-Elfattah [1] discussed the goodness of fit test for the generalized Rayleigh distribution with unknown parameters. Raqab and Madi [27] discussed the inference for the generalized Rayleigh distribution based on progressively censored data. Lio et al. [18] discussed Parameter estimations for generalized Rayleigh distribution under progressively Type-I interval censored data. Fathipour et al. [11] studied the estimating $R = P(Y < X)$ in the generalized Rayleigh distribution with different scale parameters. Al Kanani and Jasim [3] discussed non-Bayesian and Bayesian estimation for generalized Rayleigh distribution. Shrestha and Kumar[30] studied Bayesian analysis for the exponentiated Rayleigh distribution. Mahmoud and Ghazal [19] studied estimations from the exponentiated Rayleigh distribution based on generalized Type-II hybrid censored data.

The cumulative distribution function (CDF) given by

$$F(x; \alpha, \beta) = (1 - e^{-\beta x^2})^\alpha, \qquad x > 0, \ \alpha, \ \beta > 0. \tag{1.1}$$

Where α and β are the shape and scale parameters respectively, the probability density function (PDF) is

$$f(x; \alpha, \beta) = 2\alpha\beta \, x \, e^{-\beta x^2}(1 - e^{-\beta x^2})^{\alpha-1}, \qquad x > 0, \ \alpha, \ \beta > 0. \tag{1.2}$$

The Survival function $S(t)$ is given by

$$S(t) = 1 - (1 - e^{-\beta t^2})^\alpha, \qquad t > 0. \tag{1.3}$$

The hazard function $H(t)$ is given by

$$H(t) = \frac{2\alpha\beta \, t \, e^{-\beta t^2}(1 - e^{-\beta t^2})^{\alpha-1}}{1 - (1 - e^{-\beta t^2})^\alpha}, \qquad t > 0. \tag{1.4}$$

The rest of the paper is organized as follows: In Section 2, we discussed the MLEs of the unknown parameters in addition to $S(t)$ and $H(t)$. In Section 3, credible intervals based the MLEs are presented. In Section 4, confidence intervals were obtained from two parametric bootstrap procedures. In Section 5, we apply MCMC technique to obtain the confidence intervals. One data set has been analyzed for illustrative purposes in Section 6. Finally, conclusions are given in Section 7.

2. MAXIMUM LIKELIHOOD ESTIMATION

In this section, we obtained the MLEs of ER (α, β) distribution when α and β are unknown. Let (x_1, . . . , x_n) be a random sample of size n from ER(α, β) distribution, then the likelihood function for six cases of the UHCS is as follows:

$$L(\underline{x}, \theta) = \frac{n!}{(n-R)!} \Big[\prod_{i-1}^{R} f(x_i) \Big] \Big[1 - F(C) \Big]^{n-R}. \tag{2.1}$$

$$R, C) = \begin{cases} (d, T_1), & \text{for case-I,} \\ (r, x_{r:n}) & \text{for case-II and case-IV,} \\ (d_2, T_2), & \text{for case-III and for case-V,} \\ (k, x_{k:n}), & \text{for case-VI.} \end{cases}$$

Where R indicates the number of the total failures in experiment up to time C (the stopping time point) and d_1 and d_2 indicate the number of failures that occur before time points T_1 and T_2, respectively. From (1.1), (1.2) and (2.1) we get

$$L(\underline{x};\alpha,\beta) = K\alpha^R\beta^R \prod_{i=1}^{R} x_i e^{-\beta\sum_{i=1}^{R} x_i^2} \prod_{i=1}^{R}\left[1-e^{-\beta x_i^2}\right]^{\alpha-1}\left[1-\left(1-e^{-\beta C^2}\right)^\alpha\right]^{n-R}. \qquad (2.2)$$

Where $K = \dfrac{2^R n!}{(n-R)!}$.

The logarithm of the likelihood function can be written as

$$\ell(\underline{x};\alpha,\beta) = \ln(K) + R\ln(\alpha) + R\ln(\beta) + \sum_{i=1}^{R}\ln(x_i) - \beta\sum_{i=1}^{R} x_i^2$$
$$+ (\alpha-1)\sum_{i=1}^{R}\ln(1-e^{-\beta x_i^2}) + (n-R)\ln[1-(1-e^{-\beta C^2})^\alpha]. \qquad (2.3)$$

Applying the first derivative with respect to α and β in (2.3) and equating by zero, then we get the two normal equations as follows

$$\frac{\partial\ell}{\partial\alpha} = \frac{R}{\alpha} + \sum_{i=1}^{R}\ln(1-e^{-\beta x_i^2}) - \frac{(n-R)(1-e^{-\beta C^2})^\alpha\ln(1-e^{-\beta C^2})}{1-(1-e^{-\beta C^2})^\alpha} = 0, \qquad (2.4)$$

$$\frac{\partial\ell}{\partial\beta} = \frac{R}{\beta} - \sum_{i=1}^{R} x_i^2 + (\alpha-1)\sum_{i=1}^{R}\frac{x_i^2 e^{-\beta x_i^2}}{1-e^{-\beta x_i^2}} - \frac{\alpha(n-R)(1-e^{-\beta C^2})^{\alpha-1}C^2 e^{-\beta C^2}}{1-(1-e^{-\beta C^2})^\alpha} = 0. \qquad (2.5)$$

where
R and C are defined above. The MLEs of α and β can be found by solving the system of equations (2.4) and (2.5), even if the suggested estimators cannot be expressed in closed forms, we can use a suitable numerical technique to obtain the estimators. Moreover, we can obtain the MLEs of $S(t)$ and $H(t)$ after replacing α and β by their MLEs $\hat{\alpha}$ and $\hat{\beta}$ as following

$$\hat{S}_{ML}(t) = 1 - (1 - e^{-\hat{\beta}t^2})^{\hat{\alpha}}, \qquad (2.6)$$
$$\hat{H}_{ML}(t) = \frac{2\hat{\alpha}\hat{\beta}\, t\, e^{-\hat{\beta}t^2}(1 - e^{-\hat{\beta}t^2})^{\hat{\alpha}-1}}{1 - (1 - e^{-\hat{\beta}t^2})^{\hat{\alpha}}}. \qquad (2.7)$$

3. CONFIDENCE INTERVAL

The asymptotic variance-covariance of the MLEs for parameters α and β are given by elements of the inverse of the Fisher information matrix are defined as

$$I_{ij} = -E\left(\frac{\partial^2\ell}{\partial\theta_i\partial\theta_j}\right), \qquad where\, i,j=1,2, \quad \theta_1=\alpha\ and \quad \theta_2=\beta.$$

