



Subdifferential calculus for invariant linear ordered vector space-valued operators and applications

Antonio Boccuto

Dipartimento di Matematica e Informatica, University of Perugia, via Vanvitelli, 1 – I-06123 Perugia (Italy)
antonio.boccuto@unipg.it

ABSTRACT

We give a direct proof of sandwich-type theorems for linear invariant partially ordered vector space operators in the setting of convexity. As consequences, we deduce equivalence results between sandwich, Hahn-Banach, separation and Krein-type extension theorems, Fenchel duality, Farkas and Kuhn-Tucker-type minimization results and subdifferential formulas in the context of invariance. As applications, we give Tarski-type extension theorems and related examples for vector lattice-valued invariant probabilities, defined on suitable kinds of events.

Keywords

Dedekind complete partial ordered vector space, amenability, Hahn-Banach theorem, sandwich theorem, Fenchel duality theorem, Krein extension theorem, subdifferential.

Academic Discipline And Sub-Disciplines

Mathematics.

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1 INTRODUCTION

Sandwich and Hahn-Banach-type theorems are deeply studied, have several applications in different areas, and are related with several topics, among which, for example, minimization of functionals and operators. In these cases, often it is advisable to associate to a “primal” problem a “dual” problem, which is in general easier to handle, and to investigate the relations existing between them. These tools are widely used, for instance, in reconstructing images corrupted by noise and/or blur (see also [9,17]), in which it is possible to minimize the primal energy by investigating the dual energy and to deal with suitable convex approximations. There is a very wide literature concerning the Fenchel duality theorem, which are often used, for example, in problems of convex analysis, numerical analysis and Calculus of Variations (see also [1,2,37,40]).

Another kind of problems connected with sandwich-type theorems deal with nonlinear minimization programming, in which some constraints are often posed, and there are several studies on finding related necessary and/or sufficient conditions, for example saddle-type properties. Among them, we quote in particular Farkas and Kuhn-Tucker-type theorems (for a related literature and an overview, see for instance [19,27,30,46]).

Other applications and consequences of Hahn-Banach theorems can be found in separation theorems on convex sets by means of an affine manifold (see also [26]), in (Riesz) MV-algebras (see also [6,20,33]), in subdifferential calculus and its various features (see also [11,12,31]). In the literature, many studies about these topics have been extended to the context of partially ordered space-valued operators and measures (see also [8,10,14,24,32] and their bibliographies).

Another field of research related with sandwich-type theorems are extension theorems for finitely additive (and invariant) measures and probabilities, for example in exchangeable processes (see also [4,28,39]). Some theorems of this kind for invariant partially order vector space-operators were given, for instance, in [3,7,15,25,41,42]. In particular, in [3] and [7] some characterizations of amenable (semi)groups were given, in terms of these kinds of theorems, in the context of vector lattice-valued invariant operators and set functions.

Finally, in [6], some separation theorems by means of affine invariant manifolds were given, together with Farkas and Kuhn-Tucker-type theorems, in the context of partially ordered vector spaces.

In this paper, we first prove a sandwich-type theorem for invariant linear ordered vector space-valued operators in the setting of convexity, using a similar result proved in [47], when the involved linear operators are not necessarily invariant. Successively, some Fenchel duality-type theorems, subdifferential formulas, Hahn-Banach and Krein extension theorems



are proved, and their equivalence with separation, Farkas and Kuhn-Tucker-type theorems are shown. Our techniques are inspired by methods used in [21,22,34,35,36,45,46,47]. Furthermore, using Tarski-type theorems on extensions of invariant finitely additive measures, which are consequences of sandwich theorems, some applications are given, on constructing some invariant finitely additive vector lattice-valued probability measures defined on all subsets of suitable "product spaces" of the type $\prod_{t \in \mathcal{T}} B_t$, where \mathcal{T} can be viewed as the time space and the B_t 's, $t \in \mathcal{T}$, can be interpreted as "events", e.g. as logical propositions about which one knows that at the instant t (in the future or also in the past, for example in the case of lack of informations) they are true or false (see also [5]). The invariance allows to "replace" the study of the involved events related to some periods of time about which one has not enough knowledge with the corresponding events associated with other periods of time on which one has more informations.

2 PRELIMINARIES

We begin with giving the basic definitions and concepts, which will be useful for proving our results.

Let X be a real vector space. An *affine combination* of elements x_1, x_2, \dots, x_n of X is any linear combination of the form $\sum_{i=1}^n \lambda_i x_i$ with $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ and $\sum_{i=1}^n \lambda_i = 1$. An *affine manifold* of X is a nonempty subset of X , closed under affine combinations.

If $\emptyset \neq Z \subset X$, then the *affine hull* of Z is the smallest affine manifold of X which contains Z , and we denote it by $\text{span}_{\text{aff}}(Z)$ (see also [26]).

A point $x_0 \in Z$ is an *algebraic relative interior point* of Z iff for each $x \in \text{span}_{\text{aff}}(Z)$ there is $\lambda_0 > 0$ with $(1 - \lambda)x_0 + \lambda x \in Z$ for each $\lambda \in] - \lambda_0, \lambda_0[$. The set of all algebraic relative interior points of Z is denoted by $\text{int}(Z)$.

A nonempty subset D of any real vector space X is said to be *convex* iff $\lambda x_1 + (1 - \lambda)x_2 \in D$ for every $x_1, x_2 \in D$ and $\lambda \in [0,1]$.

Given any two real vector spaces X, Y and a convex set $D \subset X$, we say that a function $U: X \rightarrow Y$ is *convex* on D iff $U(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda U(x_1) + (1 - \lambda)U(x_2)$ for every $x_1, x_2 \in D$ and $\lambda \in [0,1]$. A function $U: D \rightarrow Y$ is said to be *concave* iff $-U$ is convex. We will set $D(U) := D$.

Let G be a semigroup, and $\mathcal{P}(G)$ be the family of all subsets of G . We say that G is *left* (resp. *right*) *amenable* iff there is a finitely additive measure $\mu: \mathcal{P}(G) \rightarrow [0,1]$, with $\mu(G) = 1$ and $\mu(\{hg: g \in E\})$ (resp. $\mu(\{gh: g \in E\}) = \mu(E)$) for every $E \subset G$ and $g \in G$. We say that G is *amenable* iff it is both left and right amenable. In general, left and right amenability are not equivalent, but, if G a group, then they coincide (see also [18]).

Let $G \subset X^X$ be a right amenable semigroup of (linear) homomorphisms, with $(gh)(x) = g(hx)$ for any $g, h \in G$ and $x \in X$, R be a Dedekind complete partially ordered vector space, $R^+ := \{y \in R: y \geq 0\}$, $l_b(G, R)$ be the space of all bounded R -valued functions defined on G . Given $f \in l_b(G, R)$ and $h \in G$, defined by ${}_h f(g) = f(hg)$ (resp. $f_h(g) = f(gh)$), $g \in G$. A *left* (resp. *right*) G -invariant R -mean is a linear positive function $m: L^\infty(G) \rightarrow R$ such that $m({}_h f) = m(f)$ (resp. $m(f_h) = m(f)$) for all $f \in l_b(G, R)$ and $h \in G$, and $m(y) = y$ for each $y \in R$, where y is the constant function which associates the value y to every element $g \in G$.