Unluckily, we found difficult to solve the above expectations. Therefore, we gave the approximate asymptotic variance-covariance matrix for the MLEs, which, can be obtained by

$$I^{-1}(\alpha,\beta) = \begin{pmatrix} -\frac{\partial^2\ell}{\partial\alpha^2} & -\frac{\partial^2\ell}{\partial\alpha\partial\beta} \\ -\frac{\partial^2\ell}{\partial\beta\partial\alpha} & -\frac{\partial^2\ell}{\partial\beta^2} \end{pmatrix}^{-1}_{(\hat{\alpha},\hat{\beta})} = \begin{pmatrix} \widehat{var}(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}) \\ cov(\hat{\beta},\hat{\alpha}) & \widehat{var}(\hat{\beta}) \end{pmatrix},$$

where

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{R}{\alpha^2} - \frac{(n-R)\ln(1-e^{-\beta C^2})^2(1-e^{-\beta C^2})^\alpha}{[1-(1-e^{-\beta C^2})\alpha]^2}, \tag{3.8}$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = \frac{-R}{\beta^2} + (\alpha-1)\sum_{i=1}^{R}\frac{-(1-e^{-\beta x_i^2})x_i^4 e^{-\beta x_i^2} - x_i^4 e^{-2\beta x_i^2}}{(1-e^{-\beta x_i^2})^2}$$

$$-\alpha(n-R)C^2\left[\frac{(1-(1-e^{-\beta C^2})^\alpha)[(\alpha-1)(1-e^{-\beta C^2})^{\alpha-2}C^2 e^{-2C^2\beta} - C^2 e^{-\beta C^2}(1-e^{-\beta C^2})^{\alpha-1}]}{[1-(1-e^{-\beta C^2})\alpha]^2}\right.$$

$$\left.+\frac{\alpha C^4 e^{-2\beta C^2}(1-e^{-\beta C^2})^{2\alpha-2}}{[1-(1-e^{-\beta C^2})\alpha]^2}\right], \tag{3.9}$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \alpha} = \frac{\partial^2 \ell}{\partial \alpha \partial \beta} = \sum_{i=1}^{R}\frac{x_i^2 e^{-\beta x_i^2}}{1-e^{-\beta x_i^2}} - C^2(n-R)e^{-\beta C^2}[1-e^{-\beta C^2}]^{\alpha-1}\left[\vphantom{\frac{()}{()}}\right.$$

$$\times \frac{(1-(1-e^{-\beta C^2})^\alpha)[1+\alpha\ln(1-e^{-\beta C^2})] + \alpha(1-e^{-\beta C^2})^\alpha\ln(1-e^{-\beta C^2})}{[1-(1-e^{-\beta C^2})\alpha]^2}\left.\vphantom{\frac{()}{()}}\right]. \tag{3.10}$$

The asymptotic normality of the MLEs can be used to compute the approximate confidence intervals (ACI) for parameters α and β. Therefore, $(1-\gamma)100\%$ confidence intervals for parameters α and β become

$$\left(\hat{\alpha} \pm Z_{\gamma/2}\sqrt{\widehat{var}(\hat{\alpha})}\right) \quad and \quad \left(\hat{\beta} \pm Z_{\gamma/2}\sqrt{\widehat{var}(\hat{\beta})}\right). \tag{3.11}$$

Where $Z_{\gamma/2}$ is a standard normal variate.

Moreover; to find the ACI of the $S(t)$ and $H(t)$, we need to find the variances of them. To find the approximate estimates of the variance of $\hat{S}(t)$ and $\hat{H}(t)$ we use the delta method. The delta method is a general approach for computing confidence intervals for functions of MLEs. Depending on this method, the variance of $\hat{S}(t)$ and $\hat{H}(t)$ respectively given by

$$\hat{\sigma}^2_{S(t)} = [\nabla\hat{S}(t)]^T[\hat{V}][\nabla\hat{S}(t)] \quad and \quad \hat{\sigma}^2_{H(t)} = [\nabla\hat{H}(t)]^T[\hat{V}][\nabla\hat{H}(t)], \text{ where } \nabla\hat{S}(t) \text{ and }$$

$\nabla\hat{H}(t)$ are the gradient of $\hat{S}(t)$ and $\hat{H}(t)$ respectively, with respect to α and β. And

$\hat{V} = I^{-1}(\alpha,\beta)$. Therefore, $(1-\gamma)100\%$ confidence intervals for $S(t)$ and $H(t)$ become

$$\left(\hat{S}(t) \pm Z_{\gamma/2}\sqrt{\hat{\sigma}^2_{S(t)}}\right) \quad and \quad \left(\hat{H}(t) \pm Z_{\gamma/2}\sqrt{\hat{\sigma}^2_{H(t)}}\right).$$

4. BOOTSTRAP CONFIDENCE INTERVALS

In this section, two parametric bootstrap procedures are provided to construct the bootstrap confidence intervals of α, β, $S(t)$ and $H(t)$. The first one is the percentile bootstrap (Boot-p) confidence interval proposed by Efron[9]. The second one is the bootstrap-t (Boot-t) confidence interval proposed by Hall [13]. The algorithms for estimating the confidence intervals using both methods are illustrated as follows.

4.1 The procedure of Boot-p method

1. From the original sample $\underline{x} = (x_{1;n}, x_{2;n}, ..., x_{R;n})$ compute the ML estimates of the parameters α and β from equations (2.4) and (2.5).

2. Using the values of $\hat{\alpha}$ and $\hat{\beta}$ in equations (1.3) and (1.4), we obtain the ML estimates $\hat{S}(t)$ and $\hat{H}(t)$.

3. Get a bootstrap sample $\underline{x}^* = (x_{1;n}^*, x_{2;n}^*, ..., x_{R;n}^*)$ by resampling with replacement.

4. As in step 1, based on $\underline{x}^*$ compute the bootstrap sample estimates of η where $\eta = [\alpha, \beta, S(t), H(t)]$, say $\hat{n}^* = [\hat{\alpha}^*, \hat{\beta}^*, \hat{S}^*(t), \hat{H}^*(t)]$

5. Repeat steps 3 and 4 N Boot times, and obtain $\hat{\eta}_1^*, \hat{\eta}_2^*, ..., \hat{\eta}_{NBoot}^*$, where $\hat{\eta}_i^* = [\hat{\alpha}_i^*, \hat{\beta}_i^*, \hat{S}_i^*(t), \hat{H}_i^*(t)], i = 1, 2, ... NBoot.$

6. Arrange $\hat{\eta}_i^*, i = 1, 2, ..., NBoot$ in an ascending order to obtain the bootstrap sample $(\hat{\eta}_{(1)}^*, \hat{\eta}_{(2)}^*, ..., \hat{\eta}_{(NBoot)}^*)$

7. Let $G_1(x) = p(\eta_i \leq x)$ be the CDF of η_i. Define $\eta_{iBoot} = G_1^{-1}(x)$ for given $0 < x < 1$. The approximate bootstrap 100(1-γ)% confidence interval of η_i is given by $\left[\eta_{iBoot-p}\left(\frac{\gamma}{2}\right), \eta_{iBoot-p}\left(1 - \frac{\gamma}{2}\right) \right]$.