A set $\emptyset \neq Z \subset X$ is G -invariant iff $gz \in Z$ whenever $z \in Z$. A set $\emptyset \neq A \subset X \times R$ is G -invariant iff $(gx, y) \in A$ whenever $g \in G$ and $(x, y) \in A$.

A function $L: X \rightarrow R$ is G -subinvariant (resp. G -superinvariant, G -invariant) iff $L(gx) \leq L(x)$ (resp. $L(gx) \geq L(x)$, $L(gx) = L(x)$) for every $g \in G$ and $x \in X$.

By $\mathcal{L}(X, R)$ and $\mathcal{L}(R, R)$ we denote the sets of all linear functions from X to R and from R to R , respectively. We indicate with $\mathcal{L}_{\text{inv}}(X, R)$ the set of all linear G -invariant functions $L \in \mathcal{L}(X, R)$.

A nonempty set $A \subset X \times R$ is called a *cone* with vertex $x_0 \in X \times R$ iff $\lambda(A - x_0) \subset A - x_0$ for every positive real number λ . Sometimes we associate with X a G -invariant cone $X^+ \subset X$ with vertex 0 , and the corresponding order on X defined by $x_1 \geq x_2$ if and only if $x_1 - x_2 \in X^+$. In this context we will always require that $gx_1 \geq gx_2$ whenever $g \in G$ and $x_1 \geq x_2$, without saying it explicitly. If X has such a cone X^+ , then we say that X^+ has *property \mathcal{K}* , and denote by $\mathcal{L}_{+, \text{inv}}(X, R)$ the set of all positive functions $L \in \mathcal{L}_{\text{inv}}(X, R)$.

Given $0 \neq L \in \mathcal{L}(X, R)$, $0 \neq L' \in \mathcal{L}(X, R)$ and u_0 in R , set

$$H := \{(x, y) \in X \times R: L(x) + L'(y) = u_0\}. \quad (1)$$

It is not difficult to check that the set H defined in (1) is empty or an affine manifold of $X \times R$ (see also [34]).



If A, B are two nonempty subsets of $X \times R$ and $H \neq \emptyset$ is as in (1), then we say that H separates A and B iff $A \subset H^-$ and $B \subset H^+$, where

$$H^- = \{(x, y) \in X \times R : L(x) + L'(y) \leq u_0\}, H^+ = \{(x, y) \in X \times R : L(x) + L'(y) \geq u_0\}.$$

The projection of $X \times R$ onto X is the function $P_X: X \times R \rightarrow X$ defined by $P_X(x, y) = x$ for every $(x, y) \in X \times R \rightarrow X$. Moreover, for any nonempty set $A \subset X \times R$, put

$$P_X(A) := \{x \in X : \text{there exists } y \in R \text{ with } (x, y) \in A\}.$$

It is not difficult to see that $P_X(A + B) = P_X(A) + P_X(B)$ for any two nonempty subsets $A, B \subset X \times R$.

Given a set $\emptyset \neq A \subset X \times R$, we call cone generated by A the set $C(A) := \{a \in X \times R : \text{there exist } \lambda \in \mathbb{R}_0^+ \text{ and } a' \in A \text{ with } a = \lambda a'\}$. It is not difficult to check that, if A is convex, then $C(A)$ is too, and that $A - B$ and $C(A - B)$ are G -invariant whenever A and B are G -invariant.

In proving our main results, we often will do the following assumption:

\mathcal{H}) let $U: D(U) \rightarrow R, V: D(V) \rightarrow R$ be two convex and G -subinvariant functions, where $D(U)$ and $D(V)$ are nonempty convex and G -invariant subsets of X with $\text{int}(D(U)) \cap \text{int}(D(V)) \neq \emptyset$,

and we will put $P_{U,V} := D(U) \cap D(V)$.

The G -invariant conjugate (shortly, conjugate) of U is the R -valued function U^c defined as

$$U^c(L) := \bigvee \{L(x) - U(x) : x \in D(U)\}, \quad L \in D(U^c), \quad (2)$$

where

$$D(U^c) := \{L \in \mathcal{L}_{\text{inv}}(X, R) : \bigvee \{L(x) - U(x) : x \in D(U)\} \text{ exists in } R\}, \quad (3)$$

exists in R ,

provided that $D(U^c) \neq \emptyset$. If $x_0 \in D(U)$, then we call G -invariant subdifferential (shortly, subdifferential) at x_0 the set $\partial_{\text{inv}} U(x_0)$ defined as

$$\partial_{\text{inv}} U(x_0) := \{L \in \mathcal{L}_{\text{inv}}(X, R) : L(x) - L(x_0) \leq U(x) - U(x_0)\}.$$

Any element $L \in \partial_{\text{inv}} U(x_0)$ will be called (G -invariant) subgradient of U at x_0 .

Given a nonempty set $A \subset X$ and $x_0 \in X$, we call G -invariant polar (in brief, polar) of A at x_0 the set

$$A_{\text{inv}}^*(x_0) := \{L \in \mathcal{L}_{\text{inv}}(X, R) : L(x) - L(x_0) \leq 0 \text{ for every } x \in A\}.$$

In the study of Fenchel-type duality theorem, we will consider the following programs (see also [45]).

Program I. Find $r := \bigwedge \{U(x) + V(x) : x \in P_{U,V}\}$ in R .

Program II. Find $s := \bigvee \{-U^c(L) - V^c(-L) : L \in D(U^c) \cap D(V^c)\}$ in R , provided that $D(U^c) \cap D(V^c) \neq \emptyset$.

3 THE MAIN RESULTS

We begin with the following sandwich theorem in the setting of invariance with respect to amenable semigroups of transformations and partially ordered vector spaces, extending [47, Sandwich Theorem 3.1]. Our technique is based on the existence of linear operators, not necessarily invariant, due to the corresponding classical results, and of suitable invariant partially ordered vector space-valued means (see also [15, Théorème 2]), from which it is possible to construct invariant linear operators.

Theorem 3.1. (Sandwich theorem) Let $U: D(U) \rightarrow R$ and $V: D(V) \rightarrow R$ satisfy assumption \mathcal{H} . Assume that $U(x) + V(x) \geq 0$ for every $x \in P_{U,V}$. Then there exist $L \in \mathcal{L}_{\text{inv}}(X, R)$ and $u_0 \in R$ with $L(x) - u_0 \leq U(x)$ for every $x \in D(U)$ and $L(x) - u_0 \geq -V(x)$ for each $x \in D(V)$.