4.2 The procedure of Boot-t method

1. From the original sample $\underline{x} = (x_{1;n}, x_{2;n}, ..., x_{R;n})$ compute the ML estimates of the parameters: α and β from equations (2.4) and (2.5).

2. Using the values of $\hat{\alpha}$ and $\hat{\beta}$ in equation (1.3) and (1.4), we obtain the ML estimates $\hat{S}(t)$ and $\hat{H}(t)$.

3. Get a bootstrap sample $\underline{x}^* = (x_{1;n}^*, x_{2;n}^*, ..., x_{R;n}^*)$ by resampling with replacement. $\hat{\alpha}$ and $\hat{\beta}$ in step 1 to generate a bootstrap sample.

4. As in step 1, based on $\underline{x}^*$ compute the bootstrap sample estimates of η where $\eta = [\alpha, \beta, S(t), H(t)]$, say $\hat{n}^* = [\hat{\alpha}^*, \hat{\beta}^*, \hat{S}^*(t), \hat{H}^*(t)]$

5. Compute the $T^{*\eta}$ statistic defined as

$$T^{*\eta} = \frac{\hat{\eta}^* - \hat{\eta}}{\sqrt{\widehat{var}(\hat{\eta}^*)}}.$$

Where $\mathrm{var}(\hat{\eta}^*)$ are obtained by the Fisher information matrix.

6. Repeat step 3 and 5 NBoot times and obtain $T_1^{*\eta}, T_2^{*\eta}, ..., T_{NBoot}^{*\eta}$.

7. Arrange $T_1^{*\eta}, T_2^{*\eta}, ..., T_{NBoot}^{*\eta}$, in an ascending orders and obtain the ordered sequences $(\hat{T}_{(1)}^{*\eta}, \hat{T}_{(2)}^{*\eta}, ..., \hat{T}_{(NBoot)}^{*\eta})$.

8. Let $G_2(x) = p(T^* \leq x)$ be the CDF T^*. For a given x, define $\hat{\eta}_{boot-t}(x) = \hat{\eta} + G_2^{-1}(x)\sqrt{\widehat{var}(\hat{\eta})}$. Then the 100(1-$\gamma$)% percentile confidence interval of $\eta = [\alpha, \beta, S(t), H(t)]$ is $\left[\hat{\eta}_{iBoot-p}\left(\frac{\gamma}{2}\right), \hat{\eta}_{iBoot-p}\left(1 - \frac{\gamma}{2}\right) \right]$.

5. BAYES ESTIMATION

In this section, we described how to obtain the Bayes estimators and the corresponding credible intervals of parameters α, β, $S(t)$ and $H(t)$ of $ER(\alpha, \beta)$ distribution under squared error loss, LINEX loss and general entropy loss functions, based on UHCS when α and β are supposed to be unknown. We assume that α and β have the following gamma prior distributions

$$\pi_1(\alpha) \propto \alpha^{a_1-1}e^{-b_1\alpha}, \qquad \alpha > 0, \tag{5.12}$$

$$\pi_2(\beta) \propto \beta^{a_2-1}e^{-b_2\beta}, \qquad \beta > 0. \tag{5.13}$$

Here all the hyper parameters a_1; a_2; b_1 and b_2 are assumed to be known and non-negative.

The joint prior distribution for α and β is

$$\pi(\alpha, \beta) \propto \alpha^{a_1-1}\beta^{a_2-1}e^{-(b_1\alpha+b_2\beta)}. \tag{5.14}$$

Using the joint prior distribution of α and β, we obtained the joint distribution of the data, α and β as

$$L(\underline{x}|\alpha, \beta) \times \pi_1(\alpha) \times \pi_2(\beta). \tag{5.15}$$

Based on (5.15), the joint posterior density function of α and β can be written as

$$\pi^*(\alpha, \beta|\underline{x}) = \frac{L(\underline{x}|\alpha, \beta) \times \pi_1(\alpha) \times \pi_2(\beta)}{\int_0^\infty \int_0^\infty L(\underline{x}|\alpha, \beta) \times \pi_1(\alpha) \times \pi_2(\beta)d\alpha d\beta}. \tag{5.16}$$

Therefore, the Bayes estimator of a function $u(\alpha, \beta)$ under the squared error loss function is as follows

$$\hat{u}_{BS}(\alpha, \beta) = E[u(\alpha, \beta)|\underline{x}] = \int_\alpha \int_\beta u(\alpha, \beta)\pi^*(\alpha, \beta|\underline{x})d\alpha d\beta. \tag{5.17}$$

The Bayes estimator of a function $u(\alpha,\beta)$ under the LINEX loss function is as follows

$$\hat{u}_{BL}(\alpha, \beta) = -\frac{1}{a}\ln\left(E[e^{-au(\alpha,\beta)}|\underline{x}]\right) = -\frac{1}{a}\ln\left(\int_\alpha \int_\beta e^{-au(\alpha,\beta)}\pi^*(\alpha, \beta|\underline{x})d\alpha d\beta\right). \tag{5.18}$$

The Bayes estimator of a function $u(\alpha,\beta)$ under the General entropy loss function is as follows

$$\hat{u}_{BG}(\alpha, \beta) = \left[E[u(\alpha, \beta)^{-a}|\underline{x}]\right]^{-\frac{1}{a}} = \left(\int_\alpha \int_\beta [u(\alpha, \beta)]^{-a}\pi^*(\alpha, \beta|\underline{x})d\alpha d\beta\right)^{-\frac{1}{a}}. \tag{5.19}$$

It is very difficult to compute the equations (5.17), (5.18) and (5.19) analytically. Then, we suggested using MCMC to generate samples from (5.20) and approximate these equations under square error loss, LINEX loss and general entropy loss functions. The MCMC method is used for computing the Bayes estimates of the parameter α and β in addition to $S(t)$ and $H(t)$ and also to construct the corresponding credible intervals based on the generated posterior sample. For more details about the MCMC methods see, for example, Upadhyay et al.[33] and Upadhyay and Gupta [34]. We supposed the Gibbs with in Metropolis sampler, which requires the derivation of the complete. From (5.16), the joint posterior up to proportionality can be written as

$$\pi^*(\alpha, \beta|\underline{x}) \propto \alpha^{a_1+R-1}\beta^{a_2+R-1}e^{-\beta(b_2+\sum_{i=1}^R x_i^2)}e^{-\alpha[b_1-\sum_{i=1}^R \ln(1-e^{-\beta x_i^2})]}$$

$$e^{-\sum_{i=1}^R \ln(1-e^{-\beta x_i^2})}\left[1 - (1 - e^{-\beta C^2})^\alpha\right]^{n-R}.$$