Proof. By [47, Proposition 3.9 (b)], we get $0 \in \text{int}(D(U) - D(V))$. By [47, Sandwich Theorem 3.1], there are an element $u_0 \in R$ and a function $L^* \in \mathcal{L}(X, R)$ (not necessarily G -invariant) with $L^*(x) - u_0 \leq U(x)$ for every $x \in D(U)$ and $L^*(x) - u_0 \geq -V(x)$ for each $x \in D(V)$. Pick arbitrarily $x \in X$, and define $f_x \in l_b(G, R)$ by $f_x(g) = L^*(gx)$, $g \in G$. As R is Dedekind complete and G is right amenable, by [15, Théorème 2] there is a right G -invariant R -mean $m: l_b(G, R) \rightarrow R$. Set $L(x) := m(f_x)$, $x \in X$. Since $f_{hx}(g) = L^*(ghx) = f_x(gh) = (f_x)_h(g)$ for any $g \in G$, then $L(hx) = m(f_{hx}) = m((f_x)_h) = m(f_x) = L(x)$ for every $h \in G$, and hence L is G -invariant. As $D(U)$ and $D(V)$ are G -invariant, U and V are G -subinvariant, $L^*(x) \leq u_0 + U(x)$ for every $x \in D(U)$ and $L^*(x) \geq u_0 - V(x)$ for each $x \in D(V)$, then $f_x(g) = L^*(gx) \leq u_0 + U(gx) \leq u_0 + U(x)$ for every $x \in D(U)$ and $g \in G$, and hence

$$L(x) = m(f_x) \leq m(u_0 + U(x)) = m(u_0) + m(U(x)) = u_0 + U(x)$$

for any $x \in D(U)$. Analogously it is possible to prove that $L(x) \geq u_0 - V(x)$ for every $x \in D(V)$. Moreover, if $\lambda_1, \lambda_2 \in \mathbb{R}$, $x_1, x_2 \in X$ and $g \in G$, then

$$\begin{aligned} f_{\lambda_1 x_1 + \lambda_2 x_2}(g) &= L^*(g(\lambda_1 x_1 + \lambda_2 x_2)) = L^*(\lambda_1 g x_1 + \lambda_2 g x_2) = \\ &= \lambda_1 L^*(g x_1) + \lambda_2 L^*(g x_2) = \lambda_1 f_{x_1}(g) + \lambda_2 f_{x_2}(g), \end{aligned}$$

and therefore

$$L(\lambda_1 x_1 + \lambda_2 x_2) = m(f_{\lambda_1 x_1 + \lambda_2 x_2}) =$$



$$= \lambda_1 m(f_{x_1}) + \lambda_2 m(f_{x_2}) = \lambda_1 L(x_1) + \lambda_2 L(x_2).$$

Thus, $L \in \mathcal{L}_{\text{inv}}(X, R)$. This completes the proof.

The next step is to prove the following theorems and their equivalence with Theorem 3.1, extending to invariance [22, Theorems 1 and 2].

Theorem 3.2. (Fenchel duality theorem) *Under the assumption \mathcal{H} , suppose that $r := \Lambda\{U(x) + V(x) : x \in P_{U,V}\}$ exists in R . Then Program II. has a solution L_0 , and $-U^c(L_0) - V^c(-L_0) = r$.*

Theorem 3.3. *Under the assumption \mathcal{H} , let $x_0 \in P_{U,V}$ be a solution of Program I. Then*

$$\partial_{\text{inv}} U(x_0) \cap (-\partial_{\text{inv}} V(x_0)) \neq \emptyset.$$

Theorem 3.4. *Under the assumption \mathcal{H} , let $x_0 \in P_{U,V}$, $\partial_{\text{inv}} U(x_0) \neq \emptyset$ and $\partial_{\text{inv}} V(x_0) \neq \emptyset$. Then*

$$\partial_{\text{inv}} (U + V)(x_0) = \partial_{\text{inv}} (U)(x_0) + \partial_{\text{inv}} (V)(x_0). \quad (4)$$

Theorem 3.5. *Let $A_1, A_2 \subset X$ be convex G -invariant sets with $(\text{int}(A_1)) \cap (\text{int}(A_2)) \neq \emptyset$. Then*

$$(A_1)_{\text{inv}}^*(x_0) + (A_2)_{\text{inv}}^*(x_0) = (A_1 \cap A_2)_{\text{inv}}^*(x_0)$$

for every $x_0 \in A_1 \cap A_2$.

Theorem 3.6. (Hahn-Banach) *Let $U : D(U) \rightarrow R$ be convex and G -subinvariant, let $D(U)$ be G -invariant, $U(0) = 0$, and $Z \subset X$ be a G -invariant subspace. Assume that $\text{int}(D(U)) \cap Z \neq \emptyset$. Let $L_0 \in \mathcal{L}_{\text{inv}}(Z, R)$ be such that $L_0(z) \leq U(z)$ for every $z \in D(U) \cap Z$.*

Then L_0 has an extension $L \in \mathcal{L}_{\text{inv}}(X, R)$, such that $L(x) \leq U(x)$ for each $x \in D(U)$.

Theorem 3.7. *Let U and $D(U)$ be as in Theorem 3.6. If $0 \in \text{int}(D(U))$ and $U(0) = 0$, then there exists $L \in \mathcal{L}_{\text{inv}}(X, R)$, with $L(x) \leq U(x)$ for each $x \in D(U)$.*

Theorem 3.8. *Let U and $D(U)$ be as in Theorem 3.6. If $0 \in \text{int}(D(U))$ and $U(0) \geq 0$, then there is $L \in \mathcal{L}_{\text{inv}}(X, R)$, such that $L(x) \leq U(x)$ for each $x \in D(U)$.*

Theorem 3.9. (Krein) *Let $X^+ \subset X$ be a G -invariant cone satisfying property \mathcal{K} , $Z \subset X$ be a G -invariant subspace with the order generated by $Z \cap X^+$, such that $\text{int}(X^+) \cap Z \neq \emptyset$, and $L_0 \in \mathcal{L}_{+, \text{inv}}(Z, R)$. Then there is $L \in \mathcal{L}_{+, \text{inv}}(X, R)$, with $L(z) = L_0(z)$ for every $z \in Z$.*

Theorem 3.10. (Separation theorem) *Let A, B be two G -invariant subsets of $X \times R$ such that $C(A - B)$ is convex,*

$$\text{int}(P_X(A)) \cap \text{int}(P_X(B)) \neq \emptyset \quad (5)$$

and

$$y_1 \geq y_2 \text{ whenever } (x, y_1) \in A \text{ and } (x, y_2) \in B. \quad (6)$$

Then there exist $L \in \mathcal{L}_{\text{inv}}(X, R)$ and $u_0 \in R$ such that the affine manifold

$$H := \{(x, y) \in R \times X : L(x) - y = u_0\} \quad (7)$$

separates A and B .

Theorem 3.11. (Farkas) *Let $U : D(U) \rightarrow X$ be convex and G -equivariant (that is, $U(gx) = g(U(x))$ for every $g \in G$ and $x \in D(U)$, see also [13]) $V : D(V) \rightarrow R$ be convex and G -invariant, $D(U)$ and $D(V)$ be G -invariant, assume that $X_0 := D(U) \cap D(V) \neq \emptyset$, $0 \in \text{int}(U(X_0) + X^+)$ and*

$$V(x) \geq 0 \text{ whenever } x \in X_0 \text{ and } U(x) \leq 0.$$

Then there is $L \in \mathcal{L}_{+, \text{inv}}(X, R)$ with

$$V(x) + L(U(x)) \geq 0 \text{ whenever } x \in X_0.$$

Theorem 3.12. (Kuhn-Tucker) *Under the same hypotheses and notations as in Theorem 3.11, let x_0 be a solution of the problem*

$$P1) \text{ find } x_0 \in Z_0 := \{x \in X_0 : U(x) \leq 0\} \text{ such that } V(x_0) \leq V(x) \text{ for every } x \in Z_0.$$

Then there exists $L_0 \in \mathcal{L}_{+, \text{inv}}(X, R)$ such that (x_0, L_0) is a solution of the problem

$$P2) \text{ find } x_0 \in X_0 \text{ and } L_0 \in \mathcal{L}_{+, \text{inv}}(X, R) \text{ such that}$$

$$L(U(x_0)) + V(x_0) \leq L_0(U(x_0)) + V(x_0) \leq L_0(U(x)) + V(x), \quad (8)$$

for every $x \in X_0$ and $L \in \mathcal{L}_{+, \text{inv}}(X, R)$.

Theorem 3.13. *Let $X^+ \subset X$ be a G -invariant cone having property \mathcal{K} , $Z \subset X$ be a G -invariant subspace, $L_1 \in \mathcal{L}(Z, X)$, L_1 be G -equivariant, and $L_2 \in \mathcal{L}_{\text{inv}}(Z, R)$. Suppose that $L_1(Z) \cap \text{int}(-X^+) \neq \emptyset$, and*

$$L_2(x) \geq 0 \text{ whenever } x \in Z \text{ and } L_1(x) \leq 0. \quad (9)$$



Then there is $L_0 \in \mathcal{L}_{+,inv}(X, R)$ such that

$$L_0(L_1(x)) + L_2(x) = 0 \text{ whenever } x \in Z.$$

First of all, we prove the implications (3.1) \Rightarrow (3.2) \Rightarrow (3.3) \Rightarrow (3.6) \Rightarrow (3.7) \Rightarrow (3.8) \Rightarrow (3.1).

(3.1) \Rightarrow (3.2) We extend to invariance [45, Theorem 2]. Let $\tilde{U}(x) = U(x) - r$, where $r := \bigwedge \{U(x) + V(x) : x \in P_{U,V}\}$. By hypothesis, the element r exists in R . It is not difficult to check that \tilde{U} is convex, since U is. Moreover, $P_{\tilde{U},V} = D(\tilde{U}) \cap D(V) = D(U) \cap D(V) = P_{U,V}$. For each $x \in P_{U,V}$ it is $r \leq U(x) - V(x)$, and hence $\tilde{U}(x) + V(x) \geq 0$. Thus, \tilde{U} and V satisfy the hypotheses of Theorem 3.1. Therefore, there exist $L_0 \in \mathcal{L}_{inv}(X, R)$ and $u_0 \in R$ with $L_0(x) - u_0 \leq \tilde{U}(x) = U(x) - r$ for every $x \in D(U)$ and $L_0(x') - u_0 \geq -V(x')$ for each $x' \in D(V)$. From this we get

$$L_0(x - x') = L_0(x) - L_0(x') \leq u_0 + U(x) - r - u_0 + V(x') \tag{10}$$

and hence

$$r + L_0(x) - U(x) \leq L_0(x') + V(x') \tag{11}$$

for each $x \in D(U)$ and $x' \in D(V)$. Thus,

$$\begin{aligned} r + \bigvee \{L_0(x) - U(x) : x \in D(U)\} &\leq \bigwedge \{L_0(x) + V(x) : x \in D(V)\} \\ &= -\bigvee \{-L_0(x) - V(x) : x \in D(V)\}, \end{aligned}$$

and hence $L_0 \in D(U^c) \cap D(V^c)$ and $r \leq -U^c(L_0) - V^c(-L_0)$. Furthermore, observe that

$$-U^c(L) - V^c(-L) \leq -L(x) + U(x) + L(x) + V(x) = U(x) + V(x)$$

for every $x \in P_{U,V}$ and $L \in D(U^c) \cap D(V^c)$. Taking the infimum as $x \in P_{U,V}$ and the supremum as $L \in D(U^c) \cap D(V^c)$, we obtain

$$r \leq -U^c(L_0) - V^c(-L_0) \leq s := \bigvee \{-U^c(L) - V^c(-L) : L \in D(U^c) \cap D(V^c)\} \leq r.$$

Thus, $r = -U^c(L_0) - V^c(-L_0) = s$, and the supremum in Program II. is a maximum, attained by L_0 . So, the assertion follows.

Remark 3.14. Observe that, in general, the converse of Theorem 3.2 does not hold (see also [45]).

(3.2) \Rightarrow (3.3) We characterize the solutions of Program I. in terms of G -invariant subgradients, extending to invariance [45, Theorem 3]. Let x_0 be a solution of Program I. Then, by Theorem 2, Program II. has a solution L_0 , that is an element $L_0 \in \mathcal{L}_{inv}(X, R)$ with

$$U(x_0) + V(x_0) = -U^c(L_0) - V^c(-L_0). \tag{12}$$

By definition of the conjugate function, from (12) we get

$$U(x_0) + V(x_0) \leq U(x) + V(x') - L_0(x) + L_0(x') \tag{13}$$

for every $x \in D(U)$ and $x' \in D(V)$. From (13) used with $x = x_0$ and $x' = x_0$ we obtain $-L_0 \in \partial_{inv} V(x_0)$ and $L_0 \in \partial_{inv} U(x_0)$, respectively.

Conversely, let $L \in \partial_{inv} U(x_0) \cap (-\partial_{inv} V(x_0))$. Then for every $x \in P_{U,V}$ it is

$$U(x) - U(x_0) \geq L(x) - L(x_0) \geq V(x_0) - V(x)$$

and hence $U(x) + V(x) \geq U(x_0) + V(x_0)$. Thus, x_0 is a solution of Program I.

(3.3) \Rightarrow (3.6) We extend to invariance [47, Theorem 1]. Let Z, L_0, U be as in the hypotheses of Theorem 3.6. Set $V := -L_0$. Then $D(V) = D(L_0) = Z$. Since $L_0(z) \leq U(z)$ for each $z \in D(U) \cap Z$ and $L_0(0) = U(0) = 0$, then we get

$$0 = U(0) + V(0) = \min \{U(x) + V(x) : x \in D(U) \cap Z\}.$$

By Theorem 3.3 there is $L \in \partial_{inv} U(x_0) \cap (-\partial_{inv} V(x_0))$. We have

$$L(x) = L(x) - L(0) \leq U(x) - U(0) = U(x)$$

for each $x \in D(U)$ and

$$-L(z) = -L(z) + L(0) \leq -L_0(z) + L_0(0) = -L_0(z), \tag{14}$$

that is

$$L(z) \geq L_0(z) \tag{15}$$

for each $z \in Z$. Taking $-z$ in (14) (note that $-z \in Z$), we obtain



$$L(z) = -L(-z) \leq -L_0(-z) = L_0(z) \tag{16}$$

for every $z \in Z$. From (15) and (16) it follows that $L(z) = L_0(z)$ for all $z \in Z$. Thus, the assertion follows.

(3.6) \Rightarrow (3.7) It is enough to take $Z := \{0\}$ in Theorem 3.6.