From (5.20) we get the posterior density function of α given β

$$\pi_1^*(\alpha|\beta, \underline{x}) \propto \alpha^{a_1+R-1}e^{-\alpha\left[b_1-\sum_{i=1}^R \ln(1-e^{-\beta x_i^2})\right]}. \tag{5.21}$$

Also the posterior density function of β given α can be written as

$$\pi_2^*(\beta|\alpha,\underline{x}) = \beta^{a_2+R-1}e^{-\beta(b_2+\sum_{i=1}^{R}x_i^2)}e^{(\alpha-1)\sum_{i=1}^{R}\ln(1-e^{-\beta z_i^2})}\left[1-\left(1-e^{-\beta C^2}\right)^\alpha\right]^{n-R}. \quad (5.22)$$

Therefore, the posterior density function of α given β is gamma with the shape parameter (α_1+R) and scale parameter $\left(b_1 - \sum_{i=1}^{R}\ln(1-e^{-\beta x_i^2})\right)$ and, therefore, samples of α can be easily generated using any gamma generating routine.

Furthermore, the conditional posterior distribution of β given α in (5.22) cannot be reduced analytically to well known distributions and therefore it is impossible to sample directly by standard methods. So, to generate random numbers from this distribution, we use the Metropolis-Hastings method with normal proposal distribution, see Metropolis et al. [20]). Now, we suppose the next MCMC algorithm to draw samples from the posterior density (5.20) and in turn to obtain the Bayes estimates α and β and any function of them such as $S(t)$ and $H(t)$ and the corresponding credible intervals.

Algorithm of MCMC method:

- Step 1. Take some initial guess of α and β, say $\alpha^{(0)}$ and $\beta^{(0)}$ respectively, M = burn-in.

- Step 2. Set $j = 1$.

- Step 3. Generate $\alpha^{(j)}$ from Gamma $\left(\alpha_1 + R, b_1 - \sum_{i=1}^{R}\ln(1-e^{-\beta^j x_i^2})\right)$.

- Step 4. Using Metropolis-Hastings, generate $\beta^{(j)}$ from $\pi_2^*(\beta|\alpha,\underline{x})$ with the $N(\beta^{(j-1)},\sigma^2)$ proposal distribution where σ^2 is the variance of β obtained using variance-covariance matrix.

- Step 5. Compute $S(t)$ and $H(t)$ as

$$S(t)^{(j)} = 1 - (1 - e^{-\beta^{(j)}t^2})^{\alpha^{(j)}}, \quad (5.23)$$

$$H^{(j)}(t) = \frac{2\alpha^{(j)}\beta^{(j)}te^{-\beta^{(j)}t^2}(1 - e^{-\beta^{(j)}t^2})^{\alpha^{(j)}-1}}{1 - (1 - e^{-\beta^{(j)}t^2})^{\alpha^{(j)}}}. \quad (5.24)$$

- Step 6. Set $j = j + 1$.
- Step 7. Repeat steps 3-6 N times and obtain $\alpha^{(j)}$, $\beta^{(j)}$, $S^{(j)}(t)$ and $H^{(j)}(t)$, $j = M+1$, , N.
- Step 8. To compute the credible intervals of α, β, $S(t)$ and $H(t)$, order $\alpha^{(j)}$, $\beta^{(j)}$; $S^{(j)}(t)$ and $H^{(j)}(t)$, $j = M + 1, \ldots, N$, as $(\alpha^{(1)} < \ldots, \alpha^{(N-M)})$, $(\beta^{(1)} < \ldots, < \beta^{(N-M)}, (S^{(1)}(t) < \ldots, S^{(N-M)}(t)$ and $(H^{(1)}(t) < \ldots, H^{(N-M)}(t))$. Then the 100(1 - γ)% credible interval of α, β, $S(t)$ and $H(t)$ is $\left(\zeta_{(N-M)_{\gamma/2}}, \zeta_{(N-M)_{(1-\gamma/2)}}\right)$

The Bayes estimates of $\zeta = [\alpha, \beta, S(t), H(t)]$, under squared error loss function are given by

$$\hat{\zeta}_{BS} = E[\zeta|\underline{x}] = \frac{\sum_{i=M+1}^{N}\zeta^{(i)}}{N - M}. \quad (5.25)$$

The Bayes estimates of $\zeta = [\alpha, \beta, S(t), H(t)]$, under LINEX loss function are given by

$$\hat{\zeta}_{BL} = -\frac{1}{a}\ln\left[\frac{\sum_{i=M+1}^{N}e^{-a\zeta^{(i)}}}{N - M}\right]. \quad (5.26)$$

The Bayes estimates of $\zeta = [\alpha, \beta, S(t), H(t)]$, under general entropy loss function are given by

$$\hat{\zeta}_{BG} = \left[\frac{\sum_{i=M+1}^{N}[\zeta^{(i)}]^{-a}}{N - M}\right]^{-\frac{1}{a}}. \quad (5.27)$$

6. REAL LIFE DATA

Wind power is renewable and environmentally friendly. It is an alternative clean energy source compared with the fossil fuels that pollute the lower layer of the atmosphere. Wind power is a form of solar energy, driven by the unequal heating of the earth's surface. The most important parameter of the wind power is wind speed. Statistical methods are useful for

estimating wind speed because it is a random phenomenon. For this reason, wind speed probabilities can be modelled by using probability distributions. We have taken the daily average wind speeds from 1 / 1 /2009 to 10 / 4 / 2009 for Cairo city as follows

3.5	3.1	3.8	3.2	3.2	4.5	5.6	5.7	4.9	5.7
4.3	9.4	9.3	4.4	2.7	3.8	4.9	5.4	4.9	4.2
5.4	3.3	6.9	9.8	10	8	5.6	8.2	9.4	11.3
9.4	5.5	4.9	8.6	5	4.7	3.8	4.3	6.7	7.6
13.3	8.2	5.8	5.1	7.8	10.3	9.3	4.3	7.4	13.8
10.7	12	8.9	10.6	6.8	6.6	11.1	12.5	14.4	9.9
4.8	4.2	5.5	7.3	12.4	14.7	6.4	8.7	5.2	6.8
5.6	7.5	7.7	7.1	6.1	7.6	5.8	6.3	12.2	6
3.5	9.5	8.8	5.2	5	9.8	8	7.9	6.8	5.7
7.3	6.8	4.7	5.3	9.6	10.1	7.3	6.7	5.4	5.4