(3.7) \Rightarrow (3.8) We extend to invariance [47, Corollary 2.6]. Set $\tilde{U}(x) := U(x) - U(0)$. Since $\tilde{U}(0) = 0$, \tilde{U} satisfies the hypotheses of Theorem 3.7. Therefore, there exists $L \in \mathcal{L}_{\text{inv}}(X, R)$ with $L(x) \leq \tilde{U}(x) \leq U(x)$ for every $x \in D(U)$, since $U(0) \geq 0$. This ends the proof.

(3.8) \Rightarrow (3.1) Proceeding similarly as in [47, Sandwich Theorem 3.1], let $\mathcal{D} := D(U) - D(V)$, $\mathcal{E} := \{(x_1 - x_2, U(x_1) + V(x_2) + y) \in X \times R : x_1 \in D(U), x_2 \in D(V), y \geq 0\}$, and $\mathcal{E}_x := \{y \in R : (x, y) \in \mathcal{E}\}$. Since U and V are convex, then \mathcal{E} is convex. Moreover \mathcal{E} is G -invariant, because $D(U)$ and $D(V)$ are G -invariant. Furthermore, note that $\mathcal{E}_0 \subset R^+$: indeed, if $y \in \mathcal{E}_0$, then $x_1 = x_2$, and hence $U(x_1) + V(x_2) + y \geq 0$.

Now define $p: \mathcal{E} \rightarrow R$ by $p(x) := \bigwedge \mathcal{E}_x$, $x \in \mathcal{D}$. We show that p is well-defined. Fix $x \in \mathcal{E}$. Since $0 \in \text{int}(\mathcal{E})$ (see also [47]), there is $\lambda \in]0, 1[$ small enough with $-\lambda x \in \mathcal{E}$. Pick $y' \in \mathcal{E}_{-\lambda x}$. Since \mathcal{E} is convex, then for every $y \in \mathcal{E}_x$ it is

$$\left(0, \frac{1}{1+\lambda}y' + \frac{\lambda}{1+\lambda}y\right) = \frac{1}{1+\lambda}(-\lambda x, y') + \frac{\lambda}{1+\lambda}(x, y) \in \mathcal{E},$$

that is $\frac{1}{1+\lambda}y' + \frac{\lambda}{1+\lambda}y \in \mathcal{E}_0$, and hence $\frac{1}{1+\lambda}y' + \frac{\lambda}{1+\lambda}y \geq 0$, since $\mathcal{E}_0 \subset R^+$. Thus, $\bigwedge \mathcal{E}_x$ exists in R , and hence p is well-defined.

Now we claim that p is convex. If $y_1 \in \mathcal{E}_{x_1}$, $y_2 \in \mathcal{E}_{x_2}$ and $\lambda \in [0, 1]$, then $(x_1, y_1) \in \mathcal{E}$, $(x_2, y_2) \in \mathcal{E}$, and by convexity of \mathcal{E} we get $(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \in \mathcal{E}$. Thus,

$$\lambda y_1 + (1-\lambda)y_2 \in \mathcal{E}_{\lambda x_1 + (1-\lambda)x_2}. \tag{17}$$

From (17) and arbitrariness of y_1 and y_2 we obtain

$$\bigwedge \mathcal{E}_{\lambda x_1 + (1-\lambda)x_2} \leq \lambda \bigwedge \mathcal{E}_{x_1} + (1-\lambda) \bigwedge \mathcal{E}_{x_2},$$

that is convexity of p .

Furthermore, note that $p(0) \geq 0$ because $\mathcal{E}_0 \subset R^+$, and, since $D(U)$ and $D(V)$ are G -invariant, we get $(gx, y) \in \mathcal{E}$ whenever $(x, y) \in \mathcal{E}$ and $g \in G$, namely $y \in \mathcal{E}_{gx}$ whenever $(x, y) \in \mathcal{E}$ and $g \in G$. Hence, $\bigwedge \mathcal{E}_{gx} \leq \bigwedge \mathcal{E}_x$ for each $x \in \mathcal{D}$ and $g \in G$. Thus, p is G -subinvariant.

By Theorem 3.8 there exists $L \in \mathcal{L}_{\text{inv}}(X, R)$ such that

$$L(x - x') \leq p(x - x') = \bigwedge \mathcal{E}_{x-x'} \leq U(x) + V(x'),$$

and hence

$$L(x) - U(x) \leq L(x') + V(x') \tag{18}$$

for any $x \in D(U)$ and $x' \in D(V)$. Set

$$u_0 := \bigwedge \{L(x') + V(x') : x' \in D(V)\}. \tag{19}$$

Note that $u_0 \in R$, since R is Dedekind complete. From (18) we get $L(x) - u_0 \leq U(x)$ for each $x \in D(U)$, and from (19) we obtain $L(x') - u_0 \geq -V(x')$ for every $x' \in D(V)$. Thus, the assertion follows.

Now we prove the implications (3.2) \Rightarrow (3.4) \Rightarrow (3.5) \Rightarrow (3.9).

(3.2) \Rightarrow (3.4) We extend [45, Theorem 4] to G -invariant subdifferentials. Choose arbitrarily $L' \in \partial_{\text{inv}}(U+V)(x_0)$, and set $V'(x) = V(x) - L'(x)$, $x \in D(V)$. It is not difficult to check that V' is convex. Moreover, by definition of subdifferential, it is

$$L'(x) - L'(x_0) \leq (U+V)(x) - (U+V)(x_0),$$

and hence

$$U(x_0) + V(x_0) - L'(x_0) \leq U(x) + V(x) - L'(x) = U(x) + V'(x) \tag{20}$$

for every $x \in D(U) \cap D(V)$.

Furthermore, it is

$$(V')^c(-L) = \bigvee \{L'(x) - L(x) - V(x) : x \in D(V)\} = V^c(L' - L)$$

for every $L \in D((V')^c) = D(V^c) - L'$.

By (20) and Theorem 3.2, there exists $L_0 \in \mathcal{L}_{\text{inv}}(X, R)$, with

$$U(x_0) + V(x_0) - L'(x_0) = -U^c(L_0) - (V')^c(-L_0) = -U^c(L_0) - V^c(L' - L_0),$$

that is



$$U^c(L_0) + V^c(L' - L_0) = L_0(x_0) - U(x_0) + (L' - L_0)(x_0) - V(x_0). \quad (21)$$

By definition of conjugate function, it is

$$U^c(L_0) \geq L_0(x_0) - U(x_0), \quad (22)$$

$$V^c(L' - L_0) \geq L'(x_0) - L_0(x_0) - V(x_0).$$

From (21) and (22) it follows that

$$U^c(L_0) = L_0(x_0) - U(x_0), \quad V^c(L' - L_0) = L'(x_0) - L_0(x_0) - V(x_0),$$

and hence

$$L_0(x) - U(x) \leq L_0(x_0) - U(x_0) \text{ for every } x \in D(U),$$

$$L'(x) - L_0(x) - V(x) \leq L'(x_0) - L_0(x_0) - V(x_0) \text{ for every } x \in D(V).$$

Thus, $L_0 \in \partial_{\text{inv}}(U)(x_0)$, $L' - L_0 \in \partial_{\text{inv}}(V)(x_0)$, and hence $L' \in \partial_{\text{inv}}(U)(x_0) + \partial_{\text{inv}}(V)(x_0)$. By arbitrariness of L' , it follows that

$$\partial_{\text{inv}}(U + V)(x_0) \subset \partial_{\text{inv}}(U)(x_0) + \partial_{\text{inv}}(V)(x_0).$$

The proof of the converse inclusion is straightforward.

(3.4) \Rightarrow (3.5) Let $A_1, A_2 \subset X$ be convex G -invariant sets with $(\text{int}(A_1)) \cap (\text{int}(A_2)) \neq \emptyset$, let $A_0 := A_1 \cap A_2$, and set $\mathcal{N}_{A_j}(x) := 0$ for each $x \in A_j$, $j = 0, 1, 2$. It is not difficult to see that

$$\partial_{\text{inv}} \mathcal{N}_{A_j}(x) = (A_j)_{\text{inv}}^*(x) \text{ for each } x \in A_j, j = 0, 1, 2, \quad (23)$$

and

$$\mathcal{N}_{A_0}(x) = \mathcal{N}_{A_1}(x) + \mathcal{N}_{A_2}(x) \text{ for each } x \in A_0. \quad (24)$$

Let now $x_0 \in A_0$ be fixed. From (4), (23) and (24) we get

$$\begin{aligned} (A_1)_{\text{inv}}^*(x_0) + (A_2)_{\text{inv}}^*(x_0) &= \partial_{\text{inv}} \mathcal{N}_{A_1}(x_0) + \partial_{\text{inv}} \mathcal{N}_{A_2}(x_0) = \\ &= \partial_{\text{inv}} (\mathcal{N}_{A_1} + \mathcal{N}_{A_2})(x_0) = \partial_{\text{inv}} \mathcal{N}_{A_0}(x_0) = (A_0)_{\text{inv}}^*(x_0). \end{aligned}$$

So, the assertion follows.

(3.5) \Rightarrow (3.9) We extend to invariance [47, Corollary 3.11]. Let X^+ , Z and L_0 be as in the hypotheses of Theorem 3.9. Let Z' be an *algebraic complement* of Z , namely a subspace of X with the property that every $x \in X$ can be uniquely represented as $x = z + z'$, where $z \in Z$, $z' \in Z'$. Such a subspace does exist, thanks to [29, Theorem 7.3.3]. Let us define $L_1: X \rightarrow R$ by $L_1(x) = -L_0(z)$, $x \in X$. Since $L_0 \in \mathcal{L}_{+, \text{inv}}(Z, R)$, then $L_1 \in (X^+ \cap Z)^*(0)$. By applying Theorem 3.5 with $x_0 = 0$, we find two functions $L, L' \in \mathcal{L}_{\text{inv}}(X, R)$ with $L_1(x) = L(x) + L'(x)$ for every $x \in X$, $L'(x) \leq 0$ for each $x \in X^+$ and $L(z) \leq 0$ for any $z \in Z$. Since $-z \in Z$ whenever $z \in Z$, it follows that $-L(z) = L(-z) \leq 0$, and hence $L(z) = 0$ for all $z \in Z$. If $L^* := -L'$, then $L^* \in \mathcal{L}_{+, \text{inv}}(X, R)$ and $L^*(z) = L_0(z)$ for every $z \in Z$.

The proof of (3.8) \Rightarrow (3.10) is given in [6, Theorem 3.4] (see also [47, Proposition 3.9]).

We now prove (3.10) \Rightarrow (3.6). Let Z be any G -invariant subspace of X , $L_0 \in \mathcal{L}_{\text{inv}}(Z, R)$, $U: D(U) \rightarrow R$ be a convex and G -subinvariant function, with $U(0) = 0$ and $L_0(z) \leq U(z)$ for every $z \in Z$. Let

$$\begin{aligned} A &:= \text{epi } U := \{(x, y) \in X \times R, y \geq U(x)\}, \\ B &:= \text{hypo } L_0 := \{(x, y) \in Z \times R, y \leq L_0(x)\}. \end{aligned}$$

It is not difficult to see that A and B are nonempty convex G -invariant subsets of $X \times R$, because U is convex and G -subinvariant, and L_0 is linear and G -invariant. The sets A and B are called the *epigraph* of U and the *hypograph* of L_0 , respectively. Moreover, we get $P_X(A) = X$, $P_X(B) = X$ and

$$\text{int}(P_X(A)) \cap \text{int}(P_X(B)) = \text{int}(X) \cap \text{int}(X) = X$$

(see also [21, Satz 1 (6)]). Furthermore, if $(x, y_1) \in A$, $(x, y_2) \in B$, then $x \in Z$, and $y_1 \geq U(x) \geq L_0(x) \geq y_2$.

By virtue of Theorem 3.10, there exist a function $L \in \mathcal{L}_{\text{inv}}(X, R)$ and an element $u_0 \in R$ with $L(x_1) - y_1 \leq u_0 \leq L(x_2) - y_2$ for each $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$. From this, choosing arbitrarily $x_1, x_2 \in X$ and taking $y_1 := U(x_1)$, $y_2 := L_0(x_2)$, we get

$$L(x_1) - U(x_1) \leq u_0 \leq L(x_2) - L_0(x_2), \quad (25)$$

Pick any $z \in Z$. Since $U(0) = 0$, from (25) used with $x_1 = 0$, $x_2 = z$ and with $x_1 = 0$, $x_2 = -z$, we obtain $L_0(z) \leq L(z)$, $-L_0(z) = L_0(-z) \leq L(-z) = -L(z)$, and hence $L(z) = L_0(z)$. Now, take any $x \in X$. From (25) used with $x_1 = x$ and $x_2 = 0$ we get $L(x) \leq U(x)$. This ends the proof.



The proofs of (3.10) \Rightarrow (3.11) and (3.11) \Rightarrow (3.12) are given in [6, Theorem 4.1] and [6, Theorem 4.2], respectively.

We now prove (3.12) \Rightarrow (3.13). Set $U := L_1$, $V := L_2$ and $x_0 = 0$. Note that $L_1(Z) \cap \text{int}(-X^+) = \text{int}(L_1(Z) \cap (-X^+)) \neq \emptyset$ (see also [22]). By (3.12) there is $L_0 \in \mathcal{L}_{\text{inv}}(X, R)$ with $L_0(L_1(z)) + L_2(z) \geq 0$ for each $z \in Z$. Thus we get (3.13).

We now prove (3.13) \Rightarrow (3.9). Let $L_1(z) := -z$ and $L_2(z) := L_0(z)$ for each $z \in Z$. It is readily seen that L_1 is linear and G -equivariant. From (9) we get $L_0(z) \geq 0$ for every $z \in Z \cap X^+$. By (3.13) we find a function $L \in \mathcal{L}_{+, \text{inv}}(X, R)$ with $L(-z) + L_0(z) = -L(z) + L_0(z) = 0$ for each $z \in Z$. Thus, (3.9) follows.