This data was produced by the national climatic data center (NCDC) in Asheville in the United States of America. Now, one of the most important subjects is type of distribution of any set of data will be known during statistical tests which are called the goodness of fit. We depended on Kolmogorov-Smirnov test to fit whether the data distribution as ER (α, β) distribution or not. The calculated value of the K-S test is 0.0983022 for the ER (α, β) distribution and this value is smaller than their corresponding values expected at 5% significance level, which is 0.13403 at $n = 100$. We have just plotted the empirical $S(t)$ and the fitted $S(t)$ in Figure1a. Observe that the ER (α, β) distribution can be a good model fitting this data. In Figure1b it shows that all the points of a Q-Q plot are inside the unit square, so, it can be seen that the ER (α, β) distribution fits the data very well. P-value = 0.27052, therefore, the high p-value indicates that ER (α, β) distribution can be used to analyze this data set. Now, we consider the case when the data are censored. We have six cases as following:

- Case I: $T_1 = 9.45$, $T_2 = 9.95$, $k = 70$, $r = 75$. In this case: $R = 80$, C $= T_1 = 9.45$.
- Case II: $T_1 = 9.45$, $T_2 = 9.95$, $k = 70$, $r = 83$. In this case: $R = 83$, $C = x_{r:n} = 9.8$.
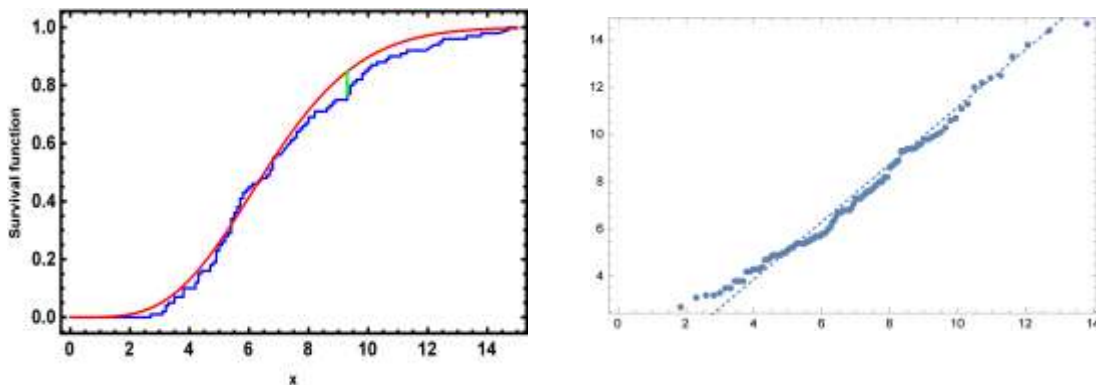
Figure 1: (a) Empirical and Fitted Survival Functions. (b) Q - Q plot compare data to a specific distribution.

- Case III: $T_1 = 9.45$, $T_2 = 9.95$, $k = 70$, r $= 86$. In this case: $R = 85$, $C =$ T2 $= 9.95$.
- Case IV: $T_1 = 8.4$, $T_2 = 10.5$, $k = 80$, r $= 87$. In this case: $R = 87$, $C = x_{r:n} = 10.1$.
- Case V: $T_1 = 8.4$, $T_2 = 10.5$, $k = 84$, r $= 90$. In this case: R $= 88$, $C =$ T2 $= 10.5$.
- Case VI: $T_1 = 8.4$, $T_2 = 9.35$, $k = 92$, r $= 93$. In this case: $R = 92$, $C = x_{k:n} = 11.3$.

In all the six cases, we estimated the unknown parameters, we computed different estimates of the parameters α and β in addition to $S(t)$ and $H(t)$. MLEs, Bootstrap confidence intervals (Boot-p, Boot-t) based on 1000 Bootstrap samples and Bayes estimates using MCMC method based on 11000 MCMC sample and discard the first 1000 values as `burn-in'. To compute the MLEs, we used the numerical method and also computed the 95% ACI; the results are given in Table (1). Also, we computed the 95% ACI Bootstrap confidence intervals. The results are given in Table (2). To compute the Bayes estimates we considered the square error loss, LINEX loss and general entropy loss functions. Under non-informative priors the hyper parameters are $a_1 = a_2 = b_1 = b_2 = 0$. The results are given in Table (3). Under the informative priors the hyper parameters are $a_1 = 0.9$, $a_2 = 0.8$, $b_1 = 0.5$ and $b_2 = 1.0$. The results are given in Table (4). Also, we computed the 95% ACI based on the MCMC samples. The results are given in Table (5). The characteristics of the unknown parameters for MCMC method under non-informative and informative priors are given in Table (6) and Table (7) respectively. Figure

(2) plot the MCMC output of α, β, $S(t)$ and $H(t)$. The histogram of α, β, $S(t)$ and $H(t)$ are given in Figure (3). In all the cases α = 2 and β = 0.026 are considered.

Table 1: Estimation of *α*, *β*, *S* and *H* and the 95% confidence interval of MLE method for six cases of UHCS

Cases	Parameters	MLE	Lower	Upper	Length
Case I	α	2.2276	1.4839	2.9714	1.4875
	β	0.0281	0.0210	0.0352	0.0142
	S	0.9646	0.9373	0.9919	0.0546
	H	0.0479	0.0231	0.0727	0.0496
Case II	α	2.2150	1.4873	2.9427	1.4554
	β	0.0279	0.0210	0.0348	0.0138
	S	0.9643	0.9370	0.9916	0.0546
	H	0.0481	0.0233	0.0728	0.0494
Case III	α	2.2426	1.5121	2.9732	1.4611
	β	0.0282	0.0214	0.0351	0.0137
	S	0.9650	0.9381	0.9918	0.0537
	H	0.0477	0.0230	0.0724	0.0494
Case IV	α	2.2724	1.5382	3.0067	1.4686
	β	0.0286	0.0218	0.0354	0.0136
	S	0.9656	0.9392	0.9920	0.0529
	H	0.0473	0.0227	0.0719	0.0493
Case V	α	2.2084	1.5001	2.9167	1.4166
	β	0.0278	0.0212	0.0344	0.0132
	S	0.9642	0.9370	0.9913	0.0543
	H	0.0482	0.0235	0.0728	0.0493
Case VI	α	2.1828	1.4959	2.8697	1.3739
	β	0.0275	0.0212	0.0339	0.0127
	S	0.9635	0.9363	0.9908	0.0545
	H	0.0485	0.0239	0.0731	0.0491

Table 2: Estimation of *α*, *β*, *S* and *H* and the 95% confidence interval of Boot-P and Boot-T methods for six cases of UHCS

Cases	Parameters	Boot-P	Lower	Upper	Length	Boot-P	Lower	Upper	Length
Case I	α	2.2648	1.8221	2.8330	1.0109	2.2461	1.8120	2.8045	0.9924
	β	0.0282	0.0242	0.0326	0.0084	0.0277	0.0245	0.0307	0.0061
	S	0.9644	0.9420	0.9830	0.0410	0.9641	0.9474	0.9897	0.0422
	H	0.0106	0.0105	0.0113	0.0008	0.0485	0.0276	0.0647	0.0371
Case II	α	2.2457	1.8314	2.8450	1.0136	2.2744	1.8247	2.8740	1.0493
	β	0.0280	0.0242	0.0326	0.0083	0.0278	0.0239	0.0311	0.0071
	S	0.9638	0.9429	0.9829	0.0400	0.9646	0.9486	0.9903	0.0416
	H	0.0106	0.0104	0.0160	0.0055	0.0479	0.0268	0.0640	0.0372
Case III	α	2.2844	1.8609	2.8806	1.0196	2.3096	1.7632	2.9003	1.1370

	β	0.0284	0.0241	0.0333	0.0092	0.0285	0.0231	0.0327	0.0095
	S	0.9648	0.9439	0.9827	0.0387	0.9668	0.9450	0.9966	0.0516
	H	0.0106	0.0100	0.0154	0.0054	0.0467	0.0217	0.0670	0.0453
Case IV	α	2.3152	1.8954	2.8630	0.9685	2.3126	1.8661	2.8512	0.9851
	β	0.0287	0.0241	0.0333	0.0092	0.0286	0.0238	0.0326	0.0088
	S	0.9658	0.9453	0.9822	0.0369	0.9685	0.9478	0.9976	0.0497
	H	0.0105	0.0065	0.0109	0.0044	0.0452	0.0210	0.0646	0.0436
Case V	α	2.2469	1.8207	2.7842	0.9634	2.2517	1.7837	2.8097	1.0260
	β	0.0281	0.0239	0.0332	0.0092	0.0281	0.0225	0.0331	0.0105
	S	0.9637	0.9416	0.9817	0.0401	0.9684	0.9440	1.0064	0.0624
	H	0.0107	0.0077	0.0111	0.0034	0.0452	0.0151	0.0677	0.0526
Case VI	α	2.2042	1.8003	2.7536	0.9533	2.2161	1.6931	2.7934	1.1002
	β	0.0275	0.0226	0.0333	0.0106	0.0275	0.0209	0.0327	0.0117
	S	0.9629	0.9418	0.9808	0.0390	0.9657	0.9427	0.9988	0.0560
	H	0.0106	0.0087	0.0171	0.0083	0.0473	0.0216	0.0691	0.0474

Table 3: Estimation of α, β, S and H for MCMC method for six cases of UHCS with non-informative prior

Cases	Parameters	SE	LINEX			GE		
			a = - 4	a = 0.5	a = 4	a = - 4	a = 0.5	a = 4
Case I	α	1.2877	1.4664	1.2688	1.1723	1.3757	1.2520	1.1667
	β	0.0194	0.0194	0.0194	0.0194	0.0205	0.0190	0.0178
	S	0.8999	0.9019	0.8996	0.8977	0.9017	0.8991	0.8970
	H	0.0837	0.0842	0.0836	0.0831	0.0885	0.0814	0.0752
Case II	α	1.3806	1.5549	1.3612	1.2603	1.4634	1.3463	1.2626
	β	0.0204	0.0204	0.0204	0.0204	0.0212	0.0200	0.0191
	S	0.9100	0.9117	0.9098	0.9082	0.9115	0.9093	0.9076
	H	0.0794	0.0800	0.0794	0.0789	0.0845	0.0771	0.0705
Case III	α	1.4912	1.7009	1.4684	1.3538	1.5822	1.4540	1.3652
	β	0.0217	0.0217	0.0217	0.0217	0.0226	0.0213	0.0203
	S	0.9193	0.9208	0.9191	0.9178	0.9207	0.9188	0.9173
	H	0.0755	0.0760	0.0754	0.0750	0.0809	0.0730	0.0660
Case IV	α	1.6093	1.8518	1.5835	1.4523	1.7044	1.5700	1.4742
	β	0.0230	0.0231	0.0230	0.0230	0.0239	0.0227	0.0216
	S	0.9279	0.9292	0.9277	0.9266	0.9290	0.9274	0.9261
	H	0.0716	0.0721	0.0715	0.0711	0.0772	0.0690	0.0617
Case V	α	1.6108	1.8425	1.5850	1.4519	1.7051	1.5714	1.4736
	β	0.0228	0.0228	0.0228	0.0228	0.0236	0.0224	0.0215
	S	0.9290	0.9303	0.9288	0.9276	0.9301	0.9285	0.9272
	H	0.0705	0.0710	0.0704	0.0700	0.0762	0.0679	0.0606
Case VI	α	1.7671	2.0264	1.7386	1.5927	1.8619	1.7274	1.6295
	β	0.0241	0.0241	0.0241	0.0240	0.0248	0.0237	0.0228
	S	0.9404	0.9414	0.9403	0.9394	0.9413	0.9401	0.9391
	H	0.0638	0.0642	0.0637	0.0633	0.0693	0.0612	0.0540

Table 4: Estimation of α, β, S and H for MCMC method for six cases of UHCS with informative prior

Cases	Parameters	SE	LINEX			GE		
			a = - 4	a = 0.5	a = 4	a = - 4	a = 0.5	a = 4
Case I	α	1.3031	1.4883	1.2841	1.1874	1.3905	1.2678	1.1829
	β	0.0196	0.0197	0.0196	0.0196	0.0206	0.0192	0.0180
	S	0.9013	0.9033	0.9011	0.8992	0.9031	0.9006	0.8985
	H	0.0832	0.0838	0.0831	0.0827	0.0882	0.0809	0.0744
Case II	α	1.4274	1.6057	1.4071	1.3012	1.5108	1.3926	1.3071
	β	0.0211	0.0211	0.0211	0.0211	0.0220	0.0207	0.0197
	S	0.9135	0.9151	0.9133	0.9118	0.9149	0.9129	0.9112
	H	0.0784	0.0789	0.0783	0.0779	0.0836	0.0761	0.0696
Case III	α	1.5250	1.7386	1.5015	1.3785	1.6153	1.4871	1.3922
	β	0.0222	0.0222	0.2220	0.0221	0.0231	0.0218	0.0207
	S	0.9215	0.9230	0.9213	0.9200	0.9228	0.9210	0.9185
	H	0.0748	0.0753	0.0747	0.0743	0.0801	0.0724	0.0657
Case IV	α	1.6407	1.8910	1.6142	1.4786	1.7362	1.6010	1.5031
	β	0.0235	0.0235	0.0235	0.0235	0.0244	0.0231	0.0221
	S	0.9296	0.9309	0.9295	0.9283	0.9307	0.9292	0.9279
	H	0.0710	0.0715	0.0709	0.0705	0.0765	0.0684	0.0612
Case V	α	1.6292	1.8782	1.6044	1.4737	1.7181	1.5917	1.4972
	β	0.0231	0.0231	0.0231	0.0231	0.0239	0.0227	0.0218
	S	0.9302	0.9314	0.9301	0.9290	0.9313	0.9298	0.9286
	H	0.0701	0.0706	0.0700	0.0696	0.0756	0.0676	0.0609
Case VI	α	1.8038	2.1435	1.7736	1.6244	1.9060	1.7629	1.6644
	β	0.0246	0.0246	0.0246	0.0246	0.0254	0.0243	0.0234
	S	0.9418	0.9427	0.9417	0.9409	0.9426	0.9415	0.9406
	H	0.0633	0.0638	0.0632	0.0628	0.0689	0.0606	0.0527