Now, let $\text{int}(R^+) \neq \emptyset$ and let us show (3.9) \Rightarrow (3.6). Let $U: D(U) \rightarrow R$ be a convex G -invariant function, with $D(U)$ G -invariant and $U(0) = 0$. Let $C := \text{epi } U := \{(x, y) \in X \times R, y \geq U(x)\}$. Since U is convex, G -subinvariant and $U(0) = 0$, it is not difficult to see that C is a G -invariant convex cone with vertex $(0, 0)$. So, in $X \times R$, we consider the order generated by the cone C . Moreover, since $\text{int}(R^+) \neq \emptyset$, by [22, Theorem 2] we get $\text{int}(\text{epi } U) \neq \emptyset$ and $(\text{int}(\text{epi } U)) \cap (Z \times R) \neq \emptyset$. Define the action of G on $X \times R$ by $g(x, y) = (gx, y)$, $(x, y) \in X \times R$. Let $L_0 \in \mathcal{L}_{\text{inv}}(Z, R)$ be such that $L_0(z) \leq U(z)$ for each $z \in D(U) \cap Z$. Define $L': Z \times R \rightarrow R$ by $L'(z, y) = -L_0(z) + y$, $(z, y) \in Z \times R$. It is not difficult to check that $L'(z, y) \geq 0$ whenever $(z, y) \in (Z \times R) \cap (\text{epi } U)$. By (3.9) there is $L'' \in \mathcal{L}_{\text{inv}}(X \times R, R)$ such that $L''(z, y) = L'(z, y) = -L_0(z) + y$ for each $(z, y) \in Z \times R$ and $L''(x, y) \geq 0$ for every $(x, y) \in \text{epi } U$. Thus, $L''(0, y) = y$ for any $y \in R$. Set $L(x) := -L''(x, 0)$, $x \in X$. It is not difficult to check that $L \in \mathcal{L}_{\text{inv}}(X, R)$. We get $L(x) = y - L''(x, y)$ for each $(x, y) \in X \times R$, and hence, when $y = U(x)$, we obtain $L(x) \leq y = U(x)$. Moreover, for every $z \in Z$ and $y \in R$, we have $L(z) = y - L''(z, y) = y - L'(z, y) = L_0(z)$. Thus, the assertion follows.

4 APPLICATIONS

Let \mathcal{T} be a partially ordered set, for instance the "time set". In [5] (see also the related bibliography, for a historical overview), we saw that an event can be viewed as a triple (E, t_0, T) , in which $t_0 \in \mathcal{T}$ is the present time instant, at which one has a suitable state of knowledge, E is a logical proposition and $\emptyset \neq T \subset \mathcal{T}$ is a set about which we know that, for each $t \in T$, either E is true at t or E is false at t . Note that, in general, it is not required that $t_0 \leq \wedge T$, because it is possible to examine not only the future, but also the past, for example when one does not have sufficient informations about. Thus, it is advisable to consider as "events" some sets of the type $\prod_{t \in T} B_t$, where the B_t 's are subsets of a fixed abstract nonempty set \tilde{R} and $B_t = \tilde{R}$ whenever t does not belong to T , where T is a suitable subset of \mathcal{T} . It is advisable to examine all subsets of B_t , without requiring, for example, that the B_t 's belong to fixed structures, like σ -algebras, and to construct finitely additive (not necessarily σ -additive) R -valued set functions. To this aim, we use the next theorem, which is a consequence of Theorem 3.1 (see also [7]).

Theorem 4.1. (Tarski theorem) *Let $\Omega \neq \emptyset$ be any set, $\mathcal{P}(\Omega)$ be the family of all subsets of Ω , \mathcal{A} be an algebra of subsets of G , $G \subset \Omega^\Omega$ be an amenable semigroup of functions, such that $g^{-1}(A) := \{\omega \in \Omega: g(\omega) \in A\} \in \mathcal{A}$ for every $A \in \mathcal{A}$ and $g \in G$, and $\psi: \mathcal{A} \rightarrow R^+$ be a finitely additive G -invariant measure. Then ψ admits a finitely additive G -invariant extension $\tilde{\psi}: \mathcal{P}(\Omega) \rightarrow R^+$.*

Proof. Let X be the space of all bounded real-valued functions defined on Ω and Z be the linear subspace of X generated by all characteristic functions χ_A , with $A \in \mathcal{A}$ (that is, $\chi_A(\omega) = 1$ if $\omega \in A$ and $\chi_A(\omega) = 0$ if $\omega \notin A$). For every $f \in X$ and $g \in G$, set $(gf)(\omega) := f(g(\omega))$. It is not difficult to check that the g 's are linear monotone homomorphisms, with respect to the above defined action of G on X . For each $f \in Z$, $f = \sum_{i=1}^n r_i \chi_{A_i}$, where $A_i \in \mathcal{A}$, $i = 1, 2, \dots, n$, put $\phi(f) = \sum_{i=1}^n r_i \psi(A_i)$. It is not difficult to see that ϕ is well-defined (that is does not depend on the representation of f), monotone, linear and G -invariant. Let $\phi^+: X \rightarrow R$, $\phi^-: X \rightarrow R$ be defined by

$$\begin{aligned} \phi^+(x) &= \bigwedge \{ \phi(z) : x \leq z, z \in Z \}, \\ \phi^-(x) &= \bigvee \{ \phi(z) : x \geq z', z' \in Z \}. \end{aligned} \tag{26}$$

Fix arbitrarily $x \in X$. By monotonicity of ϕ , for every $z, z' \in Z$ with $z' \leq x \leq z$ we get $\phi(z') \leq \phi(z)$. Taking the supremum and the infimum as in (26), we obtain $\phi^-(x) \leq \phi^+(x)$. Moreover, it is not difficult to check that $\phi^+(z) = \phi^-(z) = \phi(z)$ for each $z \in Z$.

We now prove convexity of ϕ^+ . Fix arbitrarily $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, and pick $z_1, z_2 \in Z$ with $x_i \leq z_i$, $i = 1, 2$. Then $\lambda x_1 + (1 - \lambda)x_2 \leq \lambda z_1 + (1 - \lambda)z_2$, and by linearity of ϕ it is $\phi^+(\lambda x_1 + (1 - \lambda)x_2) \leq \phi(\lambda z_1 + (1 - \lambda)z_2) = \lambda \phi(z_1) + (1 - \lambda)\phi(z_2)$. From arbitrariness of z_1 and z_2 we obtain $\phi^+(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda \phi^+(x_1) + (1 - \lambda)\phi^+(x_2)$, that is convexity of ϕ^+ . Analogously, it is possible to see that $-\phi^-$ is convex.

Now we claim that ϕ^+ is G -subinvariant. Fix arbitrarily $x \in X$ and $g \in G$. For every $z \in Z$ with $x \leq z$, we get $gx \leq gz$ and hence $\phi^+(gx) \leq \phi(gz) = \phi(z)$. Taking the infimum, we obtain $\phi^+(gx) \leq \phi^+(x)$, getting the claim. Analogously, it is possible to see that $-\phi^-$ is G -subinvariant.

By Theorem 3.1, there exist $\tilde{\phi} \in \mathcal{L}_{\text{inv}}(X, R)$ and $u_0 \in R$ with

$$\phi^-(x) \leq \tilde{\phi}(x) - u_0 \leq \phi^+(x) \tag{27}$$

for every $x \in X$. Since $\phi^+(0) = \phi^-(0) = 0$, from (27) used with $x = 0$ we obtain $u_0 = \tilde{\phi}(0) = 0$. Hence, $\tilde{\phi}(z) = \phi(z)$ for any $z \in Z$. Setting $\tilde{\psi}(C) := \tilde{\phi}(\chi_C)$, $C \subset \Omega$, we get that $\tilde{\psi}$ is the required R^+ -valued extension.

We now give our application of Theorem 4.1 to events.

Example 4.2. Let R be a Dedekind complete vector lattice, (\tilde{R}, \mathcal{B}) be a measurable space, where \mathcal{B} is an algebra of subsets of \tilde{R} , \mathcal{T} be any nonempty set, $\Omega := \tilde{R}^{\mathcal{T}}$ and set $\mathcal{D} := \{f \in \Omega : \text{there exist } t \in \mathcal{T}, B \in \mathcal{B} : f(t) \in B\}$. Note that \mathcal{D} is not an algebra, and that the algebra $\mathcal{A}(\mathcal{D})$ generated by \mathcal{D} is the family of all finite (disjoint) unions of sets of the type

$$E := \left\{ \bigcap_{i \in \Lambda} \{f \in \Omega : f(t_i) \in B_i\} : \Lambda \subset \mathcal{T} \text{ is finite} \right\} \tag{28}$$

(see also [5]). If $G \subset \mathcal{T}^{\mathcal{T}}$ is a semigroup of functions τ and $f \in \tilde{R}^{\mathcal{T}}$, then define the action of G on $\tilde{R}^{\mathcal{T}}$ by $(\tau f)(t) := f(\tau(t))$. It is not difficult to see that \mathcal{D} and $\mathcal{A}(\mathcal{D})$ are G -invariant.

Let $u \in R^+$, $u \neq 0$, and $P_t: \mathcal{B} \rightarrow R$, $t \in \mathcal{T}$, be a family of finitely additive G -invariant measures, such that $P_t(\emptyset) = 0$ and $P_t(\tilde{R}) = u$ for each $t \in \mathcal{T}$. We get that $\{P_t(B) : t \in \mathcal{T}, B \in \mathcal{B}\} \subset V[u] := \{x \in R : \text{there exists } c \in \mathbb{R}^+ \text{ such that } -cu \leq x \leq cu\}$. By Dedekind completeness of $V[u]$ and by the Kakutani representation theorem, there are a compact Hausdorff topological space Ξ and an isomorphism ι from $V[u]$ into $\mathcal{C}(\Xi) := \{f \in \mathbb{R}^{\Xi} : f \text{ is continuous}\}$, which maps u into the function 1_{Ξ} , defined on Ξ and which assumes the constant value 1 (see also [23,44]). As the P_t 's are equibounded, then they can be viewed as $V[u]$ -valued set functions. Put $P(E) := \iota^{-1}(\iota(P_{t_1}(B_1)) \cdot \iota(P_{t_2}(B_2)) \cdot \dots \cdot \iota(P_{t_q}(B_q)))$, for every set E defined as in (28). Note that, by construction, for such E 's we get $0 \leq P(E) \leq u$. If $\tilde{E} \in \mathcal{A}(\mathcal{D})$, $\tilde{E} = \bigcup_{i=1}^n B_i$, where B_i is as in (28), $i = 1, 2, \dots, n$, and the B_i 's are pairwise disjoint, then set $\tilde{P}(\tilde{E}) := \sum_{i=1}^n P(B_i)$. It is not difficult to check that \tilde{P} is a finitely additive G -invariant $V[u]$ -valued set function on $\mathcal{A}(\mathcal{D})$. Note that, if E is as in (28), then

$$P(\tau^{-1}(E)) = \iota^{-1}(\iota(P_{\tau(t_1)}(B_1)) \cdot \iota(P_{\tau(t_2)}(B_2)) \cdot \dots \cdot \iota(P_{\tau(t_q)}(B_q))).$$

By Theorem 4.1, \tilde{P} admits a finitely additive G -invariant $V[u]$ -valued extension \hat{P} , defined on the family $\mathcal{P}(\tilde{R}^{\mathcal{T}})$ of all subsets of $\tilde{R}^{\mathcal{T}}$, and in particular on the sets of the type $\prod_{t \in \mathcal{T}} B_t$, as B_t varies in the whole of $\mathcal{P}(\tilde{R})$ for every $t \in \mathcal{T}$ and does not belong necessarily to \mathcal{B} .

Some examples are when $\mathcal{T} = \mathbb{R}$, $a > 0$ and G is the group generated by the translation $\tau_a(t) := t + a$, $t \in \mathbb{R}$, or when $\mathcal{T} = \mathbb{N}$ and G is the group Σ of all permutations keeping fixed all but finitely many numbers, which is amenable but not abelian (see also [38]). Thus, in case of invariance, some (possibly) unknown data concerning some elements $t \in \mathcal{T}$ can be "replaced" by some known informations associated with other elements $t' \in \mathcal{T}$, when we study our "probabilities" P_t .

Some other applications associated with the group Σ , related to exchangeable processes, can be found, for instance, in [4] (see also [28,39]).

Remark 4.3. Observe that both Dedekind completeness and (right) amenability of G are not only sufficient, but also necessary conditions in order that the sandwich-type theorems hold (see for instance [43] and [7]).

5 CONCLUSIONS

We proved some versions of Hahn-Banach and sandwich-type theorem related to convex subinvariant operators, taking values in a partially ordered vector space R , in the setting of convexity, using a technique of construction of invariant finitely additive R -valued means, which allows to find invariant linear R -valued operators from linear operators which are not necessarily invariant, whose existence is guaranteed by the classical results. We deduced some Fenchel duality-type theorem, subdifferential formulas, Krein-type extension theorems, and showed that all these results are equivalent to separation, Farkas and Kuhn-Tucker-type minimization theorems. We used some similar classical results holding without invariance and a technique, by means to which it is possible to construct an invariant R -valued mean on all bounded R -valued functions defined on an amenable semigroup of homomorphisms and to get an invariant linear functional from a not necessarily invariant linear functional. We gave some applications to Tarski-type extension theorems for finitely additive invariant probability measures and to the study of families of "time-indexed" events, where the invariance allows to "replace" the study of the involved events connected with some periods of time on which one has not enough informations with corresponding events dealing with other periods of time about which one has a wider knowledge.

Several kinds of minimization problems, which have various applications in convex analysis, Calculus of Variations, subdifferential calculus, reconstruction of images, measure theory, probability and other branches of Mathematics, can be studied in this setting, by considering invariance or equivariance as particular constraints.

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Author biography with Photo

Antonio Boccuto was born in Catanzaro, Italy, on November 1964. He received the degree in Mathematics from the University of Perugia, Italy, in 1987, and the Ph.D. degree in Mathematical Analysis from the Mathematical Institute of the Slovak Academy of Sciences in Bratislava, Slovakia, in 2000. He received the habilitation in Mathematics to Associate Professor from the Comenius University in Bratislava in 2008. He has been a researcher in Mathematical Analysis at University of Perugia since the Academic Year 1991/1992. His research interests include Measure Theory and Integration, Real Analysis, Function Theory, Approximation Theory. He has published more than 120 papers on Mathematics in journals and conference proceedings and is a coauthor of two books on Measure and Integration Theory and Applications.