Table 5: The 95% confidence interval of MCMC for *α, β, S* and *H* for six cases of UHCS

Cases	Parameters	MCMC non informative			MCMC with informative		
		Lower	Upper	Length	Lower	Upper	Length
Case I	α	0.9372	1.7064	0.7692	0.9541	1.7245	0.7704
	β	0.0145	0.0250	0.0105	0.0146	0.0251	0.0105
	S	0.8481	0.9425	0.0944	0.8502	0.9434	0.0932
	H	0.0604	0.1083	0.0479	0.0598	0.1081	0.0483
Case II	α	0.0196	1.8074	0.7878	1.0594	1.8641	0.8047
	β	0.0157	0.0254	0.0098	0.0163	0.0264	0.0101
	S	0.8621	0.9491	0.0871	0.8674	0.9512	0.0839
	H	0.0565	0.1044	0.0479	0.0552	0.1030	0.0477
Case III	α	1.1035	1.9516	0.8481	1.1258	1.9803	0.8545
	β	0.0168	0.0270	0.0102	0.0170	0.0276	0.0107
	S	0.8748	0.9556	0.0808	0.8775	0.9562	0.0787
	H	0.0528	0.1003	0.0475	0.0523	0.0995	0.0472
Case IV	α	1.1940	2.0899	0.8960	1.2232	2.1355	0.9123
	β	0.0179	0.0285	0.0107	0.0184	0.0290	0.0106
	S	0.8862	0.9608	0.0747	0.8886	0.9616	0.0730
	H	0.0489	0.0964	0.0474	0.0483	0.0959	0.0476
Case V	α	1.1956	2.1037	0.9082	1.2190	2.0971	0.8781
	β	0.0180	0.0280	0.0101	0.0182	0.0283	0.0101
	S	0.8869	0.9623	0.0754	0.8895	0.9615	0.0720
	H	0.0477	0.0954	0.0477	0.0483	0.0950	0.0466
Case VI	α	1.3258	2.2754	0.9496	1.3662	2.3289	0.9627
	β	0.0192	0.0292	0.0101	0.0196	0.0297	0.0101
	S	0.9042	0.9687	0.0644	0.9066	0.9698	0.0632
	H	0.0423	0.0876	0.0453	0.0415	0.0870	0.0455

Table 6: The characteristics for *α, β, S* and *H* for six cases of UHCS with non- informative prior

Cases	Parameters	Mean	Median	Mode	Standard deviation	Root MS	Skewness
Case I	α	1.2878	1.2659	1.2530	0.2362	1.3092	0.5488
	β	0.0195	0.0193	0.0190	0.0032	0.0197	0.2662
	S	0.8999	0.9023	0.9153	0.0288	0.9004	- 0.5123
	H	0.0837	0.0834	0.0846	0.0146	0.0850	0.1477
Case II	α	1.3807	1.3627	1.3620	0.2386	1.4011	0.4624
	β	0.0204	0.0203	0.0203	0.0030	0.0207	0.1913
	S	0.9101	0.9126	0.9259	0.0266	0.9104	- 0.5727
	H	0.0795	0.0792	0.0824	0.0145	0.0808	0.1718
Case III	α	1.4913	1.4666	1.3370	0.2591	1.5136	0.5113
	β	0.0219	0.0216	0.0209	0.0032	0.0220	0.2348
	S	0.9194	0.9218	0.9235	0.0248	0.9197	- 0.5614
	H	0.0756	0.0750	0.0740	0.0146	0.0770	0.1914
Case IV	α	1.6094	1.5891	1.5350	0.2759	1.6329	0.4636
	β	0.0231	0.0230	0.0230	0.0032	0.0233	0.1440
	S	0.9280	0.9302	0.9321	0.0230	0.9282	- 0.6200
	H	0.0717	0.0714	0.0740	0.0145	0.0731	0.2225
Case V	α	1.6108	1.5894	1.5400	0.2756	1.6343	0.4386
	β	0.0228	0.0228	0.0228	0.0031	0.0231	0.1288
	S	0.9291	0.9315	0.9377	0.0231	0.9293	- 0.6574
	H	0.0706	0.0699	0.0677	0.0146	0.0720	0.2386
Case VI	α	1.7671	1.7467	1.6970	0.2894	1.7907	0.4213
	β	0.0241	0.0241	0.0229	0.0031	0.0243	0.1025
	S	0.9405	0.9429	0.9455	0.0198	0.9407	- 0.6924
	H	0.0638	0.0633	0.0674	0.0137	0.0653	0.2706

Table 7: The characteristics for α, β, S and H for six cases of UHCS with informative prior

| Cases	Parameters	Mean	Median	Mode	Standard deviation	Root MS	Skewness
Case I	α	1.3031	1.2827	1.2370	0.2363	1.3244	0.5675
	β	0.0197	0.0195	0.0189	0.0032	0.0199	0.3000
	S	0.9014	0.9040	0.9102	0.0286	0.9018	- 0.5398
	H	0.0833	0.0829	0.0850	0.0148	0.0846	0.1762
Case II	α	1.4275	1.4092	1.3520	0.2439	1.4482	0.4364
	β	0.0211	0.0210	0.0215	0.0031	0.0214	0.1878
	S	0.9135	0.9165	0.9249	0.0258	0.9139	- 0.6006
	H	0.0785	0.0778	0.0766	0.0145	0.0798	0.2206
Case III	α	1.5250	1.5079	104420	0.2625	1.5475	0.4247
	β	0.0222	0.0221	0.0213	0.0032	0.0224	0.1423
	S	0.9216	0.9243	0.9338	0.0243	0.9219	- 0.6532
	H	0.0748	0.0744	0.0766	0.0144	0.0762	0.2155
Case IV	α	1.6407	1.6178	1.5490	0.2792	1.6643	0.4624
	β	0.0235	0.0234	0.0229	0.0032	0.0238	0.2186
	S	0.9297	0.932	0.9366	0.0224	0.9300	- 0.644
	H	0.0710	0.0706	0.0704	0.0144	0.0725	0.2198
Case V	α	1.6292	1.6163	1.5890	0.2691	1.6513	0.4022
	β	0.0231	0.0230	0.0230	0.0031	0.0233	0.1858
	S	0.9303	0.9333	0.9349	0.0223	0.9305	- 0.7608
	H	0.0701	0.0694	0.0648	0.0142	0.0716	0.3252
Case VI	α	1.8038	1.7756	1.6670	0.2993	1.8285	0.5854
	β	0.0246	0.0245	0.0247	0.0031	0.0248	0.2061
	S	0.9419	0.9439	0.9447	0.0194	0.9421	- 0.6644
	H	0.0633	0.0629	0.0617	0.0137	0.0648	0.2263
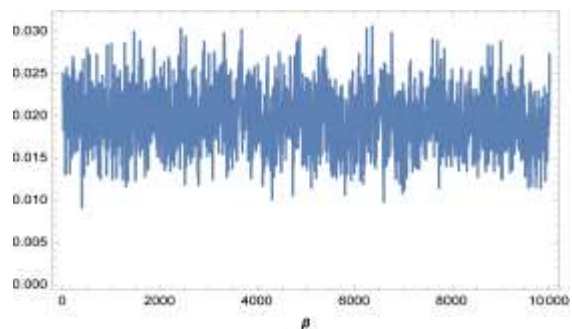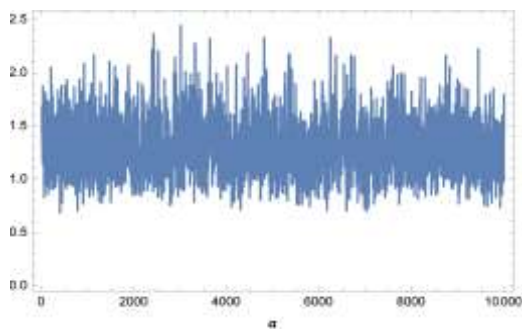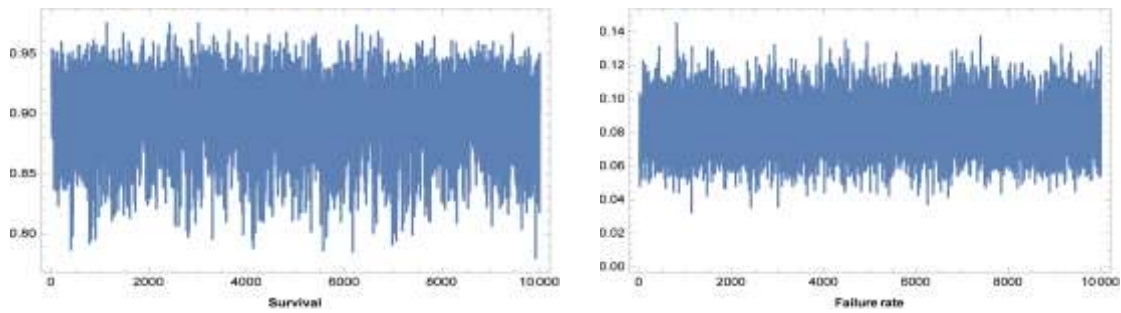
Figure 2: Trace plot of α, β, S(t) and H(t) generated by MCMC method for case I.
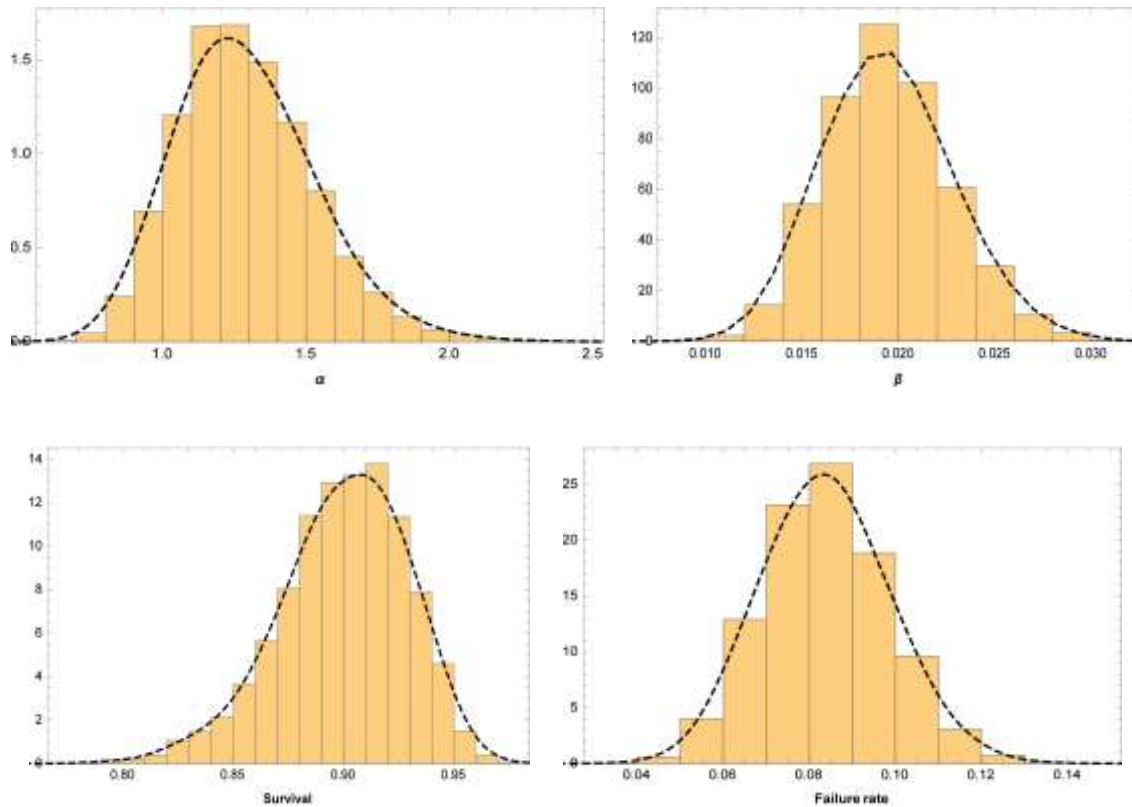

Figure 3: Histogram of *α*, *β*, *S*(*t*) and *H*(*t*) generated by MCMC method for case I.

7. CONCLUSION

In this paper, we considered the Bayes estimation of the unknown parameters of the exponentiated Rayleigh distribution when the data is collected under the unified hybrid censored data. The MLEs, the bootstrap confidence intervals and the credible intervals based on the observed Fisher information matrix have been discussed. We supposed the gamma priors for both the unknown parameters and provided the Bayes estimators under the assumptions of squared error loss, LINEX loss and general entropy loss functions. In our study, the Bayes estimates cannot be obtained in explicit form. So, we used the MCMC technique to compute the approximate Bayes estimates and the corresponding credible intervals. We have applied the developed techniques on a real data set.

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